(From TORELLI'S *Archimedes*, Oxford, 1792.)

ARISTIPPUS PHILOSOPHUS SOCRATICUS, naufragio cum ejectus ad Rhodiensium litus animadvertisset Geometrica schemata descripta, exclamavisse ad comites ita dicitur, **Bene Speremus, Hominum Enim Vestigia Video.** —VITRUV. Architect. Lib. 6, Praef.

RESEARCHES

ON THE

EVOLUTION OF THE STELLAR SYSTEMS

VOLUME II

THE CAPTURE THEORY OF COSMICAL EVOLUTION,

FOUNDED ON DYNAMICAL PRINCIPLES AND ILLUSTRATED BY PHENOMENA OBSERVED IN THE SPIRAL NEBULAE, THE PLANETARY SYSTEM, THE DOUBLE AND MULTIPLE STARS AND CLUSTERS AND THE STAR-CLOUDS OF THE MILKY WAY

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Dedicated to the Memory

of

Professor Simon Newcomb, U. S. Navy,

ILLUSTRIOUS AMERICAN ASTRONOMER,

Whose Half Century of Activity,
Consecrated to the Study of the Starry Heavens,
Shed Imperishable Lustre Upon the Service of His Country
and Contributed Greatly to the Advancement of
the Sublimest Portion of Human Knowledge.

CONTENTS.

Ταῦτα δὲ, Βασιλεῦ Γέλων, τοῖς μὲν πολλοῖς καὶ μὴ κεκοινωνηκότεσσι τῶν μαθημάτων οὐκ εὔπιστα φανήσειν ὑπολαμβάνω· τοῖς δὲ μεταλελαβηκότεσσι, καὶ περὶ τῶν ἀποστημάτων, καὶ τῶν μεγεθέων τᾶς τε γᾶς, καὶ τοῦ ἁλίου, καὶ τᾶς σελίνας, καὶ του ὅλου κόσμου πεφροντικότεσσι, πιστὰ διὰ τὰν ἀπόδειξιν ἐσσεῖθει. Διόπερ ὠήθιν καί τινας ὀυκ ἀνάρμοστον εἴη ἔτι ἐπιθεωρῆσαι ταῦτα.

I conceive that these things, King Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon, and the whole universe the proof will carry conviction. And it was for this reason that I thought the subject would be not inappropriate for your consideration.

— ARCHIMEDES, *Arenarius*, translated by T. L. HEATH.

INTRODUCTION.

THE work now offered to the public constitutes the second volume of the author's *Researches on the Evolution of the Stellar Systems*, of which the first appeared in 1896. Various circumstances have operated to retard the continuation of this work, such as the undeveloped state of the subject and the necessity of additional researches along several lines calculated to throw further light upon the Laws of Cosmical Evolution. Some delay has also arisen from the author's constant occupation with official duties and with other investigations in Theoretical and Practical Astronomy and the related branches of Natural Philosophy. But if this delay has been the means of introducing increased clearness, confidence and certainty into a subject heretofore involved in well-nigh impenetrable darkness, obviously it will not have been without some solid advantages to Science.

Heretofore so much has been written on Cosmical Evolution, and so little conclusively proved, by the establishment of rigorous criteria, based on the necessary and sufficient conditions required in mathematical discussions to ensure the validity of the reasoning, that not a few eminent mathematicians have despaired of ultimate success. And even those who have continued the search for a general Law of Cosmical Evolution have become less hopeful of finding a natural process sufficiently comprehensive to embrace within its scope the two principal types of systems observed in the actual physical universe. It has seemed scarcely admissible to postulate a Cosmical Process which would explain the planetary system, on the one hand, with its numerous very small bodies revolving in nearly circular orbits about greatly predominant central masses, and thus representing a development resulting in a mass-distribution which is essentially single; and the stellar systems, on the other, made up of double and multiple suns and having mass-distributions which are essentially double and multiple, and component stars revolving in orbits characterized by a wide range of eccentricity.

Under the circumstances it was not entirely obvious that a single Law could be discovered which would account for the development of Cosmical Systems of such remarkably opposite types. And but for the persistent labors of PROFESSOR SIR G. H. DARWIN, during the past thirty years, this effort might have been

largely or wholly abandoned. At any rate the efforts of this illustrious geometer and natural philosopher have been a constant source of inspiration to other investigators. For even when his results have turned out to be special rather than general in character, they have thrown a steady beam of light into some dark corner of the subject, because they rested on the *exact analysis of definite causes.*

The problems of Cosmical Evolution have now engaged the attention of the present writer for more than a quarter of a century; and this happens to have occurred at a time when great progress and rapid changes were taking place in nearly all branches of the Physical Sciences. To discriminate between results which were temporary and permanent, to recognize that which was mechanically sound and therefore observationally admissible, to distinguish true from deceptive appearances in the observations and photographs of celestial objects, and thus sift out the general tendencies from a multitude of special phenomena, have been far from easy problems. For in spite of much labor and research, by the most eminent mathematicians, even the origin of our own solar system has remained profoundly obscure; notwithstanding three centuries of telescopic exploration, since the time of GALILEO, with the accumulated results of many distinguished observers, from whose combined labors the planetary system is at length becoming well understood. Yet we should not, perhaps, be altogether surprised at this continued darkness and uncertainty respecting the processes of planetary genesis; for on the traditional conceptions handed down by LAPLACE, this hypothetical development of the solar system presented problems of extreme difficulty, arising from the incompleteness of the theory of gravitation. And although it is at length shown that LAPLACE's hypothesis is erroneous, it formerly was difficult to establish this defect, and still more difficult to suggest any substitute which was free from objections. Criticism which is merely destructive and not constructive is seldom effective; and therefore the darkness continued, in spite of the great importance of the subject for our general conceptions of Astronomy. So long as a solution of the problem was not forthcoming the Science of the heavens appeared at a distinct disadvantage.

As was pointed out in a recent address to the Astronomical Society of the Pacific, it has long been considered somewhat of a reproach to Astronomy that the processes of Cosmogony have remained so obscure that definite laws could not be established regarding even the mode of formation of the solar system, while still less was known about the laws for the development of other systems in space. In view of the great progress of the Physical Sciences since the time of LAPLACE, one is compelled to recognize that this criticism of the oldest and

most exact of the Physical Sciences is not wholly unjust and without a certain foundation. Not only has the failure of researches in Cosmogony effected Astronomy adversely, but it has also narrowed the field of effort in several of the related sciences. This should not, however, occasion surprise among those who study the history of the Physical Sciences. For as Cosmogony depends on the other Sciences for its fundamental data, any circumstance which has effected them adversely, would also retard the development of Cosmogony itself, and *vice versa.*

In addition to the natural difficulties inherent in the development of a complex and dependent Science like Cosmogony, another has arisen from the demoralization of spirits due to the disappointments of previous investigators. Those who have labored for years without gaining any satisfactory light on the subject may easily convince themselves that there are no definite laws of Celestial development; or imagine that such laws as exist are the outgrowth of various processes — under repulsive and even explosive forces, as well as under the more familiar attractive forces of gravitation. The investigator who has repeatedly failed will readily persuade himself that Nature has few, if any, simple laws. This mistaken tendency of many minds is strengthened and confirmed by the diversity of process apparently required to harmonize discordant phenomena; and unfortunately the love of novelty thus awakened often predominates over the simple love of truth.

Accordingly from these various causes it has come to pass that we have a multitude of theories, most of which have no foundation whatever, and should never have been advanced. The promulgation of theories wholly devoid of foundation is injurious to Science, and simply aids in the propagation of error. When nothing is known, however, almost any hypothesis which unites and harmonizes phenomena is philosophically justifiable, and may be of some value, at least for the time being; but when means exist for confronting hypotheses with observations, so as to establish contradictions with known phenomena, it is idle and vain to entertain any hypothesis which is not free from contradiction.

Heretofore most of the theories of Cosmical Evolution proposed have been involved in some inconsistency. Indeed most of them have been directly contradicted by obvious phenomena which admit of no dispute; and yet several of these baseless speculations have continued to circulate in publications of acknowledged scientific standing. The vague and chimerical theories put forth by persons of obscure mind merely serve to muddy the stream, so that very few can penetrate beneath the surface; and this leads a considerable body of observers to concentrate attention chiefly on superficial phenomena, which have the double fascination

that they are apparently obvious and at the same time may be grasped without much preliminary training and labor.

But sagacious investigators are not misled by such palpable deceptions. They are accustomed to look beneath the surface, and realize fully that hard work, often extending over many years, and far into every branch of the related Physical Sciences, is the price which must be paid for discoveries of real value. Until results can be obtained which admit of no contradiction, scrupulous workers prefer to wait for more light, and thus they have been able heretofore to make but few positive advances in developing a science of Cosmogony.

It was only after a secure observational foundation had been laid by the School of Alexandria, and tested by the mathematical criteria of the ancient geometers, that Astronomy took on the character of an exact science, which it has ever since maintained. If, with our vastly greater experience, we are able to follow the incomparable example of the Greeks, it will soon be possible to elevate Cosmogony to the rank of a real Physical Science. For my part I believe that such an advance can be made at the present time, if we properly utilize the results already accumulated by Astronomical research; but there doubtless are others of less sanguine temperament, whose faith is correspondingly doubtful. They hesitate to try the experiment in Cosmogony which TIMOCHARIS, HIPPARCHUS, and PTOLEMY carried to a successful conclusion in Astronomy. The triumph of the Geometry of ARISTARCHUS, APOLLONIUS and ARCHIMEDES will scarcely inspire the confirmed sceptic with confidence in the ultimate discovery of the Laws of Cosmical Evolution, because his scepticism is based on failure, and nothing but success will remove it.

The remarkable and somewhat unexpected success achieved by DARWIN about 1879, when he recognized the considerable part played by Bodily Tidal Friction in Cosmical Evolution, was shortly afterwards largely offset by the visible disappointment felt by POINCARÉ, over his now celebrated researches on the figures of equilibrium of rotating masses of fluid (*Acta Mathematica*, Vol. VII, 1885), which, as he justly remarked, are scarcely applicable to the solar system, because the ideal homogeneity necessarily assumed by the mathematician is essentially inconsistent with the heterogeneity characteristic of the hypothetical solar nebula. Nor was much encouragement to be derived from the complete despair of NEWCOMB, and the hopelessness of HALL. If a personal recollection may be recorded in this connection, it may be added that as far back as 1890, when the writer was a post-graduate student at the University of Berlin, NEWCOMB expressed himself in a letter as follows: "Your interest in the origin of the heavenly bodies is a very natural one, but I have little hope that any of these problems can

be definitely settled in our time. I have formerly given much attention to these questions myself, but have found the whole subject so unsatisfactory that I have entirely given it up."

Accordingly, important as were some of the concrete results already attained or then anticipated by the leading mathematicians, there was little in this situation twenty or twenty-five years ago to hold out a promising outlook to the young investigator. Still, in spite of a feeling of despair noticeable in the opinions of experienced and eminent astronomers, it always seemed to me that there was some hope left, and that years of labor might eventually be rewarded with discoveries of value.

For as a logical and necessary consequence of the establishment and verification of the law of universal gravitation, and the great development and perfection of that theory by the illustrious geometers of the past two centuries, it appeared quite clear that in all probability the next great historical problem to engage the attention of the student of the starry heavens would be a general Theory of Cosmical Evolution. Whether or not it could yet be satisfactorily solved, this was undeniably the great problem of the future; but twenty-five years ago not even an approximate solution had been attempted, and the whole process of Cosmical Evolution was involved in profound darkness. Moreover, it then seemed almost hopeless to anticipate any rapid clearing up of the subject, and few believed that solid progress in the solution of the problem ever would be achieved.

Fortunately while thus confronted with problems which often seemed utterly bewildering, the writer was able to remain more hopeful than many investigators of experience, as he labored incessantly for the introduction of principles giving greater uniformity and certainty in our Theories of Cosmogony. In the vigorous and wholesome enthusiasm of youth he was rash enough to share NEWTON's view that the ultimate laws of Nature are simple, and to feel that there surely must be a general law of Cosmical Evolution, if it could only be discovered. This investigation has now extended over a long time, but it has never seemed advisable that one should be hurried in a search after ultimate truth; for it is obvious that safety in the final conclusions is greatly to be preferred to immature results which could serve no lasting purpose. At length after twenty-two years of reflection and professional research these persistent efforts have been crowned with the discovery of a great and simple law of Nature, and the verification of its truth beyond my most sanguine expectations.

In these brief remarks it is not necessary to dwell at too great length upon the results established in this volume, but it will be seen, as was pointed out in

Astronomische Nachrichten No. 4308, and to the Astronomical Society of the Pacific, Jan. 30, 1909, that nearly all of our reasoning since the time of LAPLACE has been vitiated by a false premise, to the effect that the planets had been detached from the sun by acceleration of rotation, when the matter of this globe was originally expanded into a nebula filling their orbits, and rotating in equilibrium, under conditions of hydrostatic pressure, and that the satellites had been detached from the planets in the same way. All this reasoning, on the *shedding* of planets and satellites, under the supposed influence of the accelerated rotation of the relatively large central bodies which govern their motions, is now invalidated. And it is at last demonstrated, in more ways than one, but especially by means of the important criterion proposed by BABINET in 1861, and heretofore very generally overlooked, that the planets and satellites could never have been detached by rotation, with the existing moments of momentum, and must therefore have been captured and built up in a Resisting Medium revolving as a whirling vortex, and essentially devoid of hydrostatic pressure. In this way and in this way only can these small bodies have been formed, and their orbits reduced in size and transformed into such singular circularity.

The remarkable roundness of these orbits is a natural phenomenon of the first order of importance. And owing to the dominant influence exerted by this conspicuous circularity upon our fundamental conceptions throughout the whole History of Astronomy, it may not be lightly passed over, but obviously calls for an appropriate analysis. In the course of many centuries it has been successively remarked by PLATO and ARISTOTLE, HIPPARCHUS and PTOLEMY, COPERNICUS and TYCHO, KEPLER and GALILEO, NEWTON and LAPLACE, HILL and DARWIN. Indeed Nature's apparent preference for the circle, as illustrated by the paths of the planets, has been contemplated with wonder and astonishment from the earliest ages of Science. This property among the orbits of the five major planets known to the ancients was the principal circumstance leading to the development of the Ptolemaic System of Astronomy, which continued in use till the time of COPERNICUS. It was this singular circularity also that so greatly increased the labors of KEPLER in discovering and proving the elliptical law of the planetary motions. After several failures in tedious calculations based on observations of the bodies moving in the rounder planetary paths, this great astronomer at length found that Mars had an orbit which was sufficiently eccentric to enable him to demonstrate from the observations of TYCHO BRAHÉ that the path was a true ellipse, and not an eccentrically-placed circle, as had been supposed since the time of HIPPARCHUS.

The four large satellites of *Jupiter* discovered by GALILEO in 1610, were

soon found to exhibit in their orbital motions about that planet the same circularity that was already familiar in the case of the orbits of the major planets about the sun; and the tendency to perfect roundness in the paths of the satellites has so often been confirmed by other discoveries made during the past three centuries that this property has naturally attracted the attention of all philosophic observers. The planet *Uranus*, added to our system by HERSCHEL in 1781, along with *Neptune*, discovered from the theoretical researches of ADAMS and LEVERRIER in 1846, both preserve the striking circularity of orbit which originally called forth the admiration of the Greek geometers; and the orbits of the satellites of these remote planets have likewise proved to be perfectly circular, at least so far as may be judged from the finest observations made with the largest modern telescopes.

The *fact* of this wonderful roundness of the orbits of the principal planets and satellites excited the speculative curiosity of KEPLER and NEWTON; but at that early date the *cause* of the phenomenon necessarily remained very obscure. In due time it was noticed also with equal surprise by CLAIRAUT, EULER and LAGRANGE, but these great geometers have left us no record of their conclusions as to how the property arose. Together with the slight inclination and common direction of motion, the small eccentricity of the orbits was one of the principal points of departure in the celebrated researches of LAGRANGE, LAPLACE, and POISSON, on the stability of the solar system, with which the famous mathematicians of the 18th century were so greatly occupied.

Though all these illustrious geometers followed the Greeks in admiring Nature's apparent preference for the circle, no one of them attempted to explain the roundness of the planetary paths till LAPLACE promulgated his celebrated Nebular Hypothesis (cf. Exposition du *Système du Monde*, Note VII et DERNIÉRE, Paris, 1796), and accounted for this circularity of the orbits by a rotation of the central masses which had detached the attendant bodies quite gently and set them revolving in paths of correspondingly small eccentricity.

This explanation of LAPLACE naturally carried with it the great prestige of the illustrious author of the *Mécanique Céleste*, and has now been very generally accepted by astronomers for more than a century. And yet it may easily be proved to be untenable and altogether devoid of foundation. In fact it is demonstrated in this volume that the roundness of the orbits of the planets and satellites is to be ascribed to the secular action of the nebular resisting medium formerly pervading our solar system; and that these bodies have never been detached from the much larger central masses which now govern their motions, by acceleration of rotation, as was supposed by LAPLACE; but have all been captured, or added

from without, and have had their orbits reduced in size and rounded up under the secular action of the nebular resisting medium. It is now definitely proved that it was this resisting medium and nothing else which has given the paths of the planets and satellites that remarkably round form which has always been equally admired by the astronomer, the geometer and the natural philosopher.

Over two thousand years ago this beautiful property, still so characteristic of our system as fully explored by the most powerful telescopes of the present time, was remarked with admiration by ARISTARCHUS, APOLLONIUS, ARCHIMEDES, and other ancient geometers. It was likewise a subject of constant discussion among the natural philosophers of the classic period. But, although they were eloquent in their descriptions of the beauty and order of the Cosmos, and made this feature of the heavenly motions one of the leading doctrines in the schools of Athens and Alexandria; yet in general the Greek sages were content to follow PLATO in ascribing this property to a wise provision of the Deity, and it not unnaturally escaped their attention that there might be a profound physical reason for this almost perfect circularity of the planetary paths. In the light of modern researches on stability, it is worthy of remark that even at that early epoch, the circularity of the orbits was held to be very effective in securing the order of the system of the world, which some of the ancient philosophers believed destined to endure forever, and others for a very long period. This circularity was also the principal circumstance leading to the final adoption of the ancient system of eccentrics and epicycles, which continued in unbroken usage for fourteen centuries. It has exercised an equal fascination over the minds of the greatest modern geometers — NEWTON, EULER, LAGRANGE, LAPLACE, POISSON, GAUSS, HANSEN, NEWCOMB, HILL, DARWIN, POINCARÉ — but has always been erroneously explained on LAPLACE's original hypothesis, of a gradually accelerated rotation which detached these bodies quite gently from the central masses which now govern their motions.

Accordingly, it should not seem altogether unexpected if the discovery of the true physical cause of the circularity of the orbits of the planets and satellites should give us also the great secret of the development of our solar system. Nor would it appear inappropriate if such a long-sought discovery should prove of particular interest to the astronomer, the geometer, and the natural philosopher, each of whom has been especially favored by circumstances arising from the circularity of the planetary paths.

For in past time the labors of the practical astronomer have been greatly diminished by this simple and advantageous arrangement of the System of the World. This favorable circumstance of the small eccentricity, together with

the slight inclination of the orbits and common direction of motion, was also the essential basis of the theoretical researches of LAGRANGE, LAPLACE, and POISSON, on the Stability of the Solar System, which constitutes one of the most splendid achievements of the great mathematicians of the eighteenth century.

The pure geometer, in turn, has drawn on the circular movements of the heavenly bodies to illustrate problems in the various branches of his recondite analysis. It thus almost appears, as was long ago suggested by the Greeks, that the attention of the Deity is given to the illustration of the Science of Geometry.

Finally, the natural philosopher still sees in the planets and satellites, revolving in the depths of space, the most beautiful and magnificent models of the invisible systems of the molecules, atoms and electrons, which are introduced to account for the physical properties of matter. May we not hope that he will also gain from the processes of resistance here disclosed some conception of possible modes of molecular transformation, decay and change, which heretofore have remained so deeply mysterious?

The principle resulting from the discovery and verification of the processes of spiral development, under the action of a Resisting Medium, will not, however, prove to be more useful in the interpretation of the phenomena of molecular physics and of our solar system than in that of other systems observed in the sidereal universe. It has just been remarked, not without some measure of justice, that an important premise handed down from the days of LAPLACE, and then believed to be correct, but now shown to be false, has operated to retard our progress in Cosmical Evolution for nearly a hundred years; but in the early part of the 19th century, the time was not yet ripe for the throwing off of this fatal delusion. The results which follow from the correction of this false premise will contribute greatly to the progress of the Physical Sciences, and are sufficiently striking to be worthy of the attention of the natural philosopher.

To the investigator of the phenomena of the Physical World, no question is ever more important than that of the *correctness of the premises.* This conclusion is deserving of more emphasis than it has heretofore received. The remarkable generalization and improvement in our Theories of Earthquakes, Mountain Formation and kindred phenomena connected with the Physics of the Earth, established in four memoirs recently published in the Proceedings of the American Philosophical Society held at Philadelphia, are sufficiently impressive to afford us tangible proof of the wisdom of making the *strictest examination of the premises underlying our reasoning.* This point is too often overlooked by those who confidingly follow beaten paths.

One of the most remarkable but inevitable results now clearly established,

is the great and even paramount part played by the resistance of the Nebular Medium in the development of Cosmical Systems. *This will hereafter give the Theory of the Resisting Medium the highest importance in all researches relating to the History of the Universe.* Heretofore this subtile physical cause, which has left so profound an impress upon the size and shape of the orbits of the planets and satellites, has scarcely been thought of in connection with the origin of the solar system or of other similar systems existing in space. And yet there still exists in the anomalous motions of the satellites of *Mars*, *Jupiter*, and *Saturn* the most decisive evidence of the capture of these bodies under the action of a Resisting Medium, which has left unmistakable *survivals* to bear witness to the mode in which these systems were formed.

Tidal Friction, as developed by PROFESSOR SIR G. H. DARWIN, has been the only generally recognized cause which might have modified, in any appreciable degree, the orbits of the heavenly bodies. And as the laborious researches on Comets, carried out by various astronomers, but more especially by ENCKE, WINNECKE, VON HAERDTL, MÖLLER, BOHLIN, and BACKLUND, showed but very slight or insensible resistance in the solar system, at the present time, it was natural that the Secular Effects of a Resisting Medium as an agency of transformation in Cosmical Evolution should have been largely or wholly lost sight of. For with the theories then current no specific illustrations of the secular effects of the continued operation of a Resisting Medium readily occurred to the mathematician. *Hereafter the extreme circularity of the orbits of the planets and satellites will stand as an everlasting witness to the Secular Action of this great Physical Cause.*

The results here brought out are too far reaching to justify any present attempt at estimating the relative cosmical importance of the two antagonistic forces which are always at work in stellar systems: namely, a Resisting Medium decreasing the major axis and eccentricity of the orbits; and Tidal Friction, expanding and elongating these orbits, as was long ago shown by PROFESSOR SIR G. H. DARWIN, (Phil. Trans, and Proc. Roy. Soc., 1878–1882), and pointed out in my *Inaugural Dissertation* at the University of Berlin in 1892, and still further emphasized in the first volume of this work, with especial reference to the Double and Multiple Stars.

To have continued to overlook the great modifications produced by the Secular Action of the Resisting Medium, which is always to be conceived as composed of *cosmical dust, true physical matter, not ether*, would have left a very grave defect in our Theories of Cosmogony. It is fortunate, therefore, that in the second volume of this work we are able to recognize clearly the two antagonistic causes in their true relationship. Detailed treatment at present would be inadvisable,

and therefore we must leave to the future the further development of the Secular Effects of these two great physical causes, which have left a profound impress upon the size and shape of the orbits of all cosmical systems.

This second volume will be of especial interest to the thoughtful reader, from the light it throws upon the problem of the development of the planetary system, the solution of which has been one of the ultimate objects of Physical Science from the earliest ages; and from the enlarged philosophic view it affords us of the millions of such similar systems, with habitable planets, which may now be confidently inferred to exist in the immensity of space. Nor will the light shed upon the constitution and laws of movement in clusters be less acceptable to the student of the starry heavens. For in all probability, we could not have derived this knowledge of the spiral movements of clusters empirically, from direct observation, in less than a thousand years. The vastly greater cosmical vortices whirling with intricate helical motion in the Magellanic Clouds and in the larger Cloud Masses of the Milky Way, could hardly be disclosed by direct measurement in even a million years; and yet the modern photographic plate, by recording the *continuous lines of nebulosity* along which the streams are moving, may reveal to us the true character of these cosmical vortices, as seen in projection among the stars. Such unexpected revelations seem truly appropriate to this *saeculum mirabile*. Thrice happy those who have lived to see the light of this glorious day! An ephemeral mortal, dwelling on a tiny planet attached to a sun of inferior magnitude, may survey the uncounted millions of similar suns scattered throughout the immensity of space, and trace the progress of an orderly development extending over immeasurable æons of time!

The past ten years have brought to light the most conclusive evidence of the prominent part played by Repulsive Forces in Nature. The delicate Laboratory experiments, independently made by LEBEDOW, NICHOLS and HULL, have established the radiation pressure of light on an observational basis, and this result has been confirmed by the further theoretical researches of ARRHENIUS, SCHWARTZSCHILD, and others, along the lines originally pointed out by MAXWELL in 1873, and by BARTOLI in 1876. In fact, the radiation pressure due to the waves of light had been foreseen by the illustrious EULER as early as 1746, but the theory of this great mathematician was strongly opposed by DE MAIRAIN and others, and nearly a century and a half elapsed before it could be satisfactorily confirmed. Moreover, the observational researches now being made by various astronomers on the visible repulsion from the Sun of the matter in the tails of comets, are continually adding specific examples of cosmical repulsion based on the exact measurement and photography of celestial objects. It is worth while to recall

that the theory of the repulsion of the tails of comets was first announced by KEPLER in 1618, and thus it has taken nearly three centuries for the doctrine of Repulsive Forces to become well established.

But since we now have positive proof of the Repulsion of particles of matter due to Light and Electric Forces, our theory of the Nebulae would be wholly incomplete without some suggestion as to how these masses arise. If, then, very finely divided matter is continually being expelled from all well developed stars, and the particles becoming coarser by the recognized process of the precipitation of ions in the celestial spaces, it manifestly suffices to imagine these fine particles drifting hither and thither throughout the universe, and here and there forming immense clouds of cosmical dust. Thus have the nebulae arisen. And the forms and distribution of these cosmical clouds are to be explained on the same principles. The condensation of this cosmical dust forms stars, and its subsequent expulsion from the maturer stars again forms nebulae. In certain regions this matter will inevitably gather into fairly dense streams, and the meeting or simply the gravitational settling and coiling up of these streams of cosmical dust is what forms the spiral nebulae, and leads to the development of cosmical systems.

It always happens that new views lead to new problems, and *vice versa*, and these may eventually modify profoundly all our former conceptions of the universe. Thus it is manifest that the several new lines of thought struck out in this work may suggest many difficult problems which will call for detailed mathematical treatment by the future investigator. But until an outline of the theory as a whole is laid before the public and its essential truth clearly recognized, these investigations cannot be considered very urgent, and the treatment of such special problems must therefore be left to the future. These researches, for example, might deal with the potential of a nebulous coil of uniform density, given dimensions and constant curvature; the stability of segments of a spiral nebula; certain special cases of the problem of three bodies; the processes of growth in the building up of nuclei in such a nebula; the rates of change in the eccentricity of the orbit when the law of density of the resisting medium varies; the probability of a figure of equilibrium developing at a center, through the settling or casual impact of two streams of cosmical dust; the conditions required for the rupture of figures of equilibrium of rotating masses of fluid, etc., etc.

But such details, important as they are sure to become in time, scarcely come within the scope of the present volume, the aim of which is to lay a secure foundation, rather than to treat any one of the many problems suggested by the general theory. Our knowledge is not yet sufficiently ripe nor is there urgent

demand for the treatment of these special problems. What we need at present is not mathematical details of the treatment of individual problems, but correct outlines for the development of a rational and consistent Theory of Cosmical Evolution. When we have adequate understanding of the great laws of Nature to enable us to formulate a problem in definite terms it will not be so difficult to develop the required mathematical solution.

If the views here adopted are admissible it is clear that heretofore we have not even had a correct foundation on which to build. And hence the tracing of such a general outline is at present the most urgent desideratum of Science. The conspicuous failures of the past century, so often directly attributable to *false premises underlying a great body of reasoning*, afford a sufficiently impressive warning of the worthlessness of elaborate structural details, when the theory itself reposes on insecure grounds. Until a good foundation can be laid it is idle to dwell on the details of the superstructure.

Different persons will naturally form different estimates of the importance of the several lines of research now being prosecuted by astronomers throughout the world, and it is not well to be too intolerant of labors which seem to serve no present purpose. In some cases unexpected discoveries will follow from the simplest observations, and thus the whole trend of our thought may be greatly changed. But on the other hand, when we contemplate the vast masses of data now being accumulated by observers, and consider how difficult it will be to deduce from these records any generalizations which will aid the mind in grasping the problems of the Universe, and remember how completely such work absorbs the energies of all our observatories, the greatest as well as the smallest, one can scarcely avoid some distrust of present tendencies. We are confessedly slaves to tradition and precedent. We go on accumulating records with little regard to their present or future use, and we rarely try to use wisely those already accumulated; so that one eminent astronomer has well said that we are in danger of being crushed under the load of our observations.

One true principle gives unity and mental connection to millions of isolated facts, and it is only by means of such principles that the observed facts can be interpreted. Why not therefore give a little more attention to the discovery of principles? All the important epochs in the past history of science have been made in this way; yet this very tendency, to the development of new conceptions and the introduction of new physical laws, is the one which to-day is least encouraged. Few are supported or upheld in breaking away from the leading strings of tradition. Journals and Learned Societies are nearly all ultra conservative, and very timid about entertaining new thought. It is only daring

individuals, not aggregations of men, who have the courage to lead the way. Under the circumstances can any one be surprised that years, decades, and even centuries pass by without giving birth to one grand principle, one new physical law?

Perhaps some one will reply by repeating the well-known fallacy that the age of great discoveries has now passed; that the important discoveries of the future are to be sought in the fifth place of decimals. But in view of the discoveries in Radium and Molecular Physics made within the past ten years, such argument is not likely to make much of an impression on the modern student. Improvement in the values of our Astronomical and Physical Constants is undoubtedly important, but with vast domains of Astronomy totally unexplored, we are not justified in spending all our efforts in hair-splitting over the fifth decimal place, or in cataloguing a few more thousand of the millions of stars which stud the firmament.

It is the settled conviction that we have largely neglected our opportunities to advance the true science of the heavens, such as KEPLER, GALILEO, and NEWTON would have thought worth while, which has led to most of the advances set forth in this work. Discoveries are seldom made by those who follow beaten paths. These well trodden grounds are inviting, to be sure, and always paths of least resistance; but for this very reason not to be chosen by the sagacious and experienced explorer.

Having discovered and proved the true process by which our solar system was formed and the stellar systems greatly modified, it is impossible for the writer to entertain any doubt of the importance of the part played by the Resisting Medium in the past history of the Universe. That we should not have seen this result before will be one of the wonders of the future. But let us not go to the extreme of ascribing too great an influence to the effects of resistance.

Although the presence of resistance surpassing the effects of tidal friction produces round orbits, yet we are not on that account justified in concluding that the absence of resistance would leave the orbits nearly parabolic, and thus point to the origin of binaries from the accidental approach of single stars. The argument of POINCARÉ, carefully set forth in this volume, against the frequent close approach of separate stars, *when contrasted with the observed great abundance of double stars*, is complete and unanswerable. Beyond doubt, therefore, the double stars as a class have originated from the division of nebulae. This may arise sometimes from starlike condensation in the coils of spiral nebulae, under resistance, as in the formation of the planets, but we may still believe that it is frequently aided by the process of fission indicated by the researches of mathematicians on the

figures of equilibrium of rotating masses of fluid. To what extent the two processes are involved together in spiral nebulae must be left to the future to determine. At present it is sufficient to know that double stars arise from the development of nebulae, and not from the accidental approach of separate stars.

Accordingly, although we must at present be contented with a mere outline of Nature's mighty processes, we may feel sure that future progress will confirm and extend the laws here brought to light. And with a secure foundation already laid and confirmed both by mathematical theory and by the observation of stellar systems, nebulae, and clusters, in the most remote regions of the firmament, one may not hesitate to believe that Cosmical Evolution will soon take its place among the exact sciences.

The photographs of nebulae and clusters made by ROBERTS, KEELER, BARNARD, RITCHIE and PERRINE, and carefully reproduced in this volume, are sufficiently remarkable to be deserving of more than passing notice. Though the discovery of the prevalence of the spiral type among the nebulae is due chiefly to ROBERTS and KEELER, neither of these distinguished astronomers has explained satisfactorily how the spiral form originates. The author is not aware that any general theory of the spiral nebulae at all similar to that here outlined has been offered before, and yet he cannot doubt that it embodies the substance of an ultimate truth. The photographs afford durable and convincing illustrations of the formation of whirlpool nebulae by the *automatic winding up of two or more streams of Cosmical dust*. Indeed two such streams cannot meet or a single stream settle to equilibrium without giving rise to rotation about a center, and thus producing a whirling vortex, which eventually leads to the development of a Cosmical System.

It is difficult to imagine a simpler or more direct explanation than the one here adopted of the spiral nebulae. By studying the photographs of these remarkable objects, in connection with the known laws of motion, in such whirling systems, as brought out in this volume, we are enabled to detect various stages in the processes of Cosmical Evolution. The visible whirling of these spiral nebulae, along with the lines of spiral movement clearly traceable in certain nebulous clusters, afford overwhelming testimony to the truth of the New Theory; and the sublime spectacle presented by millions of these splendid Cosmical Vortices shining in the remotest regions of the firmament, offers the most incontestible proof of the continuation of Nature's Mighty Creative Processes. The theoretical possibility of such atomic whirlpools was dimly foreseen by LEUCIPPUS, DEMOCRITUS, and other natural philosophers among the Greeks, but the Theory was never before so beautifully illustrated as in these photographs of the gigantic systems now observed in the actual Physical Universe.

The simple theory here developed may hereafter require some extension or modification to explain phenomena not yet brought to light, but obviously we are not justified in introducing the consideration of exceptions till the theory itself is established as a whole. After the general truth is clearly recognized it will be time to consider what apparent exceptions may arise under certain special conditions.

Professor Sir G. H. Darwin has justly remarked that "the problems of Cosmical Evolution are so complicated that it is well to conduct the attack in various ways at the same time. . . . Although the several theories may seem to some extent discordant with one another, yet, as I have already said, we need not scruple to carry each to its logical conclusion." Thus we have chosen to carry this simple theory to its logical consequences, without denying that slight modifications may some day be required, but holding that in all probability they will be in degree and not in kind. A Theory of Cosmical Evolution which explains satisfactorily so many phenomena of the heavens may justly lay claim to the designation universal.

Not the least remarkable among the several interesting results which follow from the New Theory of the Origin of the Solar System is the conclusion that several unseen planets revolve beyond Neptune, some of which may be bright enough to be discovered by photography. At that great distance, however, the orbital motion will be very slow. And long exposures will be required both because of this slow motion and because of the faintness of the light, due to the probability that the discs may be smaller than that of *Neptune*, and the increased feebleness of the sun's light at the great distance of these unseen planets. The extreme circularity of *Neptune's* path points unmistakably to a resisting medium once sufficiently dense to have greatly reduced the size of that orbit and to have transformed it into an almost perfect circle; and therefore it is impossible to believe that the solar system terminates at the present known boundary. It is more likely to be found to extend to a distance approximating 100 astronomical units. A diligent and persistent search for unseen planets beyond *Neptune*, therefore offers the prospect of important discoveries, and cannot be too strongly commended to the attention of astronomers.

In the course of this volume several topics are touched upon incidentally, such as the probable spiral movement of molecules, atoms and electrons, which will be of interest to a wider range of workers in Natural Philosophy, of which Astronomy, Cosmical Physics, and Cosmical Evolution may be regarded as the sublimest parts.

During the progress of this investigation the author has had the advantage

of friendly suggestions from a number of eminent astronomers: Sir William Huggins, Professors G. W. Hill, Suante Arrhenius, Karl Bohlin, G. V. Schiaparelli, C. L. Doolittle, A. O. Leuschner, R. T. Crawford, and Mr. P. H. Cowell, regarding certain points of theoretical interest; Professors E. C. Pickering, E. E. Barnard, Max Wolf, G. W. Ritchie, C. D. Perrine, and W. W. Campbell, in relation to the photography of the nebulae—an observational inquiry in which the Director of the Lick Observatory courteously afforded opportunity for a careful examination of the whole of the unsurpassed collection of nebular photographs taken with the Crossley Reflector at Mt. Hamilton; Professors George Davidson, and Charles Burkhalter, and Mr. Otto Von Geldern, of San Francisco, and Mr. A. E. Axlund, who have given several timely suggestions. He is deeply indebted also to President Benjamin Ide Wheeler, Professors Irving Stringham, and M. W. Haskell, and Librarian J. C. Rowell, of the University of California, for the freedom of the University during the past five years, and for the kindly proffer of every facility which would contribute to the completion of this investigation. The generous spirit manifested at the University of California in the promotion of Scientific Inquiry is beyond all praise.

While all of these gentlemen and others have contributed something to the completion of the Theory here developed, the author alone is responsible for the somewhat radical departure from the beaten paths heretofore followed by most investigators, and for the defects which the published Theory may be found to contain. If it is confirmed by time and experience these unavoidable defects will not wholly obscure the considerable advance now made in our knowledge of the Geometry of the Heavens. For I venture to believe that the theory of the evolution of the orbits of the stars is Celestial Geometry of the highest interest, and will constitute a useful application of the most beautiful of the Mathematical Sciences. This seems all the more appropriate, because it was the study of the Spiral of Archimedes, as treated in the splendid edition of Torelli, 1792, which first suggested to me the idea that the spiral of the celebrated Geometer of Syracuse might find application to the figures of the nebulae, and thus led to the development of the new theory of the spiral nebulae here set forth.

Finally, it is only just to record my deep sense of obligation to Mrs. See, without whose approval, constant support and unfailing interest, this volume could hardly have been brought to a conclusion and published.

Blue Ridge on Loutre,
Montgomery City, Missouri,
May 6, 1909.

CHAPTER I.

Methods for Finding the Orbits and Spiral Paths which a Particle may Describe under Central Forces Varying as Some Power of the Distance.

§ 1. *General Considerations on the Theories of Cosmogony.*

Since the discovery of the laws of the planetary motions by Kepler (1619), the establishment of the law of universal gravitation by Newton (1687), and of the velocity of light by Roemer (1675), it has been generally believed that the greatest physical laws operate uniformly throughout the sidereal universe. Accordingly, it is not remarkable that hope has been entertained by more than one daring investigator of finding a general law of cosmical evolution. If such a cosmic process exists and is really universal in character, it should explain the development of the solar system, with its numerous planets and satellites, all revolving in nearly circular orbits about central bodies of greatly predominant mass, and at the same time account in a simple and natural way for the double and multiple stars and clusters, systems of a wholly different type, made up apparently of double and multiple suns, the luminous components of which are always equal or comparable. In the solar system the mass distribution is essentially single, for nearly all of the matter of the primordial nebula has gone into the central bodies, leaving the attendant planets and satellites relatively insignificant; whereas among the double and multiple stars disclosed by the telescope and spectroscope in the immensity of space, the mass distribution is correspondingly double and multiple. Such a fundamental difference between the types of systems now known to constitute the order of the universe may well excite the wonder of the natural philosopher.

The problem offered to our contemplation of explaining, if possible, such varied phenomena by a single physical law is undoubtedly stupendous. Indeed it has been thought by some to be one of the most difficult which has ever engaged the attention of the human mind. And since our efforts for laying a secure foundation for the Science of Cosmogony have been only partially successful, many investigators have become less hopeful of obtaining a general solution of the problem, while not a few apparently have given up the expectation of discovering

any really fixed and definite laws. Indeed this feeling of despair has been publicly expressed recently by one of the greatest mathematicians of the age. Owing to the wholly unsatisfactory outcome of the careful attention which he formerly gave to the subject, PROFESSOR NEWCOMB "still retains a little incredulity as to our power in the present state of science to reach even a high degree of probability in Cosmogony" (cf. *Popular Astronomy*, November, 1906, p. 572). Views almost identical with these were expressed by PROFESSOR NEWCOMB to the present writer in 1890, so that recent progress has not been such as to inspire him with confidence in our ability to penetrate the mystery which surrounds the origin of the planetary system. This despair was naturally somewhat augmented under the disappointment felt by POINCARÉ* and DARWIN† in their brilliant efforts to test the nebular hypothesis of LAPLACE by the calculation of the figures of equilibrium of a mass of fluid animated by rotation and subjected to the pressure and attraction of its parts. Both of these eminent investigators had found the detached mass relatively so very large that the results evidently were inapplicable to the conditions which obtained in the formation of our solar system.

While the later efforts of the present author to explain the genesis of stellar systems from nebulous masses by means of the figures of equilibrium calculated by POINCARÉ and DARWIN, and the subsequent development of these double stars under the secular action of tidal friction, appear to have inspired considerable confidence in our ability to account for the evolution of double and multiple stars, it seems to be felt that the origin of the solar system remains as mysterious as ever. Speaking with recognized authority on this subject, in his Presidential Address to the British Association at Capetown, 1905, PROFESSOR SIR G. H. DARWIN justly observes that in spite of all that has been done on the evolution of the planetary system, "nevertheless it is hardly too much to say that every stage in the supposed process presents to us some difficulty or impossibility." Though one may dissent from a few of the lines of argument adopted by DARWIN, it is undeniable that his address is by far the ablest exposition of the subject which has appeared in recent years‡.

Notwithstanding this somewhat discouraging state of affairs, as regards the progress of cosmical evolution, it must be remembered, that every physical science goes through a formative stage; and until the fundamental laws are recognized, the outlook always is very unpromising. As the processes at work among the

* "Sur l'equilibre d'une masse fluide animée d'un movement de rotation." *Acta Mathematica*, Vol. VII, 1885.

† "Figures of Equilibrium of Rotating Masses of Fluid." Phil. Trans., Roy. Soc., Vol. 178, 1887.

‡ The reader may consult also papers by Moulton, in the *Astrophysical Journal*, Vol. XI, p. 103-130, 1900; and especially Vol. XXII, No. 3, October, 1905. The present writer, however, believes this last paper to be thoroughly unsound.

stars extend over vast ages and are therefore largely hidden from our view, even when we compare the successive stages of growth seen in different objects, such a difficult science as Cosmogony should not be expected to prove an exception to this general rule. For whilst vast masses of data have been recently gathered by the researches of astronomers, these accumulated treasures have as yet given us but little insight into the great truths which undoubtedly lie hidden close beneath the surface indications of Nature.

In recent years, as observations have increased, an adequate theory of the spiral or whirlpool nebulae has become a recognized desideratum of Astronomy. The real meaning of these vast whirlpools of nebulous matter has long been perplexing to the observer, and equally bewildering to the mathematician* who is occupied with the great problems of Celestial Mechanics. Up to the present time it has been very difficult, if not impossible, to interpret the movements apparently indicated in the whirlpool nebulae; and scarcely less hazardous to attempt to imagine the connection which must naturally subsist between the spiral and the other nebulae and the systems of multiple stars scattered so abundantly throughout the immensity of space. It is obvious, however, that real continuity may be assumed to exist, even between these widely different classes of objects; and until the true order of nature is clearly made out, our theory of the development of the heavenly bodies necessarily remains very incomplete.

Some eminent investigators may doubt whether the time has yet come for an attempt at the solution of this great problem, which has more or less occupied the attention of natural philosophers from the earliest ages of science. Without underestimating the value of traditional and contemporary opinion, the writer is convinced that the occasion is auspicious for an approximate solution which will throw decided light upon some of the deepest secrets in Nature; and as this outline will stimulate research and open up new lines of thought, he does not hesitate to advance the theory at which he has arrived after a continuous series of investigations on this subject extending over many years.

§ 2. *Historical Resumé and Statement of the Problem.*

The first spiral nebula noticed in the telescope was disclosed in April, 1845, by the great reflector of LORD ROSSE, who gave considerable attention to these objects, and sketched a number of them with so much care that they have

* The celebrated French Mathematician POINCARÉ, in an address to the Astronomical Society of France (cf. Bulletin, April, 1906), has even suggested that the spiral nebulae may be other Milky Ways, and that STRATANOFF may be correct in regarding the Milky Way itself as a gigantic spiral nebula.

long figured in our handbooks of Astronomy. The great whirlpool nebula in *Canes Venatici*, Messier 51, is especially famous from LORD ROSSE's drawings; and it still remains one of the finest specimens of the spiral nebulae known. Accordingly, spiral nebulae were quite unknown to LAPLACE; for the existence of such objects had completely escaped the attention of the elder HERSCHEL in his unparalleled explorations of the heavens. And even after LORD ROSSE's unexpected discovery in 1845 and his publication of a list of fourteen of these objects five years later, no further advance of importance was made for nearly forty years. New interest, however, was awakened by DR. ISAAC ROBERTS' unexpected discovery in 1887, that the great nebula in *Andromeda* is really annular or spiral in character, with dark lanes between the whirls. DR. ROBERTS' later photographs added much to our knowledge of spiral nebulae, and gave representations of their forms, which are highly satisfactory. But it was chiefly KEELER's work at the Lick Observatory, with the Crossley Reflector, which emphasized the prevalence of the spiral form as most typical of the nebulae. Important contributions have been made to the subject also by BARNARD and RITCHIE of the Yerkes Observatory, and by MAX WOLF of Heidelberg. DR. WOLF has photographed large regions of the heavens and shown that heretofore only a small percentage of the existing nebulae have been recognized. The latest work shows that the new nebulae are both large and small, and of all possible forms, and the various investigators agree with KEELER's estimate that the number of nebulae in the sky observable by modern means is certainly not less than 120,000, and may be very much greater. Indeed PROFESSOR PERRINE, in *Lick Observatory Bulletin*, No. 64, estimates the total number of the nebulae at about a million. His account is as follows:

"PROFESSOR KEELER, soon after beginning his program of work with the Crossley Reflector, showed that the number of nebulae is very much greater than had been supposed. He conservatively placed the number within reach of that telescope at one hundred and twenty thousand. His program comprised the taking of photographs of one hundred and four of the brighter nebulae and clusters located in all parts of the sky within reach of the telescope, *i.e.*, north of declination —25°. The recent completion of this program enables us to revise his estimate.

"In fifty-seven of the regions, seven hundred and forty-five *new nebulae* have been discovered. Almost all of them are very small and faint. The regions in which no new ones were found were, as a rule, those surrounding the clusters and very large nebulae. There were one hundred and forty-two known nebulae in these regions, making the total number of nebulae observed eight hundred and eighty-seven, an average of eight and one-half per region. As it would take

sixty-two thousand such photographs to cover the entire sky, the results indicate five hundred thousand as the corresponding number of nebulae within reach of the Crossley Reflector. This assumes that the small portion observed represents fairly the entire sky. It is well known that the nebulae are much more numerous in some parts of the sky than in others. This is a tendency which, so far as we know, affects large and small nebulae alike. The fact that a considerable number of other subjects than the nebulae (presumably non-nebulous regions) are included in the program, indicates that the portion observed is fairly representative of the whole sky.

"Longer exposures, more sensitive plates, and more perfect photographs will undoubtedly reveal some nebulae which do not now appear and others which are confused with the faint stars. It seems probable, therefore, that the number of the nebulae will ultimately be found to exceed a million."

It will readily be understood that owing to the great distances of the fixed stars, small faint nebulae can not be detected even with the most powerful instruments of the present time. And nebulae situated near stars, whether they be large or small, are reduced in distinctness and not infrequently have their light largely or entirely extinguished by contrast; so that the discovery of faint nebulae becomes increasingly difficult as the stars grow in brightness and density on the background of the sky. Moreover, it has long been supposed that the radiation emitted from the stars is partially absorbed by cosmical dust pervading the regions of space which it traverses, so that the light of the remoter stars is entirely cut off. If this be true of the stars, which shine as sharp points, it must be much more true of the nebulae which are obscure, cloud-like masses of various sizes, shining with a feeble light, and in many cases certainly quite dark. Accordingly there can be scarcely any doubt that the number of the nebulae, if we could see all that really exist in the heavens, would be approximately the same as that of the stars, which, with our present means of exploration, may be taken at about two hundred million.*

Under the circumstances it becomes advisable to subject the whole theory of the nebulae to a searching examination, in the hope of obtaining additional light on the processes by which cosmical systems are formed. This will necessitate an examination of the principal laws of central forces, together with the resulting orbits and spiral paths, and a comparison of the forms of the nebulae with the possible paths of a particle moving freely under these various laws of force. In the theory of the nebulae, which are seen from but one point of view, account will have to be taken also of the effects of projection on the forms of the various curves

* In his suggestive address on the Milky Way and the Theory of Gases, above referred to, POINCARÉ estimates the total number of stars, including the dark bodies, at a thousand million.

and spirals. This difficulty has seemed so formidable to previous investigators that few have dared to believe that the true figures of the nebulae ever could be discovered. But even if the method developed in this work should not always prove adequate for attaining so desirable a result, it will at least make clear the fact that no regular geometrical figure is capable of accounting for the spiral forms of the nebulae, and therefore that they are not mathematical figures of constant form, but in fact *chance spirals*, varying more or less from one nebula to another. The introduction of figures depending on chance, or accidental circumstances in the fortuitous approach of clouds of cosmical dust, is no doubt less satisfactory to the orderly mind of the mathematician than curves of geometrical regularity; yet even this simplification will prove to be a great gain to the astronomer who has been bewildered by the confusion and hopelessness heretofore encountered in the study of the nebulae.

§ 3. *The General Differential Equations for the Laws of Force for Central Orbits and Spirals.*

The problem of central forces is a very old one. It was quite fully treated, by geometrical methods, in NEWTON's *Principia*; and it is now treated with much greater simplicity and elegance by analytical methods, which may be found in various works on *Dynamics*. It must suffice here to recall the conclusions at which mathematicians have arrived, and to determine the laws of force for the different orbits, and especially the several spirals, which might possibly represent the figures of the nebulae observed in the immensity of space. In the principal works on Dynamics it is shown that the criterion for the law of attraction P is supplied by either of the following differential equations:

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right), \tag{1}$$

$$P = \frac{h^2}{p^3}\frac{dp}{dr}, \tag{2}$$

where h = the constant of areas and represents twice the actual area swept over by the radius-vector of the particle in an element of time, r = radius-vector, $u = \frac{1}{r}$, p = perpendicular from the center of force on the tangent to the orbit, and θ = angle in polar coördinates (cf. WHITTAKER, *Analytical Dynamics*, p. 77; TAIT and STEELE, *Dynamics of a Particle*, 7th edition, pp. 123–135; ROUTH's *Dynamics of a Particle*, p. 197, et. seq.).

In the application of these general formulae to particular curves, the equations of which are known, the following formulae are very useful:

$$\left.\begin{aligned} \frac{1}{p^2} &= \frac{dr^2}{r^4 d\theta^2} + \frac{1}{r^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \\ p &= \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} \end{aligned}\right\} \qquad (3)$$

(cf. WILLIAMSON'S *Differential Calculus*, § 183).

The general solution of the problem to find the law of force when the orbit is given is perfectly satisfactory, and the above equations apply to every case which can arise. But the converse problem, given the law of central force, to find the orbit, is not capable of such simple treatment, because different initial conditions specified by the velocity and direction of the trajectory will in general give different paths. Thus in general for one law of force there may be n paths, and the problem admits of multiple solution only. In the case of gravity, however, the solution is unique, the curve being always a conic section, whatever be the initial conditions of the motion, as was shown by NEWTON in the *Principia*, in 1687.

Accordingly, it follows that the above equations give the law of the force when the body traces out any given curve, but the solutions in general are not unique; in other words, other curves may be described under the same law of force, but with different initial conditions.

It will be advisable to recall here the forms of the orbits for forces of the type $P = \frac{\mu}{r^n}$, where μ is the acceleration at unit distance.

§ 4. *First Case.* $n = -1$, $P = \mu r$.

The orbit is an ellipse, with origin of force at the centre of the curve. This problem was treated by NEWTON in the *Principia* (Lib. I, Prop. X, Prob. V.), and the solution is also unique for the converse problem.

In this first case it may be advisable to illustrate the analytical method of treatment. Under the action of a central force having the intensity μ at the unit of distance, we shall have $P = \mu r$, and the differential equations of the motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\mu x, \\ \frac{d^2y}{dt^2} &= -\mu y. \end{aligned}\right\} \qquad (4)$$

The integrals of which are

$$x = A\cos\{\sqrt{\mu}t + B\}, \tag{5}$$

$$y = A'\cos\{\sqrt{\mu}t + B'\}, \tag{6}$$

A, B, A', B' being the constants of integration, to be determined by the initial conditions of the motion. If we consider the particle projected with a velocity V, from a point on the axis of x, at the distance a from the centre, and in a direction making an angle ψ with Ox; then at the initial epoch $t = o$, and $x = a$, $y = o$,

$$\frac{dx}{dt} = V\cos\psi \quad , \quad \frac{dy}{dt} = V\sin\psi.$$

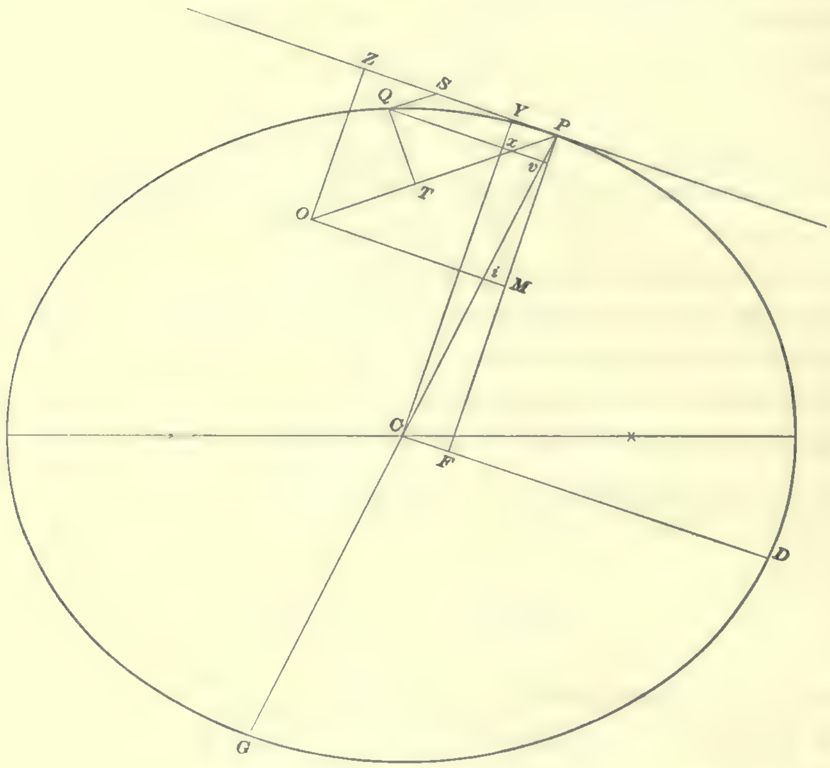


Fig. 1.

Accordingly equations (5) and (6) give

$$\left.\begin{aligned} x &= a = A\cos B, \\ y &= o = A'\cos B', \\ \frac{dx}{dt} &= V\cos\psi = -A\sqrt{\mu}\sin B, \\ \frac{dy}{dt} &= V\sin\psi = -A'\sqrt{\mu}\sin B'. \end{aligned}\right\} \tag{7}$$

When we expand the cosines in (5) and (6) and substitute the expressions in (7) involving V, for the arbitrary constants of integration, we get

$$x = \frac{V \cos \psi}{\sqrt{\mu}} \sin \sqrt{\mu} t + a \cos \sqrt{\mu} t. \qquad (8)$$

$$y = \frac{V \sin \psi}{\sqrt{\mu}} \sin \sqrt{\mu} t. \qquad (9)$$

These equations give the values of x and y for any time, and thus afford a complete solution of the problem. But to ascertain the form of the curve described, we must eliminate t, and then we have the equation of an ellipse

$$(x \sin \psi - y \cos \psi)^2 + \frac{\mu a^2}{V^2} y^2 = a^2 \sin^2 \psi. \qquad (10)$$

If now we make $\psi = 90°$, $\sin \psi = 1$, $\cos \psi = o$, which is equivalent to rotating the curve in its own plane till the apsidal points fall on the x-axis, we shall get

$$x^2 + \frac{\mu a^2}{V^2} y^2 = a^2 \quad , \quad \text{or} \quad \frac{x^2}{a^2} + \frac{\mu y^2}{V^2} = 1. \qquad (11)$$

where $\frac{\mu}{V^2} = \frac{1}{b^2}$, in the usual form of the equation for the ellipse in terms of its intercepts. It is noticeable that the values of x and y in equations (8) and (9) will be the same at the time t and $t + \frac{2\pi}{\sqrt{\mu}}$, and hence the time of a revolution is $\frac{2\pi}{\sqrt{\mu}}$; and it appears that the period is independent of the dimensions of the ellipse, but depends wholly on the intensity of the central force, the time of a revolution being inversely as the square root of μ. TAIT, whose discussions we have here followed, remarks that if μ were taken as negative in (4), so that $P = -\mu r$, the resulting curve would be an hyperbola described with O as a centre, and the particle would necessarily always remain on one branch of the curve (cf. TAIT and STEELE, *Dynamics of a Particle*, 7th edition, p. 116).

§ 5. *Second Case.* $n = o \quad , \quad P = \frac{\mu}{r^o} = \mu.$

In this case the force is constant, however the distance may vary, and we are led to GALILEO'S Theorem (cf. NEWTON'S *Principia*, Lib. I, Prop. X, Prob. V, scholium).

There is no case of constant force in Nature; but in the case of projectiles near the surface of the earth the force may be taken to be nearly constant for small

altitudes; and whatever be the direction and velocity of projection the path is a parabola. Hence we see that a constant force gives an infinite variety of parabolic motion, or an infinite system of parabolas, according to initial conditions. Here the solution is also unique. The parabolas, however, do not have a common vertex, nor do their axes coincide in position, though they are all parallel to the direction of gravity and have a common direction in space.

Galileo's Theorem or the Theory of Parabolic Motion Under Constant Acceleration. Let the axis of x and y be as shown in the figure, with the centre of the earth in the direction of the X-axis, then we have for the movement of any particle under gravity

$$\frac{d^2x}{dt^2} = g. \tag{12}$$

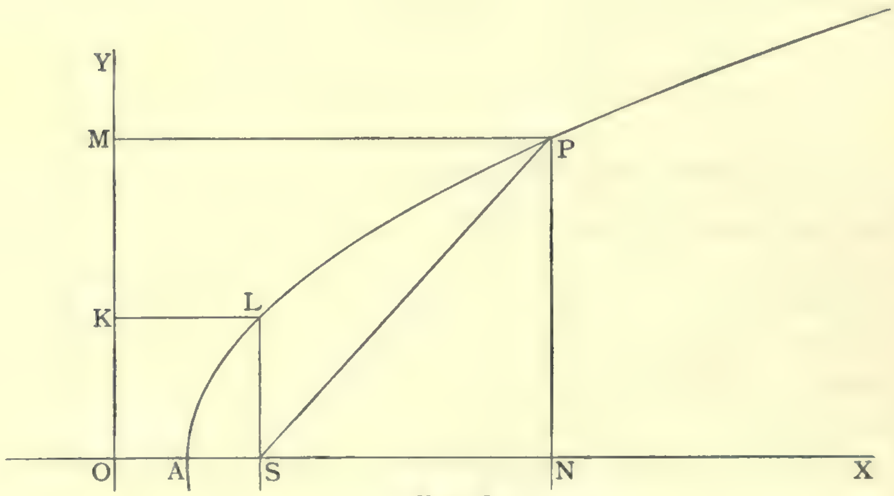


FIG. 2.

Integrating once we get

$$\frac{dx}{dt} = V = gt + C, \tag{13}$$

where C is the constant of integration. Suppose the particle projected from O downward with the velocity V, then when $t = o$, $v = V$; and hence $C = V$, so that

$$\frac{dx}{dt} = v = V + gt. \tag{14}$$

Integrating again, we have

$$x = C' + Vt + \tfrac{1}{2}gt^2. \tag{15}$$

When $t = o$, $x = a$, and thus $C' = a$, so that

$$x = a + Vt + \tfrac{1}{2} gt^2. \tag{16}$$

This is the general integral for the vertical motion under constant acceleration. If a particle be projected along the Y-axis with constant velocity v, we have $\frac{dy}{dt} = v$, $y = vt + C''$. We may take this constant C'' so that it shall vanish when $t = o$, as the particle passes the axis OX. Hence, if x be reckoned in the same way, we shall have for $t = o$, $a = o$, and $V = o$; and therefore

$$x = \tfrac{1}{2} gt^2 \quad , \quad t^2 = 2gx \quad , \quad t = \sqrt{2gx} = \frac{y}{v},$$

also

$$y^2 = 4\beta x, \tag{17}$$

where

$$\beta = gv^2.$$

Thus the equation of the motion of a particle under a constant acceleration is always a parabola

$$y^2 = 4\beta x,$$

but the position of the vertex may be varied by changing the direction of the projection from O.

§ 6. *Third Case.* $n = 1$, $P = \frac{\mu}{r}$.

Professor W. H. H. Hudson, of King's College, London, and Mr. C. G. Whitmell, of Leeds (cf. *Journal of British Astronomical Association*, Vol. XIV, No. 8, June 24, 1904), have considered this case. The polar equation to the curve is

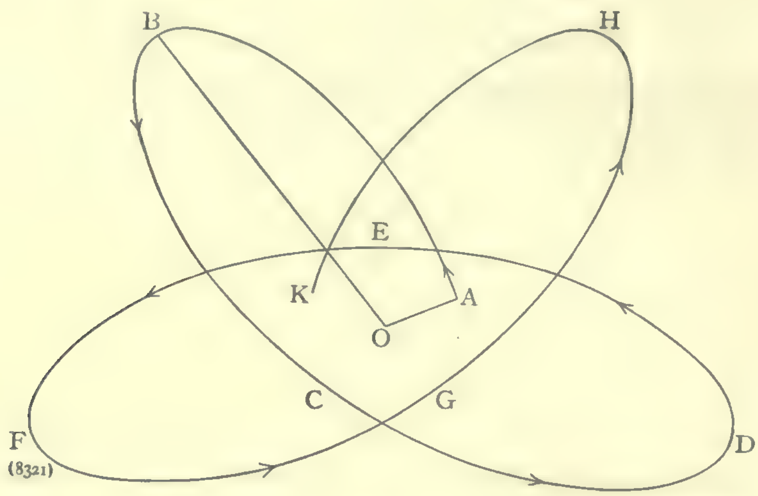
$$r = ae^{-\frac{b^2}{p^2}}. \tag{18}$$

where a and b are arbitrary constants, e the Naperian base, and p the perpendicular from the origin upon the tangent to the curve. Mr. Whitmell gives

$$\theta = \int (\lambda^2 x e^{-2x} - 1)^{-1/2} dx. \tag{19}$$

$$\text{where} \quad x = \log\left(\frac{a}{r}\right) \quad , \text{ and } \quad \lambda = \frac{a}{b} \quad \text{and} \quad p = \frac{b}{\sqrt{\log a - \log r}}\,; \tag{20}$$

which is available for tracing the path. It is obvious that r must be less than a, and p can never exceed r; so that r varies between two values, a maximum and a minimum, at both of which limits $r = p$. The path found by MR. WHITMELL has the following form:



B 2

FIG. 3.

The curve is not re-entrant unless the apsidal angle ACB is commensurable with a right angle; and in general the graphical method is not sufficiently rigorous to decide the question. By varying the limits a and b we obtain an infinite variety of curves. But as it is shown in dynamics that in a central orbit there cannot be more than two apsidal distances (cf. TAIT and STEELE, § 146), it follows that however the path may vary it will preserve the same limits throughout its course whether exactly re-entrant or not.

§ 7. *Fourth Case.* $n = 2$, $P = \frac{\mu}{r^2}$.

This is the celebrated case of Nature, corresponding to universal gravitation. The curve is always a conic section. It was fully treated by NEWTON in the *Principia,* in 1687. Here

$$P = \frac{h^2u^2}{a\,(1-e^2)} = \frac{h^2}{a\,(1-e^2)}\;\frac{1}{r^2}.$$

And the solution is unique. Since the force varies according to the law of inverse squares, only a conic section can be described; but the type of conic varies

according to the initial velocity. If μ be the acceleration at unit distance, then the conditions are:

$$\left.\begin{array}{llll} (1) & V^2 > \dfrac{2\mu}{r}, & e > 1, & \text{orbit an Hyperbola,} \\ (2) & V^2 = \dfrac{2\mu}{r}, & e = 1, & \text{orbit a Parabola,} \\ (3) & V^2 < \dfrac{2\mu}{r}, & e < 1, & \text{orbit an Ellipse.} \end{array}\right\} \begin{array}{l}\text{Centre of} \\ \text{force always} \\ \text{in the focus.}\end{array} \tag{21}$$

These results are so familiar that they do not here call for detailed discussion; but we may remark that when the force is repulsive, $P = -\frac{\mu}{r^2}$, the curve is the opposite branch of an hyperbola such as might be described under an attractive force, and obviously the particle recedes to infinity. The particle could be made to approach the repulsive focus only by projecting it in that general direction with considerable velocity; after nearing the focus till the velocity was overcome it would again recede indefinitely.

This theory is illustrated by the particles of the tails of certain comets which are falling to the sun with considerable velocity, and begin to be repelled from the sun more powerfully than they are attracted by the sun's gravity. Certain particles of the corona are carried away in the same manner, chiefly by the powerful radiation-pressure of the sun's light, which ARRHENIUS has carefully investigated in *Lick Observatory Bulletin*, No. 58. The exact shape of the path pursued depends on the intensity of the forces at work, but if the repulsive force varies inversely as the square of the distance the path is hyperbolic, with the sun in the other focus. It is supposed that the repulsive forces operating in the tails of comets depend largely on electric charges, and as these charges may become dissipated with increased distance, the hyperbola on which they are started is modified by the dissipation of the charge, or the path becomes a varying hyperbola with the sun in the other focus. If the repulsion should entirely cease and the particle did not then have a velocity sufficient to carry it indefinitely away from our sun, it might again move in an ellipse till brought into the region where the repulsive forces are predominant. Thus it appears that while much matter circulates about our sun in ellipses practically undisturbed save by the attraction of the planets, other matter is violently repelled, with such velocity that it never returns to our system; while yet other particles suffer temporary repulsion to great distances, but when the electric charge is dissipated, and the velocity not too great, they again come under the sway of central gravity and at length return to the sun.

All these varied movements are possible, and apparently verified by the phenomena witnesses among the bodies which traverse the solar system. And no doubt similar phenomena take place about other fixed stars observed in the immensity of space.

As respects intensity it may be noted that Light, Heat, Magnetic and Electric Forces, the Radiation-Pressure depending on Light, as well as Universal Gravitation — the principal natural forces operative at great distances, some attractive and some repulsive — all act according to the law of the inverse square of the distances, evidently because the expansion of space or the solid angle subtended by a body depends on this law.

The only natural forces known to obey laws different from the law of the inverse squares are chemical and molecular forces operating at very small intervals, but becoming insensible at sensible distances. It has long been known that the laws of force in such phenomena as capillarity and molecular action depend on a higher power than that of the inverse squares. MAXWELL found the equations most easily integrated when the law was taken to be inversely as the fifth power, and this has been generally assumed to hold in molecular attractions, but it cannot be said to be demonstrated.

It is shown in Chapter II that the law of attraction in many spirals is a combination of two powers, as the third and fifth, in the case of the spiral of ARCHIMEDES. Since few molecular motions are really permanent it may be that their paths are infinite coils such as the spiral of ARCHIMEDES or the other spirals of that general type. The movement of particles on such expanding or contracting paths would explain many molecular phenomena relating to stability and decay, as well as the variation of the forces with an inverse high power of the distance. We shall see later, however, that such complex laws of attraction are not at work at the great distances separating cosmical bodies in the heavenly spaces, though in some cases it may happen that the figures of the nebulae roughly resemble these spirals.

§ 8. *Fifth Case.* $n = 3$, $P = \frac{\mu}{r^3}$.

In this case the paths described are called *Cotes' Spirals*, from ROGER COTES, the friend of NEWTON, of whom the great philosopher said: "If MR. COTES had lived we should have known something." The particle is supposed to move under an attraction varying inversely as the cube of the distance, and to be projected from a given point with any velocity in any direction.

Since

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

we have

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \tag{22}$$

or

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0.$$

This differential equation of motion becomes, when $P = \mu u^3$,

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u = o, \tag{23}$$

and the integral involves exponential or circular functions according as $\frac{\mu}{h^2}$ is greater or less than unity; that is, according as the velocity at an apse is less or greater than the velocity from infinity. For $\frac{h^2}{\mu} = \frac{1}{a(1 - e^2)} = \frac{r^4\left(\frac{d\theta}{dt}\right)^2}{\mu}$, a constant; and when $e = 1$, and $a = \infty$, the velocity is that of the parabola. To classify the paths described according to the circumstances of projection, several cases must be considered. For the sake of continuity of thought a clear conception of the method is needed, and for this purpose it suffices to condense the discussion of TAIT and STEELE, which is substantially as follows:

I. Take $\frac{\mu}{h^2} > 1$, and let $\frac{\mu}{h^2} - 1 = k^2$; then our differential equation (22) becomes

$$\frac{d^2u}{d\theta^2} - k^2u = o, \tag{24}$$

and the integral involves exponentials

$$u = Ae^{k\theta} + Be^{-k\theta}. \tag{25}$$

Species I. The arbitrary constants A and B have the same sign, and

$$\frac{du}{d\theta} = k(Ae^{k\theta} - Be^{-k\theta}). \tag{26}$$

If α be the value of A corresponding to an apse, then when

$$\theta = \alpha \quad , \quad \frac{du}{d\theta} = o = Ae^{k\theta} - Be^{-k\theta}. \tag{27}$$

Suppose

$$Ae^{k\alpha} = Be^{-k\alpha} = \frac{1}{2a}; \tag{28}$$

then the integral equation (24) becomes

$$au = \tfrac{1}{2}\{e^{k(\theta-} \quad + e^{-k(\theta-\alpha)}\}. \tag{29}$$

When $A = a$, this equation gives $au = 1$, or $r = a$, the apsidal distance. As A increases, u increases, or r diminishes; and when $\theta = \infty$, $u = \infty$, or $r = o$; and hence the curve forms an infinite number of convolutions about the pole, and being symmetrical on both sides of the apse, is of the form shown in the figure.

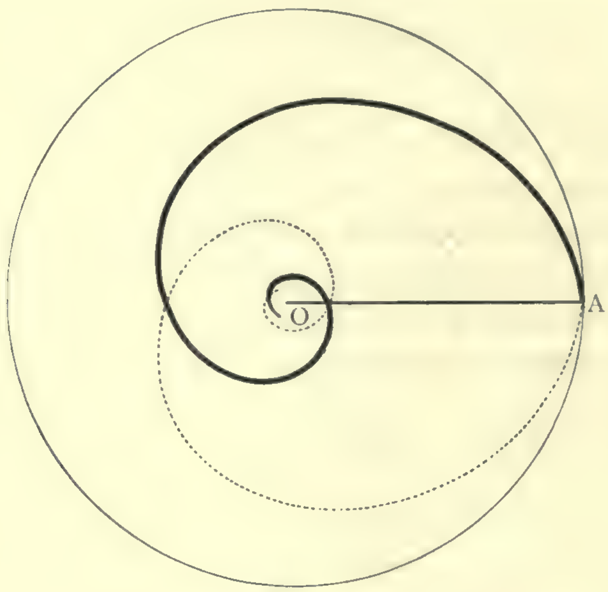


FIG. 4.

Species 2. Let $\frac{\mu}{h^2} > 1$, $B = o$, and then equation (25) becomes

$u = Ae^{k\theta} = \frac{e^{k\theta}}{2a}$, or

$$au = e^{k\theta}, \tag{30}$$

which is the equation of the logarithmic or equiangular spiral. If we had supposed $A = o$, instead of $B = o$, the nature of the resulting curve would not have been changed.

Species 3. Let $\frac{\mu}{h^2} > 1$, and B negative; then equation (25) becomes

$$u = Ae^{k\theta} - Be^{-k\theta}. \tag{31}$$

If we put $u = o$, when $\theta = \alpha$, we obtain as for species 1,

$$au = \tfrac{1}{2}\{e^{k(\theta-\alpha)} - e^{-k(\theta-\alpha)}\}. \tag{32}$$

And when $\theta = \alpha$, $u = o$, or $r = \infty$. As θ increases r decreases, and when θ is infinite $r = o$; so that the curve has an infinite number of convolutions around the pole. There is an asymptote to the curve parallel to OA at a distance $\frac{a}{k}$.

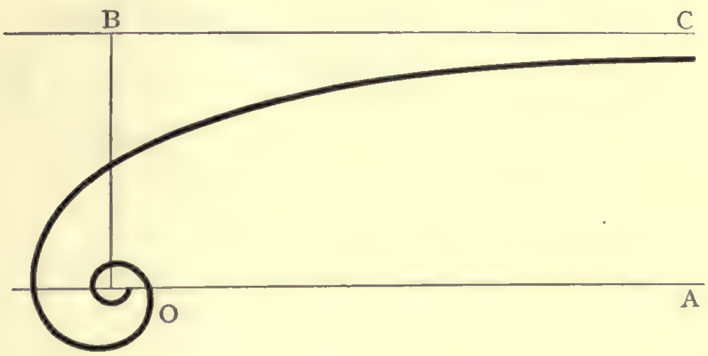


FIG. 5.

II. *Species* 4. Let $\frac{\mu}{h^2} = 1$, and the general differential equation (23) becomes

$$\frac{d^2u}{d\theta^2} = o, \tag{33}$$

the integral of which is

$$au = \theta - \alpha. \tag{34}$$

This is the equation of the reciprocal or hyperbolic spiral.

III. *Species* 5. Let $\mu < 1$, and let $1 - \frac{h^2}{\mu} = k^2$, then by equation (23) we get

$$\frac{d^2u}{d\theta^2} + k^2u = o, \tag{35}$$

the integral of which is

$$au = \cos k\,(\theta - \alpha). \tag{36}$$

If we differentiate we obtain

$$a \frac{du}{d\theta} = -k \sin k (\theta - \alpha). \tag{37}$$

It appears that α is the value of θ corresponding to an apse, and a is the apsidal distance. The curve has asymptotes and is of the form shown in the figure.

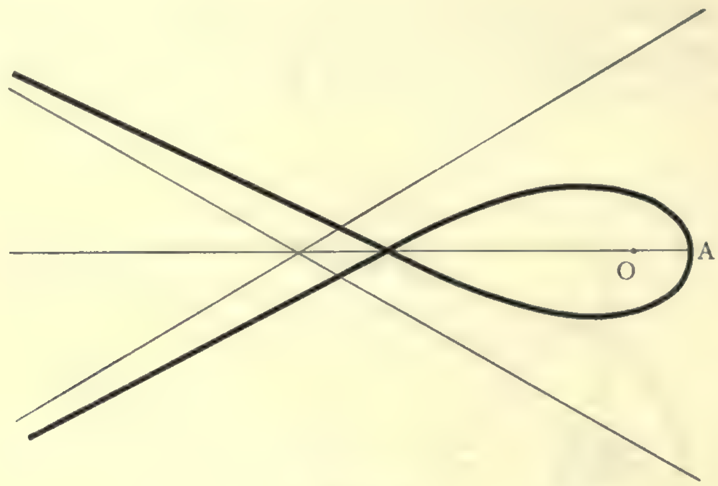


FIG. 6.

It will be seen from this discussion how varied are the paths which a particle may describe under a force varying inversely as the cube of the distance when set in motion with any velocity in any direction. These various species of COTES' spirals are by no means similar, but they do not differ among themselves much more widely than do the various types of conic sections, corresponding to central forces varying inversely as the square of the distances.

The case of the logarithmic spiral is of most interest in connection with the nebulae; but even this simple curve is found to be inapplicable.

§ 9. *Sixth Case.* $n = 4$, $P = \frac{\mu}{r^4}$.

It does not seem necessary to treat this case and the cases involving higher powers of the distances at length. We shall merely remark that the particles may describe a great variety of spiral curves, two of the simplest being:

(1) A circle with O in the centre;

(2) A cardioid with O at its cusp.

Both of these results require special conditions in the velocity and direction of the trajectory when the particle is started.

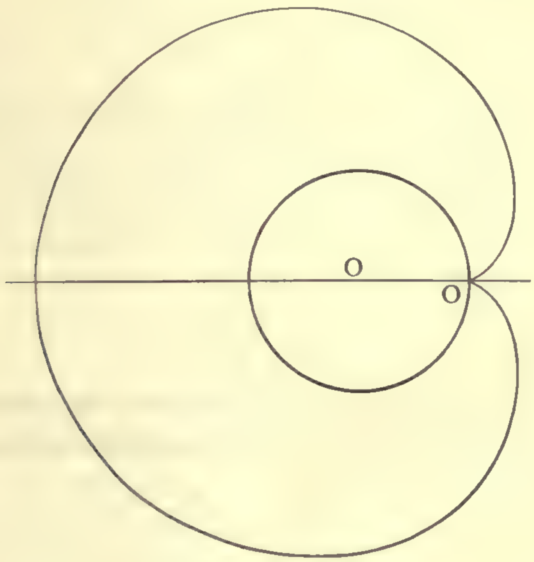


FIG. 7.

It may be remarked that for any power of the central force a circle is a possible orbit, but it is only a single curve out of the large number of paths which may be traced under various initial conditions.

§ 10. *Seventh and Ninth Cases.*

Here in $n = 5$, and $n = 7$, and

$$P = \frac{\mu}{r^5} \quad ; \quad P = \frac{\mu}{r^7}.$$

(1) In the first case, $n = 5$, it is found in general that the radius-vector r is an elliptic function of the angle θ. In one special case the path is a circle with O on the circumference. In the case of the inverse fifth power of the distance the path is generally very complex, and it is difficult to see why such orbits were assumed for the molecules of matter under conditions of average behavior.

(2) The inverse seventh power presents a great variety of results. We shall only mention the case of the Lemniscate of BERNOULI with O at the Node.

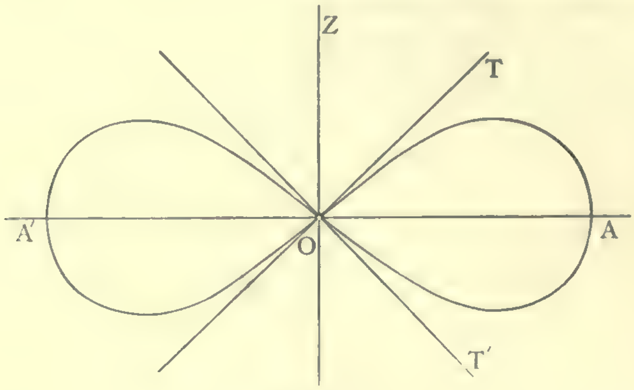


FIG. 8.

No forces acting according to any of these higher powers are known to operate in Nature, and accordingly we need not further elaborate the discussion of the paths which might be pursued.

CHAPTER II.

INVESTIGATION OF THE LAWS OF ATTRACTION FOR PARTICLES MOVING IN PARTICULAR PLANE SPIRALS; AND OF THE PROJECTION OF THESE PATHS, WITH ESPECIAL REFERENCE TO THE FIGURES OF THE NEBULAE: COMPARISON OF THEORY WITH OBSERVATIONS.

§ 11. *The Problem Presented by the Nebulae.*

IN order to ascertain whether the spiral nebulae really represent particles pursuing exact mathematical curves under given forces, it becomes necessary to inquire critically into the nature of the laws of attraction under which particles might describe these curves. After we have found the required laws of attraction corresponding to the several curves, it will remain to determine whether the curves involved are contradicted by the geometrical appearances of the nebulae, and whether these laws of attraction are also inadmissible on physical grounds. This somewhat exhaustive mode of reasoning is rendered necessary by virtue of the fact that we see the figures of the nebulae only in projection, as viewed from one point in space, and it seems unlikely that any other projection ever will be available to the inhabitants of this planet. Consequently we must exhaust all known means of involving the assumed curves and laws of attraction in contradiction with known phenomena.

Even when this method is carried out one may still imagine that he has not exhausted all possible forms of spirals, but only those known to us; and also that physical laws of attraction or repulsion may exist in the depths of space of a nature quite different from anything yet recognized upon the earth, either between large masses of matter acting on one another at considerable distances, or between very small masses acting at distances corresponding to those of our molecules and atoms, which are hopelessly below the limits of vision with any of the appliances now known to Science.

In regard to the first of these questions, it may indeed be said that other spirals may yet be discovered with properties somewhat different from those now

known; yet the species of spirals already determined are many,* and if the nebulae do not approach any of these types, or do not show by comparison with them great regularity of figure, it would manifestly be impossible that such regular properties could be given to the nebulae by new discoveries to be made hereafter. Thus the main test turns more upon the *perfect regularity* of the figures of the nebulae compared to any geometrical figure whatever, than upon our ability to exhaust the possible supply of these curves. The spirals at our disposal are so varied, regular and numerous that they may be taken as representative of all which could possibly exist; and if the nebulae depart from them all, or show conspicuous discontinuities, or breaks inconsistent with geometrical regularity, we may justly conclude that the observed spirals are not true geometrical figures, but only *chance figures* devoid of regular mathematical properties.

In regard to the second of the above questions, it suffices to say that if the laws of attraction or repulsion varied in different regions of the universe, we might expect matter, light waves in the ether, and even space itself to change with location, which is contrary to the principles of Natural Philosophy. Moreover, our own solar system itself was at one time undeniably a spiral nebula, of which the traces are now partly but not wholly obliterated; and the properties of the matter still observed in the solar system ought therefore to hold also among the other nebulae observed in the remotest regions of space. And in our solar system the chief attractive and repulsive forces at work are gravity, electric and magnetic forces, and the repulsion of light—for large distances and considerable masses; and atomic and molecular forces—for small particles acting at insensible distances. As the laws are found by experiment to be the same for matter upon the earth, and for that reaching us in the cosmical dust which continually rains down upon our planet from the heavenly spaces, we may justly conclude that they are common also to the matter observed in other parts of our planetary system, and generally to all matter whatsoever throughout the sidereal universe.

To hold that the forces dominant in the nebulae may be different from those acting at large distances in the solar system and everywhere varying as the law of the inverse squares, and to liken them rather to the forces which in our regions of space operate only between minute particles at insensible distances, would be to introduce into the science of Natural Philosophy the most contradictory principles. *It is inconceivable that the forces depending on the law of the inverse squares of the distances can be dominant among cosmical masses here, but quite dormant among the*

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* An excellent discussion of the general theory of spirals will be found in LORIA's Spezielle Algebraische und Transcendente Ebene Curven, Theorie und Geschichte, B. G. TEUBNER, Leipzig, 1902. Translated from Italian into German by SCHÜTTE.

nebulae; while the more complex forces depending on inverse higher powers of the distances known to be inactive here should be really dominant there. Accordingly, to determine the forces at work among the nebulae, it suffices to find the laws of attraction for the particular spirals which might be imagined to exist in the immensity of space, and to study these laws of attraction in combination with the theory of the projection of the spirals, the geometrical figures of which are well known.

§ 12. *The Spiral of* ARCHIMEDES.

This curve is described by a particle moving uniformly along a line that revolves uniformly about a fixed point, which thus becomes the pole. The equation of the Spiral of ARCHIMEDES is

$$r = a\theta. \tag{38}$$

In this case

$$\frac{dr}{d\theta} = a \quad ; \quad \frac{1}{p^2} = \frac{dr^2 + r^2 d\theta^2}{r^4 d\theta^2} = \frac{a^2 + r^2}{r^4};$$

$$p = \frac{r^2}{\sqrt{a^2 + r^2}} \quad ; \quad \frac{dp}{dr} = \frac{2a^2 r + r^3}{(a^2 + r^2)^{3/2}}. \tag{39}$$

Hence, by the general formula (2), we get

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}\left(1 + \frac{2a^2}{r^2}\right). \tag{40}$$

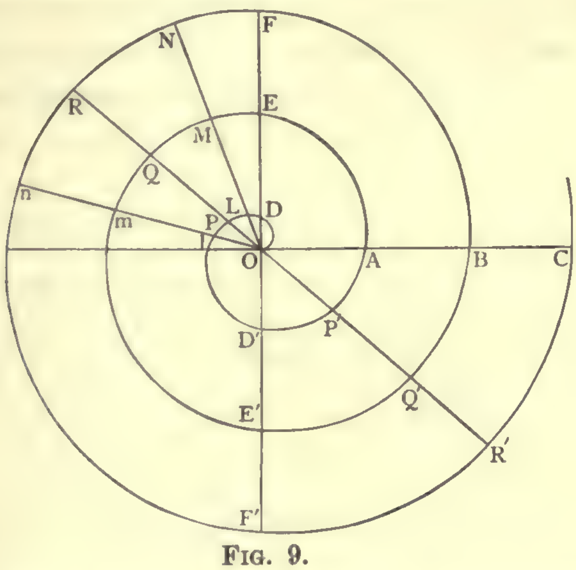

FIG. 9.

In other words, the law of attraction for a particle describing the spiral of ARCHIMEDES varies as the inverse cube and the inverse fifth power of the distance

combined. And this result for the spiral of ARCHIMEDES is quite typical of spiral movement in general. There are few spirals for which the law of attraction proves to be extremely simple. As we have already seen, the logarithmic or equiangular spiral is one of these exceptions, the law of attraction in that case varying inversely as the cube of the distance. But in nearly all the spirals which the writer has investigated, the law of central force is made up of two terms, varying as inverse powers of the distance, always higher than the second.

MAXWELL's reference of molecular forces to functions varying as the inverse fifth power of the distance and the belief among physicists that this power is too high, leads one to ask whether the law of the Archimedean spiral or some of the other related spirals might not give a better representation of the phenomena than has heretofore been obtained.

It will be shown hereafter that the figures of the spiral nebulae, so far as they can be said to be approximately of one type, correspond as nearly to the spiral of ARCHIMEDES as to any other. If the course of time should prove molecular movements to be spiral in character, and of about the same type of spiral as the average spiral observed among the nebulae, it would give remarkable unity to our conceptions of the physical universe, whether the systems concerned be vast nebulae or infinitesimal molecules, with their minuter subsystems of atoms and electrons.

So far as we can now see, the paths described by the elements of a nebula are not exact mathematical curves; neither are the motions permanent, but slowly changing. The instability of molecular mechanism, especially emphasized by the transformations made known in the researches on radio-activity, apparently point to molecular systems not unlike those seen among the nebulae scattered so abundantly throughout the immensity of space.

§ 13. *The Law of Attraction for the Logarithmic or Equiangular Spiral.*

As is well known this spiral was invented by DESCARTES. The equation of the curve, in its simplest form, is

$$r = a^\theta. \tag{41}$$

$$\frac{dr}{d\theta} = \log a . a^\theta = r \log a \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = r^2 \log^2 a \quad ; \quad \frac{1}{p^2} = \frac{\log^2 a + 1}{r^2} \quad ; \quad p = \frac{r}{(1 + \log^2 a)^{\frac{1}{2}}};$$

$$\frac{dp}{dr} = \frac{1}{(1 + \log^2 a)^{\frac{1}{2}}}. \tag{42}$$

Therefore, by means of these values, we find

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}(1 + \log^2 a)\,. \tag{43}$$

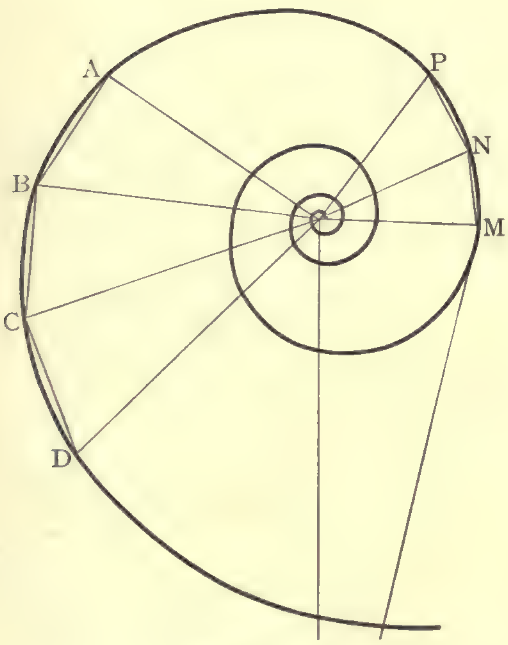

FIG. 10.

Accordingly in the logarithmic spiral the force varies inversely as the cube of the distance; and the expression for the attractive force becomes extraordinarily simple.

If ψ be the angle between the tangent to the curve and the radius-vector, we have

$$\tan\psi = r\frac{d\theta}{dr} = \frac{r}{r\log a} = \frac{1}{\log a} = m,$$

the modulus of the system of logarithms. If the base $a = e$, the Naperian base, then $\log a = 1$, $\tan\psi = 1$, $\psi = 45°$; so that in this case the radius-vector cuts the curve at the constant angle of 45°. But in the more general case the tangent of the angle between the curve and the radius-vector is equal to the modulus of the system of logarithms. By means of this property of equiangularity we have a criterion which may be applied to the nebulae, to ascertain whether their figures are logarithmic spirals.

§ 14. *The Law of Attraction for the Lituus.*

The equation of this curve is

$$r^2\theta = a^2. \tag{44}$$

And we have

$$\left.\begin{aligned} \frac{dr}{d\theta} = -\frac{r}{2\theta} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{r^2}{4\theta^2} \quad ; \quad \frac{1}{p^2} = \frac{1+4\theta^2}{r^2}; \\ p = \frac{r}{\sqrt{1+4\theta^2}} \quad ; \quad p^3 = \frac{r^3}{(1+4\theta^2)^{3/2}} \quad ; \quad \frac{dp}{dr} = \frac{1}{\sqrt{1+4\theta^2}}. \end{aligned}\right\} \tag{45}$$

By the formula for the law of attraction, we get

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{4a^2}{r^2}\right\}. \tag{46}$$

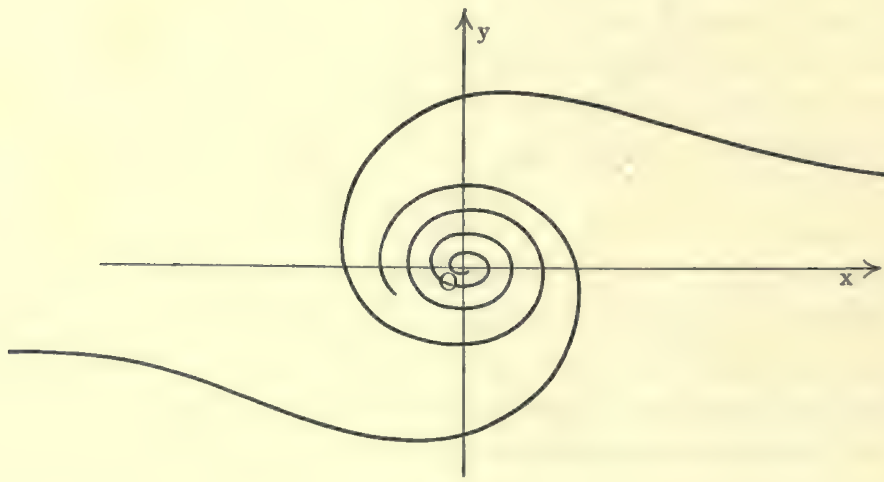

FIG. 11.

Therefore in the *Lituus* the central force varies inversely as the cube and fifth power of the distance combined. The expression for the central force here has the same general form as in the spiral of ARCHIMEDES, although neither the forms of the equations in the two cases, nor the curves themselves are very similar. This similarity in the form of the law of central force holds for a great number of spiral paths, but the coefficient for the term involving the inverse fifth power of the distance varies from one spiral to another. It is only in a few simple spirals that this term entirely disappears, and gives a law of attraction depending simply on the inverse cube of the distance.

§ 15. *The Law of Attraction for the Hyperbolic or Reciprocal Spiral.*

The equation of the curve is

$$r\theta = a. \tag{47}$$

And hence

$$\frac{dr}{d\theta} = -\frac{r^2}{a} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{a^2} \quad ; \quad \frac{1}{p^2} = \frac{r^2 + a^2}{r^2a^2};$$

$$p = \frac{ra}{\sqrt{a^2 + r^2}} \quad ; \quad \frac{dp}{dr} = \frac{a^3}{(r^2 + a^2)^{3/2}} \quad ; \quad p^3 = \frac{r^3a^3}{(a^2 + r^2)^{3/2}}. \tag{48}$$

Therefore, by the general formula, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}. \tag{49}$$

The law of attraction in this case is exceedingly simple, depending wholly on the inverse cube of the distance. The law of attraction is the same as for the equiangular or logarithmic spiral, but the constant coefficient is different in the two cases, being simply h^2 for the Hyperbolic Spiral, and $h^2\,(l + \log^2 a)$ for the Equiangular Spiral.

It is shown in works on Dynamics that *the velocity of a particle at each point of its path is inversely proportional to the perpendicular from the centre on the tangent at that point.* For if v be the velocity, s the distance passed over by the moving particle revolving through the angle θ, we have at once

$$v = \frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \frac{r^2}{p}\frac{d\theta}{dt} = \frac{h}{p}.$$

(cf. TAIT and STEELE, *Dynamics of a Particle*, § 141.) An inspection of any spiral therefore suffices to indicate whether the velocity changes rapidly, for it always varies inversely as p, and becomes very large when p is small, and *vice versa.*

§ 16. *The Law of Attraction for the Parabolic Spiral.*

The equation of the curve is

$$(r - a)^2 = 4ca\theta. \tag{50}$$

Accordingly we get

$$\frac{dr}{d\theta} = \frac{2ca + a}{r} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(2c+1)^2}{r^2} \quad ; \quad \frac{1}{p^2} = \frac{a^2(2c+1)^2 + r^4}{r^6};$$

$$p = \frac{r^3}{\sqrt{a^2(2c+1)^2 + r^4}} \quad ; \quad \frac{dp}{dr} = \frac{r^2\left[3a^2(2c+1)^2 + r^4\right]}{\left[a^2(2c+1)^2 + r^4\right]^{3/2}}; \tag{51}$$

And when we apply the general formula for the central force, we find the following law of attraction

$$P = \frac{h^2}{r^3}\left\{1 + \frac{3a^2(2c+1)^2}{r^4}\right\}. \tag{52}$$

It turns out that the law of attraction for the Parabolic Spiral is rather complex, which should not be wholly unexpected, in view of the manner in which the spiral is constructed by making the axis of the parabola the circumference of a circle of radius a, instead of a straight line.

After one convolution about the circle has been made, the spiral begins to overlap the previous convolution, so that the spaces separating the outer coils are narrower than that separating the inner one from the circle. In this respect the properties of the parabolic spiral are somewhat different from those of most spirals, which ordinarily are quite uniform throughout their length.

§ 17. *The Laws of Attraction for Spirals Resembling that of* Archimedes, *but with Exponents of the Angle Which are Greater than* 1 *and Less than* 2.

A spiral of the general form $r = a\theta^n$, where $n > 1$ and $n < 2$, may be said to resemble the Spiral of Archimedes, but the coils separate more rapidly as they recede from the pole. Spirals of this type are of considerable interest in connection with the figures of the nebulae, which usually have coils widening out with the increase of distance from the centre. We shall first consider the two cases in which $n = \frac{3}{2}$ and $n = 2$.

First case.

$$r = a\theta^{3/2}.$$

$$\frac{dr}{d\theta} = \frac{3}{2}a\theta^{\frac{1}{2}} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{9}{4}a^2\theta \quad ; \quad \frac{1}{p^2} = \frac{9}{4}\frac{a^2\theta}{r^4} + \frac{1}{r^2} = \frac{9a^2\theta + 4r^2}{4r^4};$$

$$p = \frac{2r^2}{\sqrt{9a^2\theta + 4r^2}} \quad ; \quad \frac{dp}{dr} = \frac{(36a^2\theta + 8r^2)r}{(9a^2\theta + 4r^2)^{3/2}}. \tag{53}$$

Therefore

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}\left(1 + \frac{9}{4}\frac{a^2\theta}{r^2}\right). \tag{54}$$

Second case.

$$r = a\theta^2.$$

$$\frac{dr}{d\theta} = 2a\theta \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = 4a^2\theta^2 \quad ; \quad \frac{1}{p^2} = \frac{4a^2\theta^2 + r^2}{r^4} \quad ; \quad p = \frac{r^2}{\sqrt{4a^2\theta^2 + r^2}}; \quad \frac{dp}{dr} = \frac{(8a^2\theta^2 + r^2)r}{(4a^2\theta^2 + r^2)^{3/2}}. \tag{55}$$

Therefore

$$P = \frac{h^2}{p^3}\,\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{8a^2\theta^2}{r^2}\right\}. \tag{56}$$

§ 18. *The Laws of Attraction for Spirals with Other Fractional Exponents of the Angle, Namely $\frac{4}{3}$, and $\frac{5}{4}$.*

The following two spirals closely resemble that of ARCHIMEDES, because the fractional exponents of the angle do not differ greatly from unity, which was employed by the geometer of Syracuse.

Third case.

$$r = a\theta^{4/3} \quad ; \quad \frac{dr}{d\theta} = \frac{4}{3}a\theta^{1/3} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{16}{9}a^2\theta^{2/3} \quad ;$$

$$\frac{1}{p^2} = \frac{16a^2\theta^{2/3} + 9r^2}{9r^4} \quad ; \quad p = \frac{3r^2}{\sqrt{16a^2\theta^{2/3} + 9r^2}} \quad ; \quad \frac{dp}{dr} = \frac{r(96a^2\theta^{2/3} + 27r^2)}{(16a^2\theta^{2/3} + 9r^2)^{3/2}}. \tag{57}$$

Therefore

$$P = \frac{h^2}{p^3}\,\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{96}{27}\,\frac{a^2\theta^{2/3}}{r^2}\right\}. \tag{58}$$

Fourth case.

$$\left.\begin{aligned} &r = a\theta^{5/4} \quad ; \quad \frac{dr}{d\theta} = \frac{5}{4}a\theta^{1/4} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{25}{16}a^2\theta^{1/2}; \\ &\frac{1}{p^2} = \frac{25a^2\theta^{1/2} + 16r^2}{16r^4} \quad ; \quad p = \frac{4r^2}{\sqrt{25a^2\theta^{1/2} + 16r^2}}; \\ &\frac{dp}{dr} = \frac{200a^2\theta^{1/2} + 64r^3}{(25a^2\theta^{1/2} + 16r^2)^{3/2}}. \end{aligned}\right\} \tag{59}$$

Therefore

$$P = \frac{h^2}{p^3}\,\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{200a^2\theta^{1/2}}{64r^2}\right\}. \tag{60}$$

§ 19. *The Laws of Attraction for Spirals with Angular Exponents of $\frac{6}{5}$ and $\frac{5}{3}$.*

The following spirals will suffice to conclude the theory of these curves of the general form $r = a\theta^n$, $n > 1$, $n < 2$.

Fifth case.

$$r = a\theta^{6/5} \quad ; \quad \frac{dr}{d\theta} = \frac{6}{5}a\theta^{1/5} \quad ; \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{36}{25}a^2\theta^{2/5} \quad ;$$

$$\frac{1}{p^2} = \frac{36a^2\theta^{2/5} + 25r^2}{25r^4} \quad ; \quad p = \frac{5r^2}{\sqrt{36a^2\theta^{2/5} + 25r^2}} \quad ; \quad \frac{dp}{dr} = \frac{(360a^2\theta^{2/3} + 125r^2)r}{(36a^2\theta^{2/5} + 25r^2)^{3/2}}. \tag{61}$$

Therefore

$$P = \frac{h^2}{p^3}\,\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{360a^2\theta^{2/5}}{125r^2}\right\}. \tag{62}$$

Sixth case.

$$\left.\begin{aligned} &r = a\theta^{5/3} \;;\quad \frac{dr}{d\theta} = \frac{5}{3}a\theta^{2/3} \;;\quad \left(\frac{dr}{d\theta}\right)^2 = \frac{25}{9}a^2\theta^{4/3} \;; \\ &\frac{1}{p^2} = \frac{25a^2\theta^{4/3} + 9r^2}{9r^4} \;;\quad p = \frac{3r^2}{\sqrt{25a^2\theta^{4/3} + 9r^2}} \;;\quad \frac{dp}{dr} = \frac{(150a^2\theta^{4/3} + 27r^2)r}{(25a^2\theta^{4/3} + 9r^2)^{3/2}}. \end{aligned}\right\} \tag{63}$$

Therefore

$$P = \frac{h^2}{p^3}\,\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{150a^2\theta^{4/3}}{27r^2}\right\}. \tag{64}$$

It will be seen from this series of spirals that the law of attraction in these curves has the general form

$$P = \frac{h^2}{r^3}\left\{1 + \frac{Na^2\theta^q}{r^2}\right\}, \tag{65}$$

where N is a numerical coefficient, and q is an exponent, usually a fraction, proper or improper, but always less than 2. The law of attraction in all these spirals is therefore quite complicated.

§ 20. *Other Possible Forms of Spirals.*

A great many other forms of spirals besides those here investigated could evidently be imagined. In most cases the laws of attraction would be very complex, and the results therefore of little interest in our present inquiry. In addition to the form $r = a\theta^n$ we might mention the closely analogous form $r = a^n\theta$, which differs from that above treated only in the fact that the exponent relates to the base rather than to the angle. It is, however, a matter of no practical consequence whether we have the motion from the pole uniform and the rate of motion in the angle alone variable, as in the form $r = a\theta^n$, considered in the preceding sections; or whether we take the angular motion to be uniform and the motion from the pole alone variable, as in the form $r = a^n\theta$. The result in the two cases is practically the same, for it makes no essential difference which quantity, a or θ, is subjected to successive changes of exponents.

To illustrate this we shall now consider the two simplest cases, $n = \frac{3}{2}$ and $n = 2$. In the first case,

$$r = a^{3/2}\theta. \tag{66}$$

and

$$\left.\begin{aligned} &\frac{dr}{d\theta} = a^{3/2}\ ; \quad \left(\frac{dr}{d\theta}\right)^2 = a^3\ ; \quad \frac{1}{p^2} = \frac{a^3 + r^2}{r^4}\ ; \quad p = \frac{r^2}{\sqrt{a^3 + r^2}}\ ; \\ &p^3 = \frac{r^6}{(a^3 + r^2)^{3/2}}\ ; \quad \frac{dp}{dr} = \frac{r(2a^3 + r^2)}{(a^3 + r^2)^{3/2}}. \end{aligned}\right\} \tag{67}$$

Therefore

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{2a^3}{r^2}\right\}. \tag{68}$$

In the second case,

$$r = a^2\theta. \tag{69}$$

and

$$\left.\begin{aligned} &\frac{dr}{d\theta} = a^2\ ; \quad \left(\frac{dr}{d\theta}\right)^2 = a^4\ ; \quad \frac{1}{p^2} = \frac{a^4 + r^2}{r^4}\ ; \quad p = \frac{r^2}{\sqrt{a^4 + r^2}}\ ; \\ &p^3 = \frac{r^6}{(a^4 + r^2)^{3/2}}\ ; \quad \frac{dp}{dr} = \frac{r\,(2a^4 + r^2)}{(a^4 + r^2)^{3/2}}. \end{aligned}\right\} \tag{70}$$

Therefore

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^3}\left\{1 + \frac{2a^4}{r^2}\right\}. \tag{71}$$

Consequently we see that the expression for the attractive force is always of the same general form, namely, one depending on the inverse third and inverse fifth powers of the distances, whether the equations for the spirals are of the form $r = a\theta^n$ or $r = a^n\theta$. The spirals are therefore in the two cases of the same general type, though differing somewhat in properties and mode of construction.

Thus the cases already considered seem to be the only ones of much interest, and we may proceed with our inquiry without fear that we have overlooked any important class of spirals. If the observed figures of the nebulae show no regularity compared to these standards, we may have no hesitation in concluding that their figures are naturally and inherently irregular, and follow no geometrical law whatsoever, but depend wholly on chance.

§ 21. *On the Projections of Plane Spirals in Space, and on the Mathematical Criteria for the Recognition of Particular Spirals Among the Nebulae.*

If we consider an orbit or other geometrical figure lying in a plane which cuts the plane tangent to the celestial sphere at an angle i, it will be clear that all lines in the figure drawn perpendicular to the line of nodes will be shortened by the factor cos$i$. The only radius vector of an orbit or plane spiral path which will not be shortened by projection is that coinciding with the line of nodes. Any other radius vector will be shortened by the factors $\cos i \sin u$, and become

$r \cos i \sin u$, where u is the argument of the latitude, or angular distance in the plane of the orbit, reckoned from the ascending node.

Thus the projection of any spiral will leave some geometrical elements of the figure unchanged, in certain parts of the circumference; and we may utilize the criteria thus supplied for the recognition of particular spirals, in case they should really exist among the nebulae. The uncertainty in this subject, arising mainly from the supposed difficulty of establishing decisive criteria, seems to have discouraged investigators from entering upon this comparison of the actual nebulae as seen in projection, with the mathematical figures of spirals of known properties. We shall now apply these general conceptions to some particular spirals, which will illustrate the general theory of these criteria for the rejection of spirals proposed for the explanation of the nebulae.

§ 22. *The Projection of the Spiral of* Archimedes.

We have already seen that the equation of this curve is $r = a\theta$. The property of the spiral of Archimedes most easily recognized in any projection of the path is evidently the equal intervals between the successive spires, $PQ = QR$, &c. $= 2a\pi$.

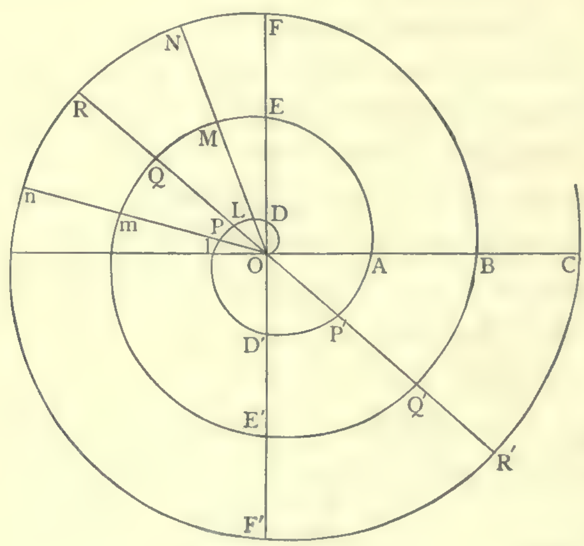

Fig. 12.

For, however this spiral be projected, one diameter will remain unchanged; and moreover the intervals between the spires in any other direction will be shortened by the same constant factor, the apparent radius vector becoming

$$r' = r\cos l = n\,.\,2a\pi\,.\cos l, \tag{72}$$

where l is the latitude, defined by the equation

$$\sin l = \sin i \sin u. \tag{73}$$

As the reduction factor $\cos l$ is common to each of the successive intercepts between the n spires seen in any direction, they would all be shortened in the same proportion, and the intercepts along any radius vector always remain of constant length, though in general they would vary along different radii vectores drawn from the centre. The intercepts coinciding with the line of nodes should have the maximum apparent length. Accordingly, if the spiral of ARCHIMEDES exists among the spiral nebulae, this criterion will serve for its immediate recognition; and we need do nothing more than compare the intercepts between the spires in any direction. The more unequal these intercepts prove to be, the more emphatically we are obliged to reject this curve as representing the figures of the nebulae.

In an attempt to test the validity of the Spiral of ARCHIMEDES, we have applied this criterion to the following nebulae: *Messier* 33 *Trianguli*, *Messier* 51 *Canum Venaticorum*, *Messier* 101 *Ursae Majoris* — as photographed by RITCHIE at the Yerkes Observatory; and to *Messier* 61, *Messier* 74 *Piscium*, *Messier* 77 *Ceti*, *Herschel* I 53 *Pegasi*, *Herschel* I 55 *Pegasi*, *Herschel* V 41, *Messier* 18 *Ursae Majoris* — all photographed by KEELER and PERRINE at the Lick Observatory; *Messier* 81 *Ceti*, and the great *Nebula of Andromeda* — photographed by ROBERTS. It was generally found that the criterion fails and that the departure of the figures of the nebulae from that of the Spiral of ARCHIMEDES is conspicuous and unmistakeable. It is only in rare cases that the intervals between successive spires are equal in any direction, and the irregularities are conspicuous in almost every nebula. Accordingly we are obliged to conclude that the convolutions in the nebulae do not correspond to the Spiral of ARCHIMEDES.

§ 23. *The Projection of the Logarithmic or Equiangular Spiral.*

The equation of this curve is $r = a^{\theta}$. The property of the curve most easily recognized is the equal angles made by the radius vector with the tangent to the spiral. This angle is always constant and the tangent of it equal to the modulus of the system of logarithms. As we see the spirals only in projection, and not as they really are, this property of equiangularity obviously will not be preserved

in the outlines traced upon the plane tangent to the celestial sphere. In some cases, however, no doubt the angle i would be small or insensible, and the protected figures of the spirals agree sensibly with the real ones.

If, for example, the radius vector be at the node, then we shall have $r' = r \cos l = r$, because the latitude is zero; but in the more general case where r' is not at the node, we have for the argument of the latitude u,

$$\tan u = \sec i \tan (p - \Omega), \tag{74}$$

where p is the position angle of the point in question. Suppose the angle between the radius vector and the tangent to the curve to be ϑ, $\tan \vartheta =$ modulus of the system of logarithms. And let the arguments of the latitude for two points of the nebula be u_1 and u_2. Then our equation is $\tan u_1 = \sec i \tan (p_1 - \Omega)$, which must be satisfied.

From the equation $r = a^\theta$, we may calculate r and θ in the real spiral; and hence we get

$$\tan u_2 = \tan (u_1 + \vartheta) = \sec i \tan (p_2 - \Omega),$$

$$\frac{\tan (u_1 + \vartheta)}{\tan u_1} = \frac{\tan (p_2 - \Omega)}{\tan (p_1 - \Omega)}. \tag{75}$$

If this equation is not satisfied, it will indicate that the spiral is not equiangular. The point p_2 must be found from $r = a^\theta$, when a and θ are known , and $r' = r \cos l_2$. When the inclination is small these tests will be easily applied, for then the inclination can be almost neglected, especially if the variations in the angle ϑ are large. For then no change in u will materially effect the observed angle ϑ'.

§ 24. *Practical Tests of Certain Spirals.*

The best way to test the Logarithmic or Equiangular Spiral is to apply it to a spiral nebula of symmetrical form as free as possible from distortion by projection. *Messier* 74 *Piscium,* as photographed at the Lick Observatory, is such a symmetrical nebula. It is found to fail in this case, and it may be presumed to fail in all other cases. It will be seen later that this spiral fails quite generally when subjected to this rigorous test, and hence we are obliged to reject the logarithmic spiral as representing the figures of the nebulae.

The Lituus of Cotes. The equation is $r^2\theta = a_2$. In this case there is an asymptote corresponding to an infinite branch, or rather two infinite branches, one positive the other negative; for when $\theta = o$, $r = \pm \infty$. As the spiral nebulae do not have infinite branches, we may reject this curve as inapplicable. It is found, moreover, that the coils of the *Lituus* do not correspond to the figures of the nebulae where they are well defined.

The Hyperbolic or Reciprocal Spiral, $r = \frac{a}{\theta}$, may be rejected for the same reason; and in general all spirals with asymptotes are excluded.

The Parabolic Spiral, with equation $(r - a)^2 = 4ca\theta$, when drawn, is found to give wider coils near the centre than further out from the origin, on account of the overlap as the successive coils are wound around the circle of radius a. And since the observed nebulae have their widest coils on the outside, we need not dwell on this curve, because it is shown by observation to be inapplicable to the nebulae.

The methods here given may be extended to any suggested spiral, and some criterion developed which will show the spiral in question to be applicable or inapplicable to the nebulae, in spite of the fact that they are always seen in projection, with only one visible outline in each case. A general objection to most complex spirals is that the law of force is very complex, and does not correspond to gravity, even as modified by the action of repulsive forces, such as those depending on the pressure of light or electric charges.

§ 25. *Further Consideration of the Spiral of* ARCHIMEDES, *with Allowance for the Uncertainty of the Photographs.*

Having now excluded the other spirals from consideration by some contradiction with observed phenomena, it remains to examine more critically the two spirals which have the closest resemblance to the observed figures of the nebulae. We shall first discuss the Spiral of ARCHIMEDES and afterwards consider the Equiangular Spiral.

The Spiral of ARCHIMEDES has equal intercepts between the successive coils, and as the parts of lines in any direction are foreshortened in the same proportion by projection, this same property is preserved in the figure of the spiral as projected on the plane tangent to the celestial sphere. Consequently unless the intercepts between the coils in any given direction are equal, though in general they will be unequal in *different directions,* owing to varying fore-

shortening in projection, we may conclude that the figures of the nebulae do not correspond to the Spiral of ARCHIMEDES.

Now in applying this simple criterion to the heavens, it is necessary to bear in mind the limits of possible errors of observation, and the fact that the photographic plate depicts actinic activity rather than mere quantity of matter, which in some places might be rather obscure, owing to lack of chemical activity. Thus a certain margin of uncertainty must be admitted to allow for the effects of unequal luminosity upon the sensitive plate. But nevertheless if the nebulae, as observed in many well authenticated cases, violate the Archimedean conditions, we shall have to reject this spiral as not representing a law of nature. We have applied these criteria to the available photographs of nebulae taken at the Lick and Yerkes Observatories and by ROBERTS at Starfield, England. And the failure of the Archimedean spiral has been so general and so conspicuous where the photographs were clear and indisputable, that we have been obliged to give up any hope that the figures of the nebulae could be thus represented.

In a few cases the criteria seem to be approximately fulfilled. But in almost all other cases there were obvious contradictions. The discordances were so decided as to be quite conspicuous, even when a wide margin for the uncertainty of the photograph was allowed; and as the contradictions are much more numerous than the agreements with the Archimedean criteria, it is obvious that we are not justified in adopting this spiral as corresponding to a law of nature.

§ 26. *Further Consideration of the Logarithmic or Equiangular Spiral with Reference to Possible Errors of Observation.*

The property of equiangularity makes it easy to test the Logarithmic Spiral where the angle of projection is insensible. In other cases the criterion is more difficult to apply, and owing to the uncertainty attaching to photographic results, in which intensity of actinic radiation rather than density of matter is depicted, there are cases which do not admit of solution with any considerable degree of confidence. But if there are a number of cases of very small or zero inclination, so that the spiral obviously is seen without appreciable distortion by projection, it becomes possible to adopt or reject the Equiangular Spiral as a curve applicable or inapplicable to the heavens. Now when we study the nebulae as depicted on the plates taken at the Lick Observatory and elsewhere, we find that several most excellent specimens of small or zero inclination of the spirals really exist, and hence these criteria can be safely applied. For example the nebulae *Messier* 51,

Messier 61, *Messier* 74; *Messier* 33, and *Messier* 101, show very little evidence of distortion by projection, and we are therefore justified in taking the observed angles as very nearly the true ones. Now these are not constant, either in different parts of the same nebulae, or in different nebulae compared to one another. The angles made by the radius vector with the tangent to the coils vary all the way from 0° to 45°, which is much too large to be ascribed to errors of observation. We are therefore justified in concluding that the angles are not equal, and the spirals are not logarithmic; though there is often a fairly close approximation to this curve for a single case. It is noticeable, however, that the angle between the radius vector and the tangent to the curve not only is not constant in any one case, but also different in different cases, as if corresponding to different systems of logarithms; so that there is evidently no correspondence to any system of logarithms whatsoever.

Thus the similarity of figure always breaks down at some point sufficiently clear and well defined to leave no doubt that the spiral cannot be considered, within admissible limits of uncertainty, to be truly equiangular in any one case, and still less is this true of all the different cases where the angles vary so greatly. Consequently the law of attraction operating in the observed spirals cannot agree with that of the inverse cube of the distance, under which alone a particle may describe the Logarithmic Spiral.

§ 27. *On the Comparison of the Other Spirals with the Photographs of the Nebulae.*

We have dwelt at some length on the observational proof that neither the Spiral of ARCHIMEDES nor the Logarithmic Spiral could be considered to apply to the figures of the nebulae. As we have seen the Archimedean Spiral represents a law of attraction varying as the inverse cube and inverse fifth power of the distance combined, and is therefore typical of the forces operating in a great number of spirals. On the other hand the Logarithmic Spiral represents a force varying simply inversely as the cube of the distance, without any term depending on the inverse fifth power; and is therefore less typical of spirals in general than is the Spiral of ARCHIMEDES, which follows a more complicated law of attraction.

Now it is tedious to dwell at length on each of a long list of spirals, and we have therefore found it more practicable to treat these spirals by types, selecting the Spiral of ARCHIMEDES as the simplest of these depending on complicated laws of attraction; and the Logarithmic Spiral as the most typical of the spirals depending

on very simple laws of attraction. As both of these types fail conspicuously to represent the actual figures of the nebulae, we may safely infer that all other spirals of the same general type would also fail, both from the nature of the curves, and from the nature of the central forces under which they might be described. If the photographs are sufficiently sharp to make entirely clear the failure of the selected types of spirals, they would also make clear the failure of the other related curves, representing both simple and complex laws of central force.

§ 28. *The General Character of Chance Spirals.*

If, therefore, we give up the expectation that any regular geometrical spirals can be found to represent the figures of the nebulae, the question at once arises whether spirals depending wholly on chance would appear to tend towards an approximate type. This is an important question, and before we can answer it we must examine the conditions of the problem very carefully. Let us first assume that the spiral nebulae most frequently arises from the approach and mutual coiling up of two streams of cosmical dust. At first they would resemble cometary nebulae, with the heads and necks curved, but with tails nearly straight. At a later stage of development the coiling would have gone on so far that only the ends of the tails would be free of bending. All the rest of the swarms would have become involved in the winding up of the two streams under the force of their mutual attraction, and by virtue of the difference of initial relative motion. As the winding up became more and more complete, the coils near the centre of the spiral would show increasingly regularity of figure, while the exterior parts would present the principal evidences of irregularity. This is what actually appears among the nebulae, in such cases as *Messier* 51, *Messier* 61, *Messier* 101, and *Messier* 33. Accordingly we may conclude that the regularity of the figure of a nebulae depends to a large extent upon its stage of development.

Now there are obviously all stages of development, from the cometic nebulae with curved necks, to the aged nebulae in which the coils have become dense and regular. In our search for spiral nebulae *we are most impressed with those which show symmetry of figure,* and approximation towards a normal type; while the unformed and but little developed spirals are scarcely noticed, because they do not tend towards a definite type.

We conclude, therefore, that the tendency towards a type noticed among the spiral nebulae is a natural outcome of chance movement, at that stage of development where the effects are conspicuous.

§ 29. *Chance Spirals Tend Towards a Type, with only an Approximate Geometrical Figure: Meaning of the Wings Shown by the Spiral Nebulae, and the Temperature of the Nebulae.*

This proposition is sufficiently clear from what precedes, and need not be dwelt upon here. It is obvious that the individual nebulae should exhibit irregularities depending on the imperfection of their development. The approximate geometrical figure towards which the nebulae conform seems to be a type combining the properties of the Spiral of Archimedes and of the Logarithmic Spiral. The Spiral of Archimedes has all its spires at equal distances, whereas in the nebulae the outer spires are frequently, if not generally, a little wider than those near the centre; on the other hand the spires as we go outward from the centre do not separate so rapidly as in the case of the Logarithmic Spiral. A mixture of these two spirals thus seems to give the type most common among the nebulae. In the older nebulae no doubt the tendency approaches the Spiral of Archimedes; in the newer nebulae it more nearly resembles the Logarithmic Spiral.

While these appear to be the average tendencies, it must be expected that chance irregularities will be very numerous, and often more conspicuous than the normal types the nebulae are supposed to approach. This state of affairs is confirmed by observation and justifies our interpretation of the figures of the nebulae as wholly the effect of chance, and of the stage of development attained in the individual case. As the coiling up of the streams of cosmical dust would sometimes give smoother figures than in others, owing to the initial conditions of motion, the stage of development could not *certainly* be inferred from the observed appearance of the coils, though this might often be done *with a high degree of probability*.

If the above view of the origin of spiral nebulae be admissible, it will follow that in the winding up of the two original streams of cosmical dust, there should appear outside of the nebulae surviving evidences of the original direction of these streams, in the form of projecting wings or ansae. These opposite wings are observed on many nebulae, and their meaning does not appear to be open to question. Before the two streams united to form a spiral nebula they were, no doubt, largely or wholly dark; but as the central mass coiled up it became luminous throughout the interior, and even a short distance beyond the borders of the last coil. Hence the visibility of the ansae or wings projecting from the spiral nebulae. Whether the luminosity depends on collisions producing light and luminescence among the particles of cosmical dust which compose a nebula, or whether electric or radio-active forces are at work and increase in intensity when the two streams

begin to wind up, we need not consider at present. Possibly both of these sources of light may be available. In any case the nebulae are certainly at very low temperature. If the temperature were appreciably above that of the celestial spaces, these rare masses would cool down in a short time to the absolute zero, because in such rare masses the heat could not be retained. This does not imply that there may not be a very considerable elevation of temperature in the central or denser parts of certain nebulae. But if the spiral nebulae be the outcome of chance swarms of great rarity, practically devoid of hydrostatic pressure in the coils, it will follow that the temperature is everywhere very low, except where centres are being developed for the evolution of planets and components of multiple stars. Planets develop in rare masses devoid of hydrostatic pressure, if we may judge by the example of the solar system; and while they may be highly heated within, the surrounding nebulosity itself is at very low temperature.

A familiar illustration of spiral movement may be noticed in the revolving spray used for watering the lawn. The movement thus arising is not exactly identical with that seen in the nebulae, but so closely analogous to it as to be worthy of attention. The water flows out under steady pressure, and two opposite twigs of the spray are curved so as to give by reaction a couple about the axis of rotation, and the reaction of the flowing stream keeps it whirling backward. As the spray rotates, it seems to carry the streams of water along with it, whereas the particles in the stream really move forward, and others rush out to take their places, so as to make the stream continuous. Notwithstanding this forward motion of the individual particles of water, the *stream itself forms backward with the whirling of the spray, and thus presents to the eye a striking analogy to the phenomena noticed in the nebulae, where streams are coiling up under their mutual attraction, and thus revolving much like the spray on the lawn.* In the heavenly spaces, as in this experiment here on the earth, the *whirling movement is in the direction which the streams take in going towards the center.*

§ 30. *Final Conclusions Regarding the Central Forces Operating in Nature.*

The treatment of Central Forces given in these two chapters, in connection with that given in Volume I of these *Researches,* is believed to be sufficient for all philosophic purposes; but it may be well to remark that this discussion does not claim to exhaust all that has been accomplished by mathematicians on this very extensive subject. It has been deemed advisable to restrict this brief account to the simpler and more practical results, which do not involve the use of the transcendental functions, such as the elliptic integrals and the elliptic functions, which

have at length become so very important in modern analysis. Those who wish to examine laws of attraction of the general type $P = \mu r^{\nu}$, other than those considered in Chapters I and II of this volume, may consult WHITTAKER'S *Analytical Dynamics*, pp. 79–86.

It is easily shown that the time is given by the integral

$$t = \frac{1}{h}\int r^2 d\theta + \text{constant} \ ;$$

and therefore the problem of motion under central forces is always soluble by quadratures, when the force is a function of the distance only, as was long ago remarked by LAPLACE in the *Mécanique Céleste* (Liv. II, Chap. I, § 2).

When $P = \mu r^{\nu}$, and $\nu = 5, 3, 1, 0, -2, -3, -5, -7$, and many fractional values besides, the problem is shown to be reducible to circular or elliptic functions. Cases with $\nu = 5, 3, 0, -4, -5, -7$, lead to elliptic integrals; but on inverting the integrals, we may obtain the solution in terms of elliptic functions. WHITTAKER gives the solution for a case in which $\nu = -5$, and finds the time in terms of the Weierstrassian elliptic functions (loc. cit., pp. 82–84). These results are of profound interest from a mathematical point of view, for they extend the domain of the higher analysis introduced by WEIERSTRASS, SCHWARTZ, and other great mathematicians; but they are too purely theoretical in character to be entered upon in this work, which must have in view chiefly application to the systems actually observed in the physical universe.

In the standard work on Dynamics just cited, WHITTAKER remarks that it has been shown by BERTRAND and KOENIGS *that of all the laws of force which give a zero force at an infinite distance, the Newtonian is the only one for which all the orbits are algebraic curves, and also the only one for which all the orbits are closed curves.* In accordance with these results, it has been shown by elaborate investigations carried out on visual binaries, in Volume I of these *Researches*, and by exact spectrographic investigations since made on many spectroscopic binaries, at the Lick Observatory and elsewhere, that the Newtonian law holds true among double and multiple stars of every known kind throughout the sidereal universe.

The spiral paths shown in the nebulae, however, are neither closed curves, nor any kind of algebraic curves; because the spirals are often broken, so as to become more or less discontinuous, and markedly irregular. Yet it is not to be inferred from this circumstance that central forces different from gravity are at work among the nebulae. On the contrary, *since gravitation is found by observation to govern the motions of the stars, and the nebulae pass by insensible gradations*

into stellar systems, as is conclusively shown in this volume, it necessarily follows that the central forces operating in the nebulae can be nothing else than universal gravitation: which is a last and sufficient proof that the observed figures of the nebulae are chance spirals, and therefore necessarily depart from any kind of geometrical regularity.

CHAPTER III.

Theory of the Motion of a System of Bodies Subjected to Their Mutual Gravitation, and of the Invariable Plane Discovered by Laplace in 1784.

§ 31. *On the Origin of Cosmical Systems from Certain Initial Conditions of Mass and Motion.*

In order to keep in mind a clear conception of the mechanism of the planetary system and of the similar systems existing in space, it becomes advisable to consider the theory of the motion of such a system of bodies and of the Invariable Plane by which it is characterized. In the planetary system all of the principal bodies revolve near a fundamental plane, which is also a plane of maximum areas; such a plane exists in every system of bodies revolving freely under the mutual gravitation of its parts, and undisturbed by extraneous influences. This plane was discovered by Laplace in 1784 (cf. *Oeuvres Complètes de* Laplace, Tome XI, pp. 548–551), and is called the Invariable Plane of the system, because no mutual action of the attracting bodies can ever disturb it. When our whole solar system is transported along in its secular motion through space, this plane remains rigorously parallel to its original position among the fixed stars.

It will be seen below that the position of the Invariable Plane is determined by the masses, their mutual distances and the initial velocities and directions with which they are started; so that the variation of these initial elements would shift the position of the plane of maximum areas; but when they are once given in the starting of the system, its position is thereafter forever fixed, and nothing but external forces can ever disturb it. As the universe is constructed on an immense scale, with vast spaces intervening between the systems which compose it, each system carries with it an invariable plane which remains parallel to its original position.

An examination of the theory of the motion of an infinite system of bodies subjected to their mutual gravitation will therefore throw light upon the development of cosmical systems from spiral nebulae. These whirling vortices originate

by the chance approach or settling of streams of nebulosity drifting hither and thither in the general movements of the universe. We shall, therefore, give this mathematical theory with some care, because a correct understanding of it is essential for the interpretation of the phenomena presented by the solar system and by the partially developed systems seen in the spiral nebulae.

§ 32. *On the Differential Equations of the Motions of a System of Bodies Subjected to their Mutual Gravitation.*

Suppose a system of $n + 1$ rigid bodies, whose masses are respectively M_0, $m_1, m_2, m_3 \ldots m_n$, revolving without constraint and subject to the action of their mutual gravitation. Let $O\xi$, $O\eta$, $O\zeta$ be a system of rectangular axes fixed in space and ξ_i, η_i, ζ_i the coördinates of the centers of gravity of these masses referred to this origin.

We shall designate the distance between any two masses m_i and m_j by $\triangle_{i,j}$; and therefore

$$\triangle^2_{i,j} = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2 \tag{76}$$

When $i = 0$, the mass will be M_0, and its coördinates ξ_0, η_0, ζ_0. The forces which any mass m_1 exert upon the mass M_0 are evidently

$$\frac{M_0 m_1}{\triangle_{0,1}^2} \cdot \frac{\xi_1 - \xi_0}{\triangle_{0,1}} \quad ; \quad \frac{M_0 m_1}{\triangle_{0,1}^2} \cdot \frac{\eta_1 - \eta_0}{\triangle_{0,1}} \quad ; \quad \frac{M_0 m_1}{\triangle_{0,1}^2} \cdot \frac{\zeta_1 - \zeta_0}{\triangle_{0,1}}. \tag{77}$$

We have therefore the following differential equations for the motion of M_0 under the action of the system of bodies

$$\left.\begin{aligned}
M_0\frac{d^2\xi_0}{dt^2} &= M_0 m_1 \frac{\xi_1 - \xi_0}{\triangle_{0,1}^3} + M_0 m_2 \frac{\xi_2 - \xi_0}{\triangle_{0,2}^3} + \ldots + M_0 m_n \frac{\xi_n - \xi_0}{\triangle_{0,n}^3}, \\
M_0\frac{d^2\eta_0}{dt^2} &= M_0 m_1 \frac{\eta_1 - \eta_0}{\triangle_{0,1}^3} + M_0 m_2 \frac{\eta_2 - \eta_0}{\triangle_{0,2}^3} + \ldots + M_0 m_n \frac{\eta_n - \eta_0}{\triangle_{0,n}^3}, \\
M_0\frac{d^2\zeta_0}{dt^2} &= M_0 m_1 \frac{\zeta_1 - \zeta_0}{\triangle_{0,1}^3} + M_0 m_2 \frac{\zeta_2 - \zeta_0}{\triangle_{0,2}^3} + \ldots + M_0 m_n \frac{\zeta_n - \zeta_0}{\triangle_{0,n}^3}.
\end{aligned}\right\} \tag{78}$$

In like manner the action of the system upon the mass m_1 would be

$$\left.\begin{aligned}
m_1\frac{d^2\xi_1}{dt^2} &= m_1 M_0 \frac{\xi_0 - \xi_1}{\triangle_{1,0}^3} + m_1 m_2 \frac{\xi_2 - \xi_1}{\triangle_{1,2}^3} + m_1 m_3 \frac{\xi_3 - \xi_1}{\triangle_{1,3}^3} + \ldots + m_1 m_n \frac{\xi_n - \xi_1}{\triangle_{1,n}^3}, \\
m_1\frac{d^2\eta_1}{dt^2} &= m_1 M_0 \frac{\eta_0 - \eta_1}{\triangle_{1,0}^3} + m_1 m_2 \frac{\eta_2 - \eta_1}{\triangle_{1,2}^3} + m_1 m_3 \frac{\eta_3 - \eta_1}{\triangle_{1,3}^3} + \ldots + m_1 m_n \frac{\eta_n - \eta_1}{\triangle_{1,n}^3}, \\
m_1\frac{d^2\zeta_1}{dt^2} &= m_1 M_0 \frac{\zeta_0 - \zeta_1}{\triangle_{1,0}^3} + m_1 m_2 \frac{\zeta_2 - \zeta_1}{\triangle_{1,2}^3} + m_1 m_3 \frac{\zeta_3 - \zeta_1}{\triangle_{1,3}^3} + \ldots + m_1 m_n \frac{\zeta_n - \zeta_1}{\triangle_{1,n}^3}.
\end{aligned}\right\} \tag{79}$$

And in general

$$m_i \frac{d^2\xi_i}{dt^2} = m_i M_0 \frac{\xi_0 - \xi_i}{\triangle_{i,0}^3} + m_i m_2 \frac{\xi_2 - \xi_i}{\triangle_{i,2}^3} + m_i m_3 \frac{\xi_3 - \xi}{\triangle_{i,3}^3} + \ldots\ldots + m_i m_n \frac{\xi_n - \xi_i}{\triangle_{i,n}^3},$$

with corresponding equations for the other coordinates η_i, ζ_i.

LAGRANGE has simplified the treatment of these equations by the introduction of the *force function:*

$$\left.\begin{array}{rll} U = \dfrac{M_0 m_1}{\triangle_{0,1}} + \dfrac{M_0 m_2}{\triangle_{0,2}} + \dfrac{M_0 m_3}{\triangle_{0,3}} + \ldots\ldots + \dfrac{M_0 m_n}{\triangle_{0,n}}, & i = 0 \ , \ j = 1 \ , \ \ldots\ldots \ j = n\,, \\ \dfrac{m_1 m_2}{\triangle_{1,2}} + \dfrac{m_1 m_3}{\triangle_{1,3}} + \ldots\ldots + \dfrac{m_1 m_n}{\triangle_{1,n}}, & i = 1 \ , \ j = 2 \ , \ \ldots\ldots \ j = n\,, \\ \dfrac{m_2 m_3}{\triangle_{2,3}} + \ldots\ldots + \dfrac{m_2 m_n}{\triangle_{2,n}}, & i = 2 \ , \ j = 3 \ , \ \ldots\ldots \ j = n\,, \\ \ldots\ldots\ldots\ldots & \\ \dfrac{m_{n-1} m_n}{\triangle_{n-1,n}}. & i = n-1 \ , \ j = n\,. \end{array}\right\} \tag{80}$$

$$= \sum_{i=0}^{i=n-1} \sum_{j=1}^{j=n} \frac{m_i m_j}{\triangle_{i,j}}.$$

Now since

$$\frac{\partial \triangle_{0,1}}{\partial \xi_0} = \frac{\xi_0 - \xi_1}{\triangle_{0,1}} \ , \quad \frac{\partial \triangle_{0,2}}{\partial \xi_0} = \frac{\xi_0 - \xi_2}{\triangle_{0,2}} \ , \quad \frac{\partial \triangle_{0,3}}{\partial \xi_0} = \frac{\xi_0 - \xi_3}{\triangle_{0,3}} \ , \quad \ldots\ldots \quad \frac{\partial \triangle_{0,n}}{\partial \xi_0} = \frac{\xi_0 - \xi_n}{\triangle_{0,n}}, \tag{81}$$

we have

$$\frac{\partial U}{\partial \xi_0} = -\frac{M_0 m_1}{\triangle_{0,1}^2}\frac{\partial \triangle_{0,1}}{\partial \xi_0} - \frac{M_0 m_2}{\triangle_{0,2}^2}\frac{\partial \triangle_{0,2}}{\partial \xi_0} - \frac{M_0 m_3}{\triangle_{0,3}^2}\frac{\partial \triangle_{0,3}}{\partial \xi_0} - \ldots\ldots - \frac{M_0 m_n}{\triangle_{0,n}^2}\frac{\partial \triangle_{0,n}}{\partial \xi_0}. \tag{82}$$

Whence

$$\frac{\partial U}{\partial \xi_0} = M_0 m_1 \frac{\xi_1 - \xi_0}{\triangle_{0,1}^3} + M_0 m_2 \frac{\xi_2 - \xi_0}{\triangle_{0,2}^3} + M_0 m_3 \frac{\xi_3 - \xi_0}{\triangle_{0,3}^3} + \ldots\ldots + M_0 m_n \frac{\xi_n - \xi_0}{\triangle_{0,n}^3}. \tag{83}$$

Therefore equation (77) gives

$$\left.\begin{array}{lll} M_0 \dfrac{d^2\xi_0}{dt^2} = \dfrac{\partial U}{\partial \xi_0} \ , & M_0 \dfrac{d^2\eta_0}{dt^2} = \dfrac{\partial U}{\partial \eta_0} \ , & M_0 \dfrac{d^2\zeta_0}{dt^2} = \dfrac{\partial U}{\partial \zeta_0}, \\ m_1 \dfrac{d^2\xi_1}{dt^2} = \dfrac{\partial U}{\partial \xi_1} \ , & m_1 \dfrac{d^2\eta_1}{dt^2} = \dfrac{\partial U}{\partial \eta_1} \ , & m_1 \dfrac{d^2\zeta_1}{dt^2} = \dfrac{\partial U}{\partial \zeta_1}, \\ m_2 \dfrac{d^2\xi_2}{dt^2} = \dfrac{\partial U}{\partial \xi_2} \ , & m_2 \dfrac{d^2\eta_2}{dt^2} = \dfrac{\partial U}{\partial \eta_2} \ , & m_2 \dfrac{d^2\zeta_2}{dt^2} = \dfrac{\partial U}{\partial \zeta_2}, \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ m_n \dfrac{d^2\xi_n}{dt^2} = \dfrac{\partial U}{\partial \xi_n} \ , & m_n \dfrac{d^2\eta_n}{dt^2} = \dfrac{\partial U}{\partial \eta_n} \ , & m_n \dfrac{d^2\zeta_n}{dt^2} = \dfrac{\partial U}{\partial \zeta_n}. \end{array}\right\} \tag{84}$$

The integration of this system of $3n + 3$ differential equations of the second order would give the motion of the $n + 1$ bodies; but in spite of the labors of the greatest geometers of the past two centuries, this has not yet been accomplished with entire rigor and generality, except in the simple case where n equals 1. In this case of the problem of two bodies the result is undisturbed motion in a conic section, which NEWTON obtained in his *Philosophiae Naturalis Principia Mathematica*, 1687.

§ 33. *Integrals for the Motion of the Center of Gravity.*

If we differentiate the force function,

$$U = \sum_{i=0}^{i=n-1} \sum_{j=1}^{j=n} \frac{m_i m_j}{\triangle_{i,j}},$$

we shall obtain

$$\left.\begin{aligned} \frac{\partial U}{\partial \xi_i} &= m_i \sum_{j=1}^{j=n} \frac{\xi_j - \xi_i}{\triangle_{i,j}^3}, \\ \frac{\partial U}{\partial \eta_i} &= m_i \sum_{j=1}^{j=n} \frac{\eta_j - \eta_i}{\triangle_{i,j}^3}; \end{aligned}\right\} \tag{85}$$

whence

$$\xi_i \frac{\partial U}{\partial \eta_i} - \eta_i \frac{\partial U}{\partial \xi_i} = m_i \sum_{j=1}^{j=n} m_j \frac{\xi_i \eta_j - \eta_i \xi_j}{\triangle_{i,j}^3}. \tag{86}$$

These equations give

$$\left.\begin{aligned} \sum_{i=0}^{i=n-1} \frac{\partial U}{\partial \xi_i} &= \sum_{i=0}^{i=n-1} \sum_{j=1}^{j=n} m_i m_j \frac{\xi_j - \xi_i}{\triangle_{i,j}^3}, \\ \sum_{i=0}^{i=n-1} \left(\xi_i \frac{\partial U}{\partial \eta_i} - \eta_i \frac{\partial U}{\partial \xi_i}\right) &= \sum_{i=0}^{i=n-1} \sum_{j=1}^{j=n} m_i m_j \frac{\xi_i \eta_j - \eta_i \xi_j}{\triangle_{i,j}^3}. \end{aligned}\right\} \tag{87}$$

If in the second members of these equations we change i into j, and conversely, the elementary terms become equal and of contrary sign, and thus the summation vanishes,

$$\left.\begin{aligned}
&\sum_{i=0}^{i=n-1}\frac{\partial U}{\partial \xi_i}=0 \quad , \quad \sum_{i=0}^{i=n-1}\left(\xi_i\frac{\partial U}{\partial \eta_i}-\eta_i\frac{\partial U}{\partial \xi_i}\right)=0\,,\\
&\sum_{i=0}^{i=n-1}\frac{\partial U}{\partial \eta_i}=0 \quad , \quad \sum_{i=0}^{i=n-1}\left(\eta_i\frac{\partial U}{\partial \zeta_i}-\zeta_i\frac{\partial U}{\partial \eta_i}\right)=0\,,\\
&\sum_{i=0}^{i=n-1}\frac{\partial U}{\partial \zeta_i}=0 \quad , \quad \sum_{i=0}^{i=n-1}\left(\zeta\frac{\partial U}{\partial \xi_i}-\xi_i\frac{\partial U}{\partial \zeta_i}\right)=0\,.
\end{aligned}\right\} \tag{88}$$

Using these relations in equations (84) we get

$$\sum_{i=0}^{i=n} m_i\frac{d^2\xi_i}{dt^2}=0 \quad , \quad \sum_{i=0}^{i=n} m_i\frac{d^2\eta_i}{dt^2}=0 \quad , \quad \sum_{i=0}^{i=n} m_i\frac{d^2\zeta_i}{dt^2}=0\,. \tag{89}$$

Also

$$\left.\begin{aligned}
&\sum_{i=0}^{i=n} m_i\left(\eta_i\frac{d^2\zeta_i}{dt^2}-\zeta_i\frac{d^2\eta_i}{dt^2}\right)=0\,,\\
&\sum_{i=0}^{i=n} m_i\left(\zeta_i\frac{d^2\xi_i}{dt^2}-\xi_i\frac{d^2\zeta_i}{dt^2}\right)=0\,,\\
&\sum_{i=0}^{i=n} m_i\left(\xi_i\frac{d^2\eta_i}{dt^2}-\eta_i\frac{d^2\xi_i}{dt^2}\right)=0\,.
\end{aligned}\right\} \tag{90}$$

If we integrate equations (89), denoting the arbitrary constants of integration by a_1, a_2; b_1, b_2; c_1, c_2; we shall find successively

$$\left.\begin{aligned}
&\sum_{i=0}^{i=n} m_i\frac{d\xi_i}{dt}=a_1 \quad ; \quad \sum_{i=0}^{i=n} m_i\frac{d\eta_i}{dt}=b_1 \quad ; \quad \sum_{i=0}^{i=n} m_i\frac{d\zeta_i}{dt}=c_1\,;\\
&\sum_{i=0}^{i=n} m_i\xi_i=a_1t+a_2 \quad ; \quad \sum_{i=0}^{i=n} m_i\eta_i=b_1t+b_2 \quad ; \quad \sum_{i=0}^{i=n} m_i\zeta_i=c_1t+c_2\,.
\end{aligned}\right\} \tag{91}$$

These six equations show that the motion of the center of gravity is rectilinear and uniform, and are known as the integrals of the motion of the center of gravity. They may be put under the following form

$$\left.\begin{aligned}
a_2&=\sum_{i=0}^{i=n} m_i\xi_i-t\sum_{i=0}^{i=n} m_i\frac{d\xi_i}{dt}\,,\\
b_2&=\sum_{i=0}^{i=n} m_i\eta_i-t\sum_{i=0}^{i=n} m_i\frac{d\eta_i}{dt}\,,\\
c_2&=\sum_{i=0}^{i=n} m_i\zeta_i-t\sum_{i=0}^{i=n} m_i\frac{d\zeta_i}{dt}\,;
\end{aligned}\right\} \tag{92}$$

which exhibits the constants a_2, b_2, c_2, as functions of the time. Accordingly we see that the integrals of (84) are of the required type, namely,

$$\text{constant} = F\left(\xi_0,\ \eta_0,\ \zeta_0,\ \ldots\ \frac{d\xi_0}{dt},\ \frac{d\eta_0}{dt},\ \frac{d\zeta_0}{dt}\ \ldots\right). \tag{93}$$

It is easily recognized that equations (91) have the following physical significance: That however the masses of the system may interact upon one another, the center of gravity is not thereby affected and must preserve its state of rest or of uniform motion in a right line.

§ 34. *General Integrals of the Areas and of the Vis Viva for a System of Bodies.*

If we multiply equations (90) by dt and integrate, denoting the arbitrary constants by a_3, b_3, c_3, we shall get

$$\left.\begin{aligned} a_3 dt &= \sum_{i=0}^{i=n} m_i\left(\eta_i\frac{d\zeta_i}{dt} - \zeta_i\frac{d\eta_i}{dt}\right)dt, \\ b_3 dt &= \sum_{i=0}^{i=n} m_i\left(\zeta_i\frac{d\xi_i}{dt} - \xi_i\frac{d\zeta_i}{dt}\right)dt, \\ c_3 dt &= \sum_{i=0}^{i=n} m_i\left(\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right)dt; \end{aligned}\right\} \tag{94}$$

which are the *General Integrals of the Areas.* Hence the important theorem that the sums of the areas described on the coördinate planes by vectors drawn from the origin to the $n+1$ bodies of the system multiplied by their respective masses are proportional to the time. If now we multiply the equations (84) by

$$\frac{2d\xi_0}{dt},\ \frac{2d\eta_0}{dt},\ \frac{2d\zeta_0}{dt};\ \frac{2d\xi_1}{dt},\ \frac{2d\eta_1}{dt},\ \frac{2d\zeta_1}{dt};\ \frac{2d\xi_2}{dt},\ \frac{2d\eta_2}{dt},\ \frac{2d\zeta_2}{dt},\ \text{etc.},$$

and add the results, observing that the Force Function U contains explicitly only the coördinates ξ_0, η_0, ζ_0; ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; etc.; so that

$$\left.\begin{aligned} \frac{dU}{dt} = \frac{\partial U}{\partial\xi_0}\frac{d\xi_0}{dt} + \frac{\partial U}{\partial\eta_0}\frac{d\eta_0}{dt} + \frac{\partial U}{\partial\zeta_0}\frac{d\zeta_0}{dt} + \frac{\partial U}{\partial\xi_1}\frac{d\xi_1}{dt} + \frac{\partial U}{\partial\eta_1}\frac{d\eta_1}{dt} + \frac{\partial U}{\partial\zeta_1}\frac{d\zeta_1}{dt} + \ldots \\ + \frac{\partial U}{\partial\xi_n}\frac{d\xi_n}{dt} + \frac{\partial U}{\partial\eta_n}\frac{d\eta_n}{dt} + \frac{\partial U}{\partial\zeta_n}\frac{d\zeta_n}{dt}; \end{aligned}\right\} \tag{95}$$

we shall find

$$\left.\begin{aligned} M_0\left(\frac{2d\xi_0}{dt}\frac{d^2\xi_0}{dt^2} + \frac{2d\eta_0}{dt}\frac{d^2\eta_0}{dt^2} + \frac{2d\zeta_0}{dt}\frac{d^2\zeta_0}{dt^2}\right) + m_1\left(\frac{2d\xi_1}{dt}\frac{d^2\xi_1}{dt^2} + \frac{2d\eta_1}{dt}\frac{d^2\eta_1}{dt^2} + \frac{2d\zeta_1}{dt}\frac{d^2\zeta_1}{dt^2}\right) + \ldots \\ + m_n\left(\frac{2d\xi_n}{dt}\frac{d^2\xi_n}{dt^2} + \frac{2d\eta_n}{dt}\frac{d^2\eta_n}{dt^2} + \frac{2d\zeta_n}{dt}\frac{d^2\zeta_n}{dt^2}\right). \end{aligned}\right\} \tag{96}$$

This expression may be written in the unexpanded form

$$\frac{d}{dt}\sum_{i=0}^{i=n} m_i\left(\frac{d\xi_i^2}{dt^2} + \frac{d\eta_i^2}{dt^2} + \frac{d\zeta_i^2}{dt^2}\right) = \frac{2dU}{dt}. \tag{97}$$

If we multiply this expression by dt, and then integrate, adding an arbitrary constant h, we shall obtain a new general integral of the system

$$2h = \sum_{i=0}^{i=n} m_i\left(\frac{d\xi_i^2}{dt^2} + \frac{d\eta_i^2}{dt^2} + \frac{d\zeta_i^2}{dt^2}\right) - 2U. \tag{98}$$

This is the integral of the *living force* or *Vis Viva.*

§ 35. *The Ten Complete General Integrals for the Motion of a System of Bodies.*

These ten general integrals, viz: six for the motion for the center of gravity, three for the conservation of areas, and one for the *Vis Viva*, are the only rigorous general integrals hitherto obtained for the motion of a system of $n + 1$ bodies. As the differential equations are of the second order, there would be required in the general problem of $n + 1$ bodies $6n + 6$ integrals of which only the ten here mentioned are known; so that even in the simplest case of three bodies, the general problem is insoluble, though a few special solutions have been discovered. One of these is LAGRANGE's solution with the three bodies at the corners of an equilateral triangle revolving with uniform velocity. As the eighteen integrals required for the complete solution of the problem of three bodies cannot be obtained, geometers have had to restrict themselves to particular solutions depending on special conditions of mass, distance or motion in space.

In his important discussion of the planetary integrals in the *Acta Mathematica*, Vol. XI, PROF. H. BRUNS, of Leipzig, has proved that there are no more general integrals of an algebraic character. POINCARÉ afterwards demonstrated in a prize memoir published in the *Acta Mathematica*, Vol. XIII, and somewhat extended in *Les Methodes Nouvelles de la Mécanique Céleste*, Tome I, Chap. V., that the problem of three bodies has no new uniform transcendental integrals, even when the restriction is introduced that the masses of two of the bodies shall be quite small with respect to the third or central body of the system. At present, therefore, mathematicians have little hope of finding any more general integrals, but a great number of special devices are introduced for throwing light on the nature of the movement in certain cases. Probably there is no general solution of the problem of three bodies, but an infinite number* of particular solutions, depending on the initial conditions.

* In his *Mécanique Céleste*, Tome I, p. 101, POINCARÉ concludes that even in the restricted problem of three bodies, where one of them is a particle, and the other two revolve in circles, there is a quadruple infinity of periodic solutions, depending on the period of the infinitesimal body, the constant of energy, the moment of conjunction, and the longitude of conjunction. DARWIN has shown, however, that this quadruple infinity may be reduced to a single infinity (cf. *Periodic Orbits*, p. 100). When the problem is not restricted, the total number of solutions would probably be an infinity of infinite order (cf. also WHITTAKER, *Analytical Dynamics*, pp. 380–381). The word *infinite* used above is to be understood in the general sense, without regard to the order of the infinity. The restricted problem of three bodies is further treated in Chapters VIII–X.

§ 36. LAPLACE'S *Invariable Plane in Any System of Bodies.*

If we examine the elements $\eta_i \frac{d\zeta_i}{dt} - \zeta_i \frac{d\eta_i}{dt}$, etc., included under the summation sign in the second members of (94), we shall find that these elements represent the double areal velocity of the masses m_i, as projected on the coördinate planes. The arbitrary constants a_3, b_3, c_3, represent the sums of these areas projected on the coördinate planes, each multiplied by the corresponding masses, for all the bodies of the system; and these constants are proportional to the time. The areas described by the projection of the radii vectores of the several bodies on the coördinate planes are the projections of areas actually described by the radii vectores of the bodies in space. Now the system of coördinates may be revolved into any position, and therefore so as to make two of the constants in (94) vanish and the third equal to $\sqrt{a_3^2 + b_3^2 + c_3^2}$, as was first remarked by LAPLACE. Hence we may imagine a resultant or principal plane of projection such that the areas traced by the projection of each radius vector on this new plane when again projected on the coördinate planes, each areal velocity being multiplied by the corresponding mass, will be respectively equal to the second members of the equations (94).

Let α, β, γ be the angle formed by the normal to this new plane with the coördinate axes $0\xi, 0\eta, 0\zeta$ and let K denote the sum of the areas traced on this plane in a unit of time by the radii vectores, each area being multiplied by the corresponding mass; then it will be found that the sum K is a maximum, and its projections on the coördinate planes for the element of time dt will be

$$a_3dt = \Pi_{\xi,\eta} = K \cos \alpha dt \quad ; \quad b_3dt = \Pi_{\eta,\zeta} = K \cos \beta dt \quad ; \quad c_3dt = \Pi_{\zeta,\xi} = K \cos \gamma dt \qquad (99)$$

Therefore we have

$$a_3 = K \cos \alpha \quad ; \quad b_3 = K \cos \beta \quad ; \quad c_3 = K \cos \gamma \quad ; \quad K^2 = a_3^2 + b_3^2 + c_3^2 . \qquad (100)$$

The angles α, β, γ are defined by the relations

$$\cos \alpha = \frac{a_3}{\sqrt{a_3^2 + b_3^2 + c_3^2}} \quad ; \quad \cos \beta = \frac{b_3}{\sqrt{a_3^2 + b_3^2 + c_3^2}} \quad ; \quad \cos \gamma = \frac{c_3}{\sqrt{a_3^2 + b_3^2 + c_3^2}} . \qquad (101)$$

As these angles are constant it follows that the principal plane of projection remains parallel to itself, whatever be the motion of the system of bodies. This remarkable plane was discovered by LAPLACE, A.D. 1784, and is called the Invariable Plane of the system. Such invariable planes characterize the planetary system and the other similar systems existing in the immensity of space; and are the only known geometrical elements of a perfectly invariable nature yet disclosed in the changing physical universe.

Since the positions of our coördinate planes are arbitrary, we may suppose that of ξ, η to coincide with the invariable plane of the system, this will give $\cos\beta = 0$, $\cos\gamma = 0$, and therefore $b_3 = 0$, $c_3 = 0$. And since K is not altered by shifting the coördinate planes, the vanishing of b_3, c_3 from the second member of the equation

$$K^2 = a_3^2 + b_3^2 + c_3^2\ ,$$

renders a_3 a maximum. Thus it appears that the invariable plane is also a plane of maximum areas (*Mécanique Céleste*, Liv. VII, Chap. XVII). Moreover, since the position of the axes of ξ, η in this plane are arbitrary, it follows that for every plane perpendicular to the invariable plane, the sum of the areas traced thereon by the projection of their radii vectores, each multiplied by the corresponding masses, is zero.

§ 37. *Transformation of the Equations for the Invariable Plane.*

It is therefore natural to take this plane as the plane of ξ, η; and to place the origin of coördinates at the center of gravity of the system. If the center of gravity of the system has a uniform rectilinear motion in space the origin of coördinates will move with it, and at any time we shall have

$$\Xi = \Xi_0 + At \quad ; \quad H = H_0 + Bt \quad ; \quad Z = Z_0 + Ct \quad ;$$

where the subscripts zero denote the coördinates at the epoch t_0, and A, B, C are the component velocities of the new origin moving with the center of gravity relatively to the old origin fixed in space. As the origin of the coördinates 0ξ, 0η, 0ζ is now at the center of gravity of the system, it is easy to put the preceding equations under a somewhat different form, which seems desirable. Let us put for brevity

$$K_i = \frac{\xi_i d\eta_i - \eta_i d\xi_i}{dt};$$

then we have

$$\sum_{i=0}^{i=n-1} m_i\left(\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right) = M_0K_0 + m_1K_1 + m_2K_2 + \;.\;.\;.\;. + m_{n-1}K_{n-1}\,. \tag{102}$$

And

$$\sum_{i=0}^{i=n-1} m_i\left(\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right)\sum_{j=1}^{j=n} m_j = (M_0K_0 + m_1K_1 + m_2K_2 + \;.\;.\;.\;. m_nK_n)(M_0 + m_1 + m_2 + \;.\;.\;.\;. + m_n) \tag{103}$$

The part of the right member of this expression depending on the factor M_0 is

$$M_0K_0(M_0 + m_1 + m_2 + \;.\;.\;.\;. + m_{n-1}) + M_0(m_1K_1 + m_2K_2 + m_3K_3 + \;.\;.\;.\;. + m_{n-1}K_{n-1}) \tag{104}$$

and a similar expression holds for the part depending on any mass m_i. We shall now show by expansion that whatever be the origin of coördinates we shall have identically

$$\left.\begin{aligned}\sum_{i=0}^{i=n-1} m_i\left(\xi_i\frac{d\eta_i}{dt}-\eta_i\frac{d\xi_i}{dt}\right)\sum_{j=1}^{j=n} m_j = \sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j\left\{\frac{(\xi_j-\xi_i)(d\eta_j-d\eta_i)-(\eta_j-\eta_i)(d\xi_j-d\xi_i)}{dt}\right\}\\ +\sum_{i=0}^{i=n-1} m_i\xi_i\sum_{j=1}^{j=n} m_j\frac{d\eta_i}{dt}-\sum_{i=0}^{i=n-1} m_i\eta_i\sum m_j\frac{d\xi_j}{dt}.\end{aligned}\right\}\quad(105)$$

The first term of the second member of this expression, when expanded, becomes

$$\left.\begin{aligned}M_0m_1[\xi_1, \xi_0]+M_0m_2[\xi_2,\xi_0]+M_0m_3[\xi_3,\xi_0]+\;\ldots\;+M_0m_n[\xi_n,\xi_0]\quad,\quad(i=0)\,,\\ m_1m_2[\xi_2,\xi_1]+m_1m_3[\xi_3,\xi_1]+\;\ldots\;+m_1m_n[\xi_n,\xi_1]\quad,\quad(i=1)\,,\\ m_2m_3[\xi_3,\xi_2]+\;\ldots\;+m_2m_n[\xi_n,\xi_2]\quad,\quad(i=2)\,,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +m_{n-1}m_n[\xi_n,\xi_{n-1}]\quad,\quad(i=n-1)\,.\end{aligned}\right\}\quad(106)$$

where the bracket $[\xi_j, \xi_i]$ is an abbreviation for the long brace in (105). The last two terms of (105) become

$$\left.\begin{aligned}(M_0\xi_0+m_1\xi_1+\;\ldots\;+m_{n-1}\xi_{n-1})\left(M_0\frac{d\eta_0}{dt}+m_1\frac{d\eta_1}{dt}+\;\ldots\;+m_{n-1}\frac{d\eta_{n-1}}{dt}\right)\\ -(M_0\eta_0+m_1\eta_1+\;\ldots\;+m_{n-1}\eta_{n-1})\left(M_0\frac{d\xi_0}{dt}+m_1\frac{d\xi_1}{dt}+\;\ldots\;+m_{n-1}\frac{d\xi_{1-n}}{dt}\right).\end{aligned}\right\}\quad(107)$$

The part (106) and (107) depending on M_0 is

$$\left.\begin{aligned}&M_0^2\left(\xi_0\frac{d\eta_0}{dt}-\eta_0\frac{d\xi_0}{dt}\right)+M_0\Big\{m_1[\xi_1,\xi_0]+m_2[\xi_2,\xi_0]+\ldots+m_{n-1}[\xi_{n-1},\xi_0]\Big\}\\ &+M_0\left\{\xi_0\left(m_1\frac{d\eta_1}{dt}+m_2\frac{d\eta_2}{dt}+\ldots+m_{n-1}\frac{d\eta_{n-1}}{dt}\right)-\eta_0\left(m_1\frac{d\xi_1}{dt}+m_2\frac{d\xi_2}{dt}+\ldots+m_{n-1}\frac{d\xi_{n-1}}{dt}\right)\right\}\\ &+M_0\left\{\frac{d\eta_0}{dt}\left(m_1\xi_1+m_2\xi_2+\ldots+m_{n-1}\xi_{n-1}\right)-\frac{d\xi_0}{dt}\left(m_1\eta_1+m_2\eta_2+\ldots+m_{n-1}\eta_{n-1}\right)\right\}.\end{aligned}\right\}\quad(108)$$

The part of this expression depending on $M_0\,m_1$ is

$$M_0m_1\left\{[\xi_1,\xi_0]+\frac{(\xi_0d\eta_1+\xi_1d\eta_0-\eta_0d\xi_1-\eta_1d\xi_0)}{dt}\right\}.\quad(109)$$

Using the full expression for $[\xi_1\,,\,\xi_0]$ this becomes

$$\left.\begin{aligned}M_0m_1\left\{\frac{(\xi_1-\xi_0)(d\eta_1-d\eta_0)-(\eta_1-\eta_0)(d\xi_1-d\xi_0)+\xi_0d\eta_1+\xi_1d\eta_0-\eta_0d\xi_1-\eta_1d\xi_1}{dt}\right\}\\ =M_0m_1\left\{\frac{(\xi_0d\eta_0-\eta_0d\xi_0)+(\xi_1d\eta_1-\eta_1d\xi_1)}{dt}\right\}=M_0m_1(K_0+K_1)\,.\end{aligned}\right\}\quad(110)$$

In like manner the parts depending on $M_0\,m_2$, $M_0\,m_3$, ... $M_0\,m_{n-1}$ are easily seen to be

$$M_0m_2(K_0+K_2)\quad;\quad M_0m_3(K_0+K_3)\quad;\quad\ldots\quad M_0m_{n-1}(K_0+K_{n-1})\,.$$

Consequently the whole expression depending on M_0 becomes

$$\left.\begin{aligned} &M_0^2K_0 + M_0m_1(K_0+K_1) + M_0m_2(K_0+K_2) + \ldots + M_0m_{n-1}(K_0+K_{n-1}) \\ &= M_0K_0(M_0+m_1+m_2+\ldots+m_{n-1}) + M_0(m_1K_1+m_2K_2+\ldots+m_{n-1}K_{n-1})\,. \end{aligned}\right\} \quad (111)$$

This is identical with the right member of (104) and hence the equation (105) is established. This equation (105) holds for any origin of coördinates, and by fixing the origin at the center of gravity, we have

$$\sum_{i=0}^{i=n-1} m_i \frac{d\xi_i}{dt} = 0 \quad , \quad \sum_{i=0}^{i=n-1} m_i \frac{d\eta_i}{dt} = 0 \; ; \qquad (112)$$

so that the last terms of (105) disappear. And on putting $C_1 = \sum_{i=0}^{i=n-1} (\xi_i d\eta_i - \eta_i d\xi_i)$, with similar expressions for C_2, C_3, in the first member, we have

$$\left.\begin{aligned} C_1\sum_{j=1}^{j=n} m_j &= \sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j \left\{ \frac{(\xi_j-\xi_i)(d\eta_j-d\eta_i)-(\eta_j-\eta_i)(d\xi_j-d\xi_i)}{dt} \right\}, \\ C_2\sum_{j=1}^{j=n} m_j &= \sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j \left\{ \frac{(\eta_j-\eta_i)(d\zeta_j-d\zeta_i)-(\zeta_j-\zeta_i)(d\eta_j-d\eta_i)}{dt} \right\}, \\ C_3\sum_{j=1}^{j=n} m_j &= \sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j \left\{ \frac{(\zeta_j-\zeta_i)(d\xi_j-d\xi_i)-(\xi_j-\xi_i)(d\zeta_j-d\zeta_i)}{dt} \right\}. \end{aligned}\right\} \quad (113)$$

§ 38. *Geometrical Significance of These Equations.*

In deriving these equations LAPLACE observes (*Mécanique Céleste*, Liv. I, Chap. V, § 22) that the second members multiplied by dt express the sums of the projections of the elementary areas described by each of the $\frac{n(n+1)}{2}$ right lines connecting the $\frac{n(n+1)}{2}$ pairs of bodies in a system made up of $n+1$ masses; one of the bodies being supposed to move about the other considered at rest, and each area being multiplied by the products of the two masses connected by the right lines. It is easy to show, as LAPLACE pointed out, that the plane passing constantly through any one of the bodies of the system relatively to which the function

$$\sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j \left\{ \frac{(\xi_j-\xi_i)(d\eta_j-d\eta_i)-(\eta_j-\eta_i)(d\xi_j-d\xi_i)}{dt} \right\}.$$

is a *maximum*, preserves its parallelism during the motion of the system, and that this plane is parallel to the plane passing through the center of gravity, and rel-

atively to which the function $\sum_{i=0}^{i=n} m_i\left(\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right)$ is a maximum. The second members of (113) are zero for all planes passing through the same body perpendicular to the plane of maximum areas. These two functions, which represent areas, and relative areas respectively, each multiplied by the corresponding masses, thus have exactly analogous properties.

§ 39. *Integrals for the Absolute Motion in Space.*

Since we can observe only the relative motion of the heavenly bodies, it is usual to refer their motions to the center of the sun. We shall therefore determine the relative motions of $m_1, m_2, m_3 \ldots m_n$ about M_0 considered as the centre of their motions. Let x, y, z, be the coördinate of M_0 and $\xi_1, \xi_2, \ldots \xi_n$; $\eta_1, \eta_2, \ldots \eta_n$; $\zeta_1, \zeta_2, \zeta_3 \ldots \zeta_n$ those of $m_1, m_2, m_3 \ldots m_n$. Then we shall have $x + \xi_1$; $y + \eta_1$; $z + \zeta_1$; for the coördinates of m, and similar expressions for any other body. Let $r_1, r_2, r_3 \ldots r_n$ be the distances of the bodies $m_1, m_2, m_3 \ldots m_n$ from M_0; then

$$r_1 = \sqrt{\xi_1^2 + \eta_1^2 + \zeta_1^2} \;;\; r_2 = \sqrt{\xi_2^2 + \eta_2^2 + \zeta_2^2} \;;\; r_i = \sqrt{\xi_i^2 + \eta_i^2 + \zeta_i^2}.$$

And we shall have for the motion of M_0,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - \sum_{i=1}^{i=n}\frac{m_i\xi_i}{r_i^3} &= 0, \\ \frac{d^2y}{dt^2} - \sum_{i=1}^{i=n}\frac{m_i\eta_i}{r_i^3} &= 0, \\ \frac{d^2z}{dt^2} - \sum_{i=1}^{i=n}\frac{m_i\zeta_i}{r_i^3} &= 0. \end{aligned}\right\} \tag{114}$$

The action of M_0 on any body m_j taken in a direction opposite to the origin will be

$$- M_0\frac{\xi_j}{r_j^3}.$$

The sum of the reactions of the n-bodies $m_1, m_2 \ldots m_n$ upon any mass m_j will be given by

$$\frac{1}{m_j}\frac{\partial U}{\partial \xi_j} \;;\; \frac{1}{m_j}\frac{\partial U}{\partial \eta_j} \;;\; \frac{1}{m_j}\frac{\partial U}{\partial \zeta_j}.$$

Accordingly, any body m_j will have its motion defined by the differential equations:

$$\left.\begin{aligned}
&\frac{d^2(x+\xi_i)}{dt^2}+\frac{M_0\xi_j}{r_j^3}-\frac{1}{m_j}\frac{\partial U}{\partial \xi_j}=0\,,\\
&\frac{d^2(y+\eta_i)}{dt^2}+\frac{M_0\eta_j}{r_j^3}-\frac{1}{m_j}\frac{\partial U}{\partial \eta_j}=0\,,\\
&\frac{d^2(z+\zeta_i)}{dt^2}+\frac{M_0\zeta_j}{r_j^3}-\frac{1}{m_j}\frac{\partial U}{\partial \zeta_j}=0\,.
\end{aligned}\right\}\tag{115}$$

Substituting for

$$\frac{d^2x}{dt^2}\,,\quad\frac{d^2y}{dt^2}\,,\quad\frac{d^2z}{dt^2}$$

their values in (115), we shall have

$$\left.\begin{aligned}
&\frac{d^2\xi_j}{dt^2}+\frac{M_0\xi_i}{r_j^3}+\sum_{i=1}^{i=n}\frac{m_i\xi_i}{r_i^3}-\frac{1}{m_j}\frac{\partial U}{\partial \xi_j}=0\,,\\
&\frac{d^2\eta_j}{dt^2}+\frac{M_0\eta_j}{r_j^3}+\sum_{i=1}^{i=n}\frac{m_i\eta_i}{r_i^3}-\frac{1}{m_j}\frac{\partial U}{\partial \eta_j}=0\,,\\
&\frac{d^2\zeta_j}{dt^2}+\frac{M_0\zeta_j}{r_j^3}+\sum_{i=1}^{i=n}\frac{m_i\zeta_i}{r_i^3}-\frac{1}{m_j}\frac{\partial U}{\partial \zeta_j}=0\,.
\end{aligned}\right\}\tag{116}$$

If we multiply the n differential equations in ξ_j, by $m_1, m_2, \ldots m_n$ and add the products, observing that by the nature of the function U,

$$\sum_{i=1}^{i=n}\frac{\partial U}{\partial \xi_i}=\frac{\partial U}{\partial \xi_1}+\frac{\partial U}{\partial \xi_2}+\ldots+\frac{\partial U}{\partial \xi_n}=0\,,\quad\text{and similarly}\sum_{i=1}^{i=n}\frac{\partial U}{\partial \eta_i}=0\,,\quad\sum_{i=1}^{i=n}\frac{\partial U}{\partial \zeta_i}=0\,,$$

we shall find

$$\left.\begin{aligned}
&\sum_{i=1}^{i=n}m_j\frac{d^2\xi_j}{dt^2}+M_0\sum_{j=1}^{j=n}\frac{m_j\xi_j}{r_j^3}+\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\xi_i}{r_i^3}=0\,,\\
&\sum_{i=1}^{i=n}m_j\frac{d^2\eta_j}{dt^2}+M_0\sum_{j=1}^{j=n}\frac{m_j\eta_j}{r_j^3}+\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\eta_i}{r_i^3}=0\,,\\
&\sum_{i=1}^{i=n}m_j\frac{d^2\zeta_j}{dt^2}+M_0\sum_{j=1}^{j=n}\frac{m_j\zeta_j}{r_j^3}+\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\zeta_i}{r_i^3}=0\,.
\end{aligned}\right\}\tag{117}$$

Multiplying equations (114) by $M_0+\sum\limits_{j=1}^{j=n}m_j$ we get

$$\left.\begin{aligned}
&\Big(M_0+\sum_{j=1}^{j=n}m_j\Big)\frac{d^2x}{dt^2}-M_0\sum_{i=1}^{i=n}\frac{m_i\xi_i}{r_i^3}-\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\xi_i}{r_i^3}=0\,,\\
&\Big(M_0+\sum_{j=1}^{j=n}m_j\Big)\frac{d^2y}{dt^2}-M_0\sum_{i=1}^{i=n}\frac{m_i\eta_i}{r_i^3}-\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\eta_i}{r_i^3}=0\,,\\
&\Big(M_0+\sum_{j=1}^{j=n}m_j\Big)\frac{d^2z}{dt^2}-M_0\sum_{i=1}^{i=n}\frac{m_i\zeta_i}{r_i^3}-\sum_{j=1}^{j=n}m_j\sum_{i=1}^{i=n}\frac{m_i\zeta_i}{r_i^3}=0\,.
\end{aligned}\right\}\tag{118}$$

If we add (117) and (118) we shall obtain

$$\left.\begin{aligned}
\left(M_0 + \sum_{j=1}^{j=n} m_j\right)\frac{d^2x}{dt^2} + \sum_{j=1}^{j=n} m_j \frac{d^2\xi_j}{dt^2} &= 0\,,\\
\left(M_0 + \sum_{j=1}^{j=n} m_j\right)\frac{d^2y}{dt^2} + \sum_{j=1}^{j=n} m_j \frac{d^2\eta_j}{dt^2} &= 0\,,\\
\left(M_0 + \sum_{j=1}^{j=n} m_j\right)\frac{d^2z}{dt^2} + \sum_{j=1}^{j=n} m_j \frac{d^2\zeta_j}{dt^2} &= 0\,.
\end{aligned}\right\} \tag{119}$$

The integration of these equations gives

$$\left.\begin{aligned}
x &= a + bt - \frac{\sum\limits_{j=1}^{j=n} m_j\xi_j}{M_0 + \sum\limits_{j=1}^{j=n} m_j}\,,\\
y &= a' + b't - \frac{\sum\limits_{j=1}^{j=n} m_j\eta_j}{M_0 + \sum\limits_{j=1}^{j=n} m_j}\,,\\
z &= a'' + b''t - \frac{\sum\limits_{j=1}^{j=n} m_j\zeta_j}{M_0 + \sum\limits_{j=1}^{j=n} m_j}\,.
\end{aligned}\right\} \tag{120}$$

The quantities $a, a', a'', b, b', b'', c, c', c''$ are arbitrary constants of integration, and these integrals give the absolute motion of M_0 in space, when the relative motions of $m_1, m_2, m_3, \ldots m_n$ about it are known.

§ 40. *Rigorous Integrals of the Areas in Their Simplest Form.*

If we multiply the n equations in ξ_j (115) by

$$- m_j\eta_j + \frac{m_j \sum\limits_{j=1}^{j=n} m_j\eta_j}{M_0 + \sum\limits_{j=1}^{j=n} m_j}\,,$$

and the n equations in η_j by

$$m_j\xi_j - \frac{m_j \sum\limits_{j=1}^{j=n} m_j\xi_j}{M_0 + \sum\limits_{j=1}^{j=n} m_j}\;;$$

and then put in succession $j = 1,\ j = 2, \ldots j = n,$ and add the results, observing that by the nature of the function

$$\sum_{i=1}^{i=n}\xi_i\frac{\partial U}{\partial \eta_i}-\sum_{i=1}^{i=n}\eta_i\frac{\partial U}{\partial \xi_i}=0\quad,\quad \sum_{i=1}^{i=n}\frac{\partial U}{\partial \xi_i}=0\quad,\quad \sum_{i=1}^{i=n}\frac{\partial U}{\partial \eta_i}=0\,, \tag{121}$$

we shall obtain

$$\left.\begin{aligned}
&\sum_{j=1}^{j=n}m_j\left\{\frac{\xi_jd^2\eta_j-\eta_jd^2\xi_j}{dt^2}\right\}-\frac{\sum\limits_{j=1}^{j=n}m_j\xi_j}{M_0+\sum\limits_{j=1}^{j=n}}\sum_{j=1}^{j=n}m_j\frac{d^2\eta_j}{dt^2}+\frac{\sum\limits_{j=1}^{j=n}m_j\eta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d^2\xi_j}{dt^2}=0\,,\\
&\sum_{j=1}^{j=n}m_j\left\{\frac{\eta_jd^2\zeta_j-\zeta_jd^2\eta_j}{dt^2}\right\}-\frac{\sum\limits_{j=1}^{j=n}m_j\eta_j}{M_0+\sum\limits_{j=1}^{j=n}}\sum_{j=1}^{j=n}m_j\frac{d^2\zeta_j}{dt^2}+\frac{\sum\limits_{j=1}^{j=n}m_j\zeta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d^2\eta_j}{dt^2}=0\,,\\
&\sum_{j=1}^{j=n}m_j\left\{\frac{\zeta_jd^2\xi_j-\xi_jd^2\zeta_j}{dt^2}\right\}-\frac{\sum\limits_{j=1}^{j=n}m_j\zeta_j}{M_0+\sum\limits_{j=1}^{j=n}}\sum_{j=1}^{j=n}m_j\frac{d^2\xi_j}{dt^2}+\frac{\sum\limits_{j=1}^{j=n}m_j\xi_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d^2\zeta_j}{dt^2}=0\,.
\end{aligned}\right\} \tag{122}$$

The integrals of these equations are easily shown to be

$$\left.\begin{aligned}
c_1&=\sum_{j=1}^{j=n}m_j\frac{(\xi_jd\eta_j-\eta_jd\xi_j)}{dt}-\frac{\sum\limits_{j=1}^{j=n}m_j\xi_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\eta_j}{dt}+\frac{\sum\limits_{j=1}^{j=n}m_j\eta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\xi_j}{dt}\,,\\
c_2&=\sum_{j=1}^{j=n}m_j\frac{(\eta_jd\zeta_j-\zeta_jd\eta_j)}{dt}-\frac{\sum\limits_{j=1}^{j=n}m_j\eta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\zeta_j}{dt}+\frac{\sum\limits_{j=1}^{j=n}m_j\zeta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\eta_j}{dt}\,,\\
c_3&=\sum_{j=1}^{j=n}m_j\frac{(\zeta_jd\xi_j-\xi_jd\zeta_j)}{dt}-\frac{\sum\limits_{j=1}^{j=n}m_j\zeta_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\xi_j}{dt}+\frac{\sum\limits_{j=1}^{j=n}m_j\xi_j}{M_0+\sum\limits_{j=1}^{j=n}m_j}\sum_{j=1}^{j=n}m_j\frac{d\zeta_j}{dt}\,.
\end{aligned}\right\} \tag{123}$$

If we multiply the expressions by $M_0+\sum\limits_{j=1}^{j=n}m_j$ and put C_1, C_2, C_3 for the products of the constant terms by $M_0+\sum\limits_{j=1}^{j=n}m_j$, we shall get

$$\left.\begin{aligned}
C_1&=M_0\sum_{j=1}^{j=n}m_j\frac{(\xi_jd\eta_j-\eta_jd\xi_j)}{dt}+\sum_{j=1}^{j=n}m_j\frac{(\xi_jd\eta_j-\eta_jd\xi_j)}{dt}\sum_{j=1}^{j=n}m_j-\sum_{j=1}^{j=n}m_j\xi_j\sum_{j=1}^{j=n}m_j\frac{d\eta_j}{dt}+\sum_{j=1}^{j=n}m_j\eta_j\sum_{j=1}^{j=n}m_j\frac{d\xi_j}{dt}\,,\\
C_2&=M_0\sum_{j=1}^{j=n}m_j\frac{(\eta_jd\zeta_j-\zeta_jd\eta_j)}{dt}+\sum_{j=1}^{j=n}m_j\frac{(\eta_jd\zeta_j-\zeta_jd\eta_j)}{dt}\sum_{j=1}^{j=n}m_j-\sum_{j=1}^{j=n}m_j\eta_j\sum_{j=1}^{j=n}m_j\frac{d\zeta_j}{dt}+\sum_{j=1}^{j=n}m_j\zeta_j\sum_{j=1}^{j=n}m_j\frac{d\eta_j}{dt}\,,\\
C_3&=M_0\sum_{j=1}^{j=n}m_j\frac{(\zeta_jd\xi_j-\xi_jd\zeta_j)}{dt}+\sum_{j=1}^{j=n}m_j\frac{(\zeta_jd\xi_j-\xi_jd\zeta_j)}{dt}\sum_{j=1}^{j=n}m_j-\sum_{j=1}^{j=n}m_j\zeta_j\sum_{j=1}^{j=n}m_j\frac{d\xi_j}{dt}+\sum_{j=1}^{j=n}m_j\xi_j\sum_{j=1}^{j=n}m_j\frac{d\zeta_j}{dt}\,.
\end{aligned}\right\} \tag{124}$$

But we have already found in (105) and (113) that expressions of the form

$$\left.\begin{aligned}&\sum_{j=1}^{j=n} m_j\left(\xi_j\frac{d\eta_j}{dt}-\eta_j\frac{d\xi_j}{dt}\right)\sum_{i=1}^{i=n} m_i=\sum_{j=1}^{j=n}\sum_{i=1}^{i=n} m_j m_i\left\{\frac{(\xi_j-\xi_i)(d\eta_j-d\eta_i)-(\eta_j-\eta_i)(d\xi_j-d\xi_i)}{dt}\right\}\\&+\sum_{j=1}^{j=n} m_j\xi_j\sum_{i=1}^{i=n} m_i\frac{d\eta_i}{dt}-\sum_{j=1}^{j=n} m_j\eta_j\sum_{i=1}^{i=n} m_i\frac{d\xi_i}{dt}.\end{aligned}\right\}\tag{125}$$

Using this value in the second member of (124) we find

$$\left.\begin{aligned}C_1&=M_0\sum_{j=1}^{j=n} m_j\frac{(\xi_j d\eta_j-\eta_j d\xi_j)}{dt}+\sum_{j=1}^{j=n}\sum_{i=1}^{i=n} m_j m_i\left\{\frac{(\xi_j-\xi_i)(d\eta_j-d\eta_i)-(\eta_j-\eta_i)(d\xi_j-d\xi_i)}{dt}\right\},\\C_2&=M_0\sum_{j=1}^{j=n} m_j\frac{(\eta_j d\zeta_j-\zeta_j d\eta_j)}{dt}+\sum_{j=1}^{j=n}\sum_{i=1}^{i=n} m_j m_i\left\{\frac{(\eta_j-\eta_i)(d\zeta_j-d\zeta_i)-(\zeta_j-\zeta_i)(d\eta_j-d\eta_i)}{dt}\right\},\\C_3&=M_0\sum_{j=1}^{j=n} m_j\frac{(\zeta_j d\xi_j-\xi_j d\zeta_j)}{dt}+\sum_{j=1}^{j=n}\sum_{i=1}^{i=n} m_j m_i\left\{\frac{(\zeta_j-\zeta_i)(d\xi_j-d\xi_i)-(\xi_j-\xi_i)(d\zeta_j-d\zeta_i)}{dt}\right\}.\end{aligned}\right\}\tag{126}$$

These integrals are entirely rigorous and of the utmost generality.

§ 41. *Practical Application of the Theory of the Invariable Plane to the Solar System.*

We shall now put these expressions in suitable form for computation of the position of the invariable plane of the solar system. The following discussion is based on the author's well known paper in *Astronomische Nachrichten*, No. 3923. Any areal element of the expression

$$M_0\sum_{j=1}^{j=n} m_j\frac{(\xi_j d\eta_j-\eta_j d\xi_j)}{dt},$$

occurring in the first term of the second member of (126) may be given in terms of the elements of the orbit of the body about the sun. Suppose the semi-axis major to be denoted by a_j, and the eccentricity by e_j, the mass by m_j; then whatever be the position of the axes 0ξ, 0η, we shall have

$$\frac{\xi_j d\eta_j-\eta_j d\xi_j}{dt}=\sqrt{(M_0+m_j)\,a_j(1-e_j^2)}.\tag{127}$$

If we multiply the second member of this equation by $\cos\chi$, where χ is the inclination of the plane of the orbit to the fixed plane, we shall obtain the area projected on this plane; and if the axes be taken in this plane

$$\frac{\xi_j d\eta_j - \eta_j d\xi_j}{dt} = \sqrt{(M_0+m_j)\, a_j\, (1 - e_j^2)} \;\cdot\; \cos\chi_j\,. \tag{128}$$

Now we may put $\sqrt{(M_0 + m_j)\, a_j} = n_j\, a_j^2$, where n_j is the *mean motion* of the planet, and $1 - e_j^2 = \cos^2\varphi_j$, where $e = \sin\varphi_j$. Any element of the first term of the second member of (126) is therefore of the form

$$M_0 m_j (\xi_j d\eta_j - \eta_j d\xi_j) = M_0 m_j \sqrt{(M_0 + m_j)\, a_j} \cos\varphi_j \cos\chi_j = M_0 m_j n_j a_j^2 \cos\varphi_j \cos\chi_j\,. \tag{129}$$

The elements projected on the other coördinate planes would be

$$\left.\begin{aligned} M_0 m_j (\eta_j d\zeta_j - \zeta_j d\eta_j) &= M_0 m_j n_j a_j^2 \cos\varphi_j \sin\chi_j \cos\psi_j\,, \\ M_0 m_j (\zeta_j d\xi_j - \xi_j d\zeta_j) &= M_0 m_j n_j a_j^2 \cos\varphi_j \sin\chi_j \sin\psi_j\,. \end{aligned}\right\} \tag{130}$$

Here ψ_j is the longitude of the ascending node of the planet's orbit on the ecliptic at a particular epoch. The second members of (129) and (130) are rigorously exact on the hypothesis that the elements vary slowly in consequence of the secular inequalities. To compute the terms the elements would have to be known for a particular epoch. The sun's mass M_0 is usually taken to be unity and hence the elements of the first terms of (129) and (130) have the form

$$\left.\begin{aligned} m_j (\xi_j d\eta_j - \eta_j d\xi_j) &= m_j n_j a_j^2 \cos\varphi_j \cos\chi_j\,, \\ m_j (\eta_j d\zeta_j - \zeta_j d\eta_j) &= m_j n_j a_j^2 \cos\varphi_j \sin\chi_j \cos\psi_j\,, \\ m_j (\zeta_j d\xi_j - \xi_j d\zeta_j) &= m_j n_j a_j^2 \cos\varphi_j \sin\chi_j \sin\psi_j\,. \end{aligned}\right\} \tag{131}$$

These equations include all terms of the order m_j, that is, of the first order with respect to the planetary masses. The remaining terms of (126) include terms of the order $m_j \,.\, m_i$, or of the second order. These terms are very minute, and may be neglected, because their total effects are smaller than the uncertainty attaching to the first order terms, as will more fully appear below. Thus the maximum term of the second order will be that depending on *Jupiter* and *Saturn*. The best available masses of *Jupiter* and *Saturn* are the following

$$m_{\text{v}} = \frac{1}{1047.35 \pm 0.10} \text{ (Newcomb)} \quad ; \quad m_{\text{vi}} = \frac{1}{3500 \pm 2.0} \text{ (See, } A.N.\text{ 3923)}.$$

Accordingly, $m_{\text{v}} \,.\, m_{\text{vi}}$ equals $\frac{1}{3,665,725}$, which is of the second order of smallness as respects the uncertainty still attaching to the mass of *Saturn*, *viz*: $\frac{1}{1750}$. The uncertainty attaching to the mass of *Jupiter* is only $\frac{1}{10473.5}$, so that a great degree of precision in the determination of its mass has already been attained. The most probable masses of *Uranus* and *Neptune* appear to be

$$m_{\text{vii}} = \frac{1}{22780 \pm 50} \quad ; \quad m_{\text{viii}} = \frac{1}{19313 \pm 25} \quad \text{(See, } A.N.\text{ 3923)}.$$

The uncertainty attaching to these masses, therefore, in the case of *Uranus*, amounts

to $\frac{1}{300}$ of the whole, and in the case of *Neptune*, to $\frac{1}{200}$. The other masses of the planetary system may be neglected in comparison with the uncertainty attaching to the masses of the outer planets. It is therefore evident that terms of the second order in the determination of the position of the invariable plane will be wholly insensible. In view of the practical difficulty of obtaining better masses of the outer planets, it is probable that for a long time to come astronomers will have to neglect terms of the second order in determining the position of the invariable plane. In the tables given in the following section we have derived the position of the invariable plane, and also indicated the uncertainty attaching to its position on account of the uncertainty of the masses of the planets. This is about all that can be done in the present state of our knowledge.

If other bodies of our system should be discovered which have sensible masses, the position there assigned would be subject to corresponding alteration. It is certain that the masses of the comets and asteroids will exert no sensible influence on the position of the invariable plane. For their masses collectively are smaller than the uncertainty still attaching to the masses of the major planets, and moreover at any given epoch the masses will be so distributed with regard to the invariable plane of the system as to exert no influence on its position.

The only other causes which could modify the position of this plane, are the rotations of the sun and planets, and the orbital motions of the satellite. The rotations will suffer no change within historical time, and may therefore be ignored, as exerting no influence on the position of the invariable plane, so that the planets may be regarded as material points.

§ 42. *Determination of the Position of the Invariable Plane of the Planetary System, and of the Degree of Uncertainty Still Attaching to This Plane.*

The following tabular data and discussion are taken from the author's well known paper "On the Degree of Accuracy Attainable in Determining the Position of Laplace's Invariable Plane of the Planetary System," *Astronomische Nachrichten*, No. 3923, a part of which is here reproduced without material change.

Table I. Planetary Masses, with Other Elements from the Theories of G. W. Hill.

Planet	i	m_i	u_i	n_i	$\log\left(\frac{n_i}{n_3}\right)$	$\log a_i$	e_i	χ_i	ψ_i
				$''$				° ′ ″	° ′ ″
Mercury	1	1:14868548±743427	$\frac{1}{20}$	5381016.2925	0.6182669	9.5878217−10	0.20560478	7 0 7.71	46 33 8.6
Venus	2	1:408134±8163	$\frac{1}{50}$	2106641.3980	0.2109932	9.8593378−10	0.00684331	3 23 34.83	75 19 52.2
Earth	3	1:328715±328	$\frac{1}{1000}$	1295977.41516	0.0000000	0.0000000	0.01677110	0 0 0.0	—
Mars	4	1:3089967±10300	$\frac{1}{300}$	689050.8014	9.7256538−10	0.1828971	0.09326113	1 51 2.28	48 23 53.0
Jupiter	5	1:1047.35±0.10	$\frac{1}{10000}$	109256.62552	8.9258504−10	0.7162375	0.048255511	1 18 42.10	98 56 19.79
Saturn	6	1:3500±2.0	$\frac{1}{1750}$	43996.21506	8.5308179−10	0.9794956	0.05606025	2 29 40.19	112 20 49.05
Uranus	7	1:22780±76	$\frac{1}{300}$	15425.752	8.0756489−10	1.2831044	0.0469236	0 46 20.54	73 14 8.0
Neptune	8	1:19313±96	$\frac{1}{200}$	7864.935	7.7830978−10	1.4781414	0.0084962	1 47 1.68	130 7 31.83

TABLE II. DEDUCTION OF INTEGRALS OF AREAS FROM ADOPTED ELEMENTS.

(Units of the 6th decimal).

$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \cos\chi_i$	$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \sin\chi_i \cos\psi_i$	$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \sin\chi_i \sin\psi_i$
0.040645	+ 0.003433	0.003624
2.080146	+ 0.031231	0.119309
3.041721	—	—
0.397531	+ 0.008528	0.009605
2175.773000	− 7.740952	49.214443
880.330610	−14 581735	35.471106
192.214430	+ 0.747426	2.481157
283.991120	− 5.699903	6.762740
$\Sigma_1 = 3537.869203 = c_1$	$\Sigma_2 = -27.231972 = c_2$	$\Sigma_3 = 94.061984 = c_3$

$\Omega = 106° \; 8' \; 46''.688$; $\gamma = 1° \; 35' \; 7''.745$ $\left\{ \begin{array}{l} \text{Ecliptic and Mean Equinox} \\ \text{1850 January 0.0, Greenwich M.T.} \end{array} \right.$

TABLE III. INTEGRALS OF AREAS AS MODIFIED BY ALTERING THE MASSES BY THE AMOUNT OF THEIR UNCERTAINTY, SO AS TO DISPLACE THE INVARIABLE PLANE (cf. page 81).

(Units of the 6th decimal).

$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \cos\chi_i$	$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \sin\chi_i \cos\psi_i$	$m_i\left(\frac{n_i}{n_3}\right) a_i^2 \cos\varphi_i \sin\chi_i \sin\psi_i$
0.038710	+ 0.003269	0.003451
2.039357	+ 0.030618	0.116969
3.044760	—	—
0.398860	+ 0.008557	0.009637
2175.565000	− 7.740213	49.209740
879.827800	−14.573406	35.450850
192.857840	+ 0.749928	2.489463
285.409700	− 5.728244	6.796506
$\Sigma_1' = 3539.182027 = c_1'$	$\Sigma_2' = -27.136177 = c_2'$	$\Sigma_3' = 94.076616 = c_3'$

$\Omega = 106° \; 5' \; 24''.20$; $\gamma = 1° \; 35' \; 4''.906$; $d\Omega = \pm 202''.49$; $d\gamma = \pm 2''.839$.

Let γ denote the inclination of the Invariable Plane and Ω the longitude its ascending node on the fixed ecliptic of 1850.0; then we shall have

$$c_1 \tan\gamma \sin\Omega = c_3 \, ; \quad c_1 \tan\gamma \cos\Omega = c_2 \, . \qquad [1]$$

The values of c_1, c_2, c_3, found in Table II give

$$\gamma = 1° \; 35' \; 7''.745 \, ; \quad \Omega = 106° \; 8' \; 46''.688 \, . \qquad [2]$$

The previous determinations of the position of this plane are given in Table IV. These several determinations are not referred to a common ecliptic and mean

equinox, and such a reduction is hardly worth while in view of the diversity of elements and masses employed by the several investigators, which would also greatly affect the resulting position of the invariable plane. Moreover, STOCKWELL's investigation is the only one in which account is taken of the existence of the planet *Neptune*, which was discovered subsequent to the researches of LAPLACE and PONTÉCOULANT.

TABLE IV. PREVIOUS DETERMINATIONS OF THE POSITION OF THE INVARIABLE PLANE.

☊	γ	Epoch	Authority	Source
° ′ ″	° ′ ″			
102 57 29	1 35 31	Ecliptic and Mean Equinox of 1750.0	Laplace, 1802	Mécanique Céleste, Liv. VI, ch. XVII, § 46
102 57 15	1 35 31	Ecliptic of 1750, but elements of Planets for 1950.0	Laplace, 1802	Mécanique Céleste, Liv. VI, ch. XVII, § 46
103 8 45 103 8 50	1 34 16 1 34 15	Ecliptic and Mean Equinox of 1800.0. In second value elements of Planets are for 2000.0	Pontécoulant, 1834	Théorie Analytique du Système du Monde, Tome III, Liv. VI, ch. XXII
106 14 6.00	1 35 19.376	Ecliptic and Mean Equinox of 1850.0	Stockwell, 1872	Smithsonian Contributions to Knowledge, No. 232, p. 166

UNCERTAINTY ATTACHING TO THE POSITION OF THE INVARIABLE PLANE.

In concluding this discussion it seems desirable to ascertain the degree of probable uncertainty still attaching to the inclination and node of the invariable plane in the present state of our knowledge respecting the masses and elements.

POISSON has observed (*Traité de Mécanique*, Paris, 1811, Tome I, p. 281) that the invariable plane is a *plane of moments;* and hence if the several masses on one side of this plane at a given epoch be increased by the amount of the uncertainty attaching to each mass, while those on the other side are decreased by the amount of their uncertainty, the effect will be a maximum displacement of the computed position of the invariable plane due to the alterations in the masses of the planets. For any other increase or decrease of the masses, within the limits of uncertainty assigned, will effect the position of the plane less than that here suggested, where the increments are all positive on one side of the plane, and negative on the other. Accordingly we have computed the position of each orbit with reference to the invariable plane by means of the formulae:

$$\left.\begin{aligned} \sin\chi_i^0 \sin\psi_i^0 &= \sin\chi_i \sin(\psi_i - \Omega)\,, \\ \sin\chi_i^0 \cos\psi_i^0 &= \cos\gamma \sin\chi_i \cos(\psi_i - \Omega) - \sin\gamma\cos\chi_i\,. \end{aligned}\right\} \qquad [3]$$

TABLE V. LONGITUDES OF ASCENDING NODES AND INCLINATIONS OF ORBITS ON THE INVARIABLE PLANE, THE LONGITUDES BEING RECKONED FROM DESCENDING NODE OF ECLIPTIC OF 1850 ON INVARIABLE PLANE.

Planet	(Longitude of Ep.) L	(Longitude of Perih.) π	ψ_i^0	χ_i^0	ψ_i^0 (Stockwell)	χ_i^0 (Stockwell)
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Mercury	323 11 23.53	75 7 13.78	288 1 33.03	6 20 52.67	287 54 5.12	6 20 58.08
Venus	243 57 44.34	129 27 14.36	307 24 41.17	2 11 15.11	307 14 8.10	2 11 13.57
Earth	99 48 18.67	100 21 21.34	180 0 0.00	1 35 7.745	180 0 0.00	1 35 19.376
Mars	83 9 16.93	333 17 53.49	249 6 13.50	1 40 30.85	248 56 21.45	1 40 43.70
Jupiter	159 56 24.98	11 54 31.67	210 4 32.84	0 19 42.09	210 7 35.44	0 19 59.674
Saturn	14 49 38.09	90 6 41.37	16 45 35.15	0 56 2.79	16 34 26.66	0 55 30.924
Uranus	28 25 17.05	168 15 6.7	204 7 30.31	1 1 36.19	204 12 33.78	1 1 45.27
Neptune	335 5 38.91	43 17 30.3	86 29 28.26	0 43 34.35	286 39 55.10*	0 43 24.845

*Probably a misprint for 86° 39′ 55″.10.

The values of χ_i^0 and ψ_i^0 are given in Table V, which also contains the ordinary elements L and π omitted from Table I. From this it will be seen that at the epoch 1850.0, the *Earth, Mars, Uranus,* and *Neptune* were north of the invariable plane, while *Mercury, Venus, Jupiter* and *Saturn* were south of it. To shift the computed position of the invariable plane by a maximum amount for the uncertainty attaching to the planetary masses, we take u_i with positive sign for the planets *Earth, Mars, Uranus* and *Neptune,* and negative sign for the planets *Mercury, Venus, Jupiter* and *Saturn.* The result of this computation is given in Table III. Hence it appears that the variations in the position of the plane are

$$d\Omega = \pm 202''.49 \quad ; \quad d\gamma = \pm 2''.839.$$

This shows that the invariable plane may now be determined with very considerable accuracy. The actual shifting of the plane due to improvements in the masses is likely to be about one-third of the maximum variation here computed, and hence we may, I think, conclude that the inclination is uncertain to the extent of about 1″ and the node by about 1′.

If serious efforts were made during the next half century, probably the masses of the planets and their elements could be found with such precision, that the invariable plane would become known with all the accuracy required in Practical Astronomy. Already the inclination of the plane is known with a degree of accuracy approximating that of our knowledge of the ecliptic and equator, to which the planets and fixed stars are referred. If the inclination of the invariable plane were certainly known within the limits of ±0″.20, a degree of precision would be attained which would leave very little to be desired. Considering the progress of Practical Astronomy since the time of BESSEL, this improvement might be easily affected during the present century. In that case LAPLACE's expectations

of finding an immovable plane of reference equally good for all ages and serviceable alike for the planets, comets and fixed stars, eventually would be realized.

It is hardly necessary to state that the satellites, asteroids and comets are too small and too symmetrically distributed to exert a sensible influence on the position of this plane. But it is likely that some other planet as yet undiscovered would need to be taken into account. LAPLACE has shown (cf. *Mécanique Céleste*, Liv. VI, Ch. XVIII, § 47) that the great distance of the fixed stars renders their perturbing action upon the planets so very minute that it seems unlikely ever to give this fundamental plane of our system a precessional motion among the stars which could be perceived by the inhabitants of the terrestrial globe. Accordingly, when the elements of the principal planets and their masses are known with the required accuracy, the resulting position of the plane may be regarded as rigidly invariable.

As transformation to such a plane in practice would be somewhat troublesome, it is not likely to be so useful in the theory of the fixed stars, as in the theory of the planets and comets, where the orbits undergo great periodic oscillations depending on the secular variations of the elements under the action of universal gravitation.

In conclusion it remains to add that we shall see hereafter how a spiral nebula gradually develops into a sun and a system of planets like our solar system; and therefore this careful examination of the theory of the motions of our system will alone give an adequate idea of the perfect order and harmony and stability which may result from the chaos of nebulosity governed wholly by chance. The streams of whirling nebulosity define a fundamental plane into which the bodies are gradually drawn, and there alone permanent planets develop, while the resisting medium in which they move incessantly reduces the size and eccentricity of their orbits, till at length the planetary paths are as round as those witnessed in our solar system. These orbits are such perfect approximations to exact circles that they have always been equally admired by the astronomer, the geometer, and the natural philosopher.

CHAPTER IV.

Theory of the Rotation of a Condensing Mass, and of the Formation of the Spiral or Whirlpool Nebulae.

§ 43. *When the Bodies of the System are Congealed into One Rigid Mass Endowed with a Rotatory Motion About an Axis, the Invariable Plane Becomes the Plane of the Equator.*

If now we imagine the bodies forming such a system as we have discussed to be instantly congealed into a solid mass so as to form a *rigid* instead of a *changing* system, it is clear that the rigid body thus constituted would also have the same invariable plane as before. The system would become a rigid body with the invariable plane as its equator. It would rotate about an axis perpendicular to the invariable plane, or the equator, and the axis would remain fixed with respect to the particles of the body and also with respect to space. External disturbance of the system might give the equatorial plane a precessional motion like that of the equinoxes; and internal disturbances might derange the pole from coincidence with one of the principal axes, and it would oscillate around it in a given period, like the polar motion in the variation of terrestrial latitude. But if the figure of the system of bodies was oblate like that of the earth, with maximum moment of inertia about the axis of rotation, the motion about the polar axis would be stable and no large departure from that axis would ever be possible. For a similar reason there is no secular change in the position of the axis in the body of the earth.

If, therefore, a system of bodies subjected to their mutual gravitation has an invariable plane, and on becoming congealed would have an equator of similar character, it follows that in any system of particles subjected to their mutual gravitation, these geometrical elements are already determined by the initial conditions of the motions of the separate bodies. Variations in the masses, distances and velocities and directions of projection of the several bodies would give infinite variety in the resultant rotation and invariable plane of the system.

Whatever changes take place, the rotation of the system may always be represented geometrically by a vector of definite length and direction.

In order to develop this theory a little more fully, suppose ω to be the velocity of this rotation of the system or body at the epoch t; and let the direction of the axis be OI. On the line OI, in a direction from the origin to be determined by the sense of the rotation, we take once for all $OI = \omega$; then the projections of the point I upon the axes Ox_1, Oy_1, Oz_1, fixed in and rotating with the body, are three auxiliary variables which may be introduced. We may represent them by $p = Ox_1$, $q = Oy_1$, $r = Oz_1$. If we decompose the rotation ω about the line OI into three others about the coördinate axes, which is equivalent to taking the projections of OI along the axes Ox, Oy, Oz, the velocities of the component rotations are exactly equal to p, q, r.

If ψ, θ, ϕ be the angles introduced by EULER into the theory of a rotating body, reckoned from axes fixed in space as shown in the figure,

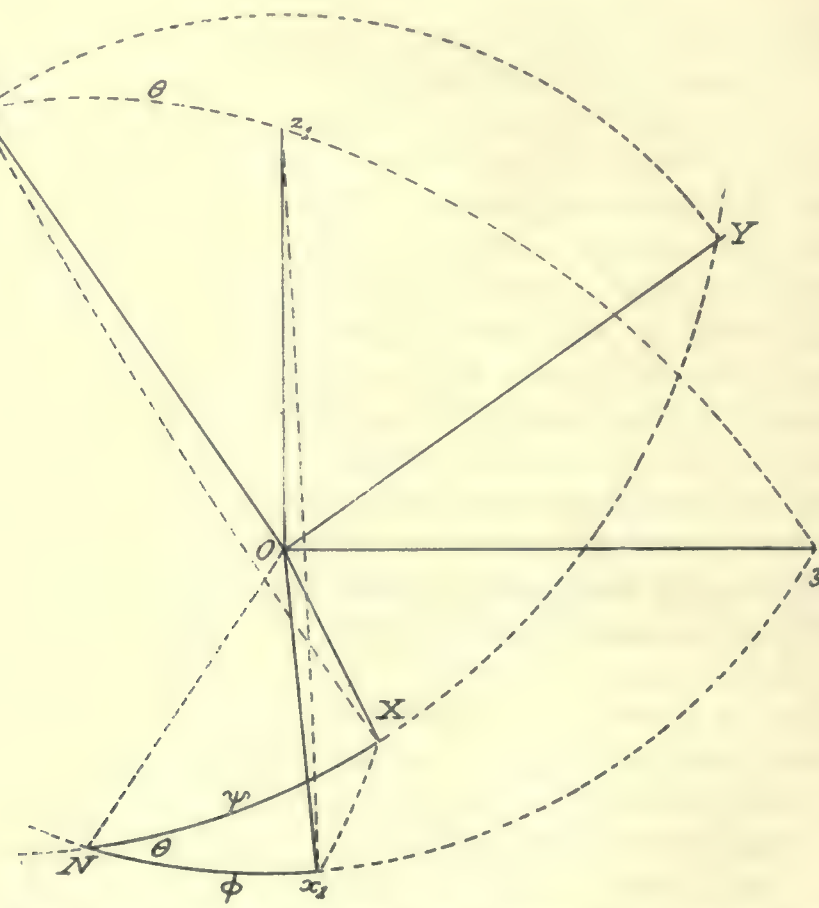

FIG. 13. EULER'S THREE ANGLES, ψ, θ, ϕ, FOR SPECIFYING THE DISPLACEMENT OF A RIGID BODY ROTATING ABOUT A POINT O.

then we have the following well known expressions for $p, q, r,$ (cf. TISSERAND'S *Mécanique Céleste,* Tome II, Chap. XXII, p. 373; THOMSON and TAIT'S *Natural Philosophy,* Vol. I; Part I, § 101; POISSON'S *Traité de Mécanique,* second edition, 1833, Tome II, Chaps. II–IV, pp. 42–178).

$$\left.\begin{aligned} p &= \frac{d\psi}{dt}\sin\theta\sin\phi - \frac{d\theta}{dt}\cos\phi\,, \\ q &= \frac{d\psi}{dt}\sin\theta\cos\phi + \frac{d\theta}{dt}\sin\phi\,, \\ r &= \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta\,. \end{aligned}\right\} \tag{136}$$

And the cosines $\lambda, \mu, \nu,$ of the angles which define the directions of the instantaneous axis of rotation are given by the expressions —

$$\left.\begin{aligned} \lambda &= \frac{p}{\omega} = \cos(x_1\, OI) = \frac{p}{\sqrt{p^2+q^2+r^2}}\,, \\ \mu &= \frac{q}{\omega} = \cos(y_1\, OI) = \frac{q}{\sqrt{p^2+q^2+r^2}}\,, \\ \nu &= \frac{r}{\omega} = \cos(z_1\, OI) = \frac{r}{\sqrt{p^2+q^2+r^2}}\,, \\ \omega &= \sqrt{p^2+q^2+r^2}\,. \end{aligned}\right\} \tag{137}$$

The living force of the body becomes

$$2T = \sum_{i=0}^{i=i} m_i\left[\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]. \tag{a}$$

If the axes $Ox_1, Oy_1, Oz_1,$ are the principal axes of inertia of the point O, and $A, B, C,$ the principal moments of inertia; that is, if

$$\left.\begin{aligned} &\Sigma m y_1 z_1 = 0 \quad ; \quad \Sigma m z_1 x_1 = 0 \quad ; \quad \Sigma m x_1 y_1 = 0 \quad ; \\ &\Sigma m(y_1^2 + z_1^2) = A \quad ; \quad \Sigma m(z_1^2 + x_1^2) = B \quad ; \quad \Sigma m(x_1^2 + y_1^2) = C\,; \end{aligned}\right\} \tag{b}$$

then we shall have

$$2\,T = Ap^2 + Bq^2 + Cr^2\,. \tag{c}$$

If by the point m_i we draw a line equal and parallel to the magnitude of the motion of m_i, but in the opposite direction, we obtain a couple. There will be as many couples as there are points m_i, and all these couples may be compounded into a single *resultant couple* of the magnitude of the movement. Let OG be the axis of the couple in magnitude and in direction, and then we shall have —

$$\left.\begin{aligned}\cos(x_1OG) &= \frac{Ap}{\sqrt{A^2p^2 + B^2q^2 + C^2r^2}},\\ \cos(y_1OG) &= \frac{Bq}{\sqrt{A^2p^2 + B^2q^2 + C^2r^2}},\\ \cos(z_1OG) &= \frac{Cr}{\sqrt{A^2p^2 + B^2q^2 + C^2r^2}},\\ G &= \sqrt{A^2p^2 + B^2q^2 + C^2r^2}.\end{aligned}\right\} \tag{d}$$

We may calculate the angle between the instantaneous axis of rotation OI and the axis of the resultant couple OG; thus from (137) and (d) we get

$$\left.\begin{aligned}\cos(GOI) &= \frac{Ap^2 + Bq^2 + Cr^2}{\sqrt{p^2 + q^2 + r^2}\,\sqrt{A^2p^2 + B^2q^2 + C^2r^2}},\\ \sin^2(GOI) &= \frac{(C - B)^2q^2r^2 + (A - C)^2r^2p^2 + (B - A)^2p^2q^2}{(p^2 + q^2 + r^2)(A^2p^2 + B^2q^2 + C^2r^2)}.\end{aligned}\right\} \tag{e}$$

§ 44. *The Observed Spirals Due to Movements Toward Centers.*

The more we study the forms of nebulae, and compare the spirals with known laws of force, the more evident it becomes that the nebulae represent movement of condensation towards centers. No other tenable theory of these phenomena can be formed. The question then arises: How does the spiral movement originate?

Let us consider a mass m projected into the sphere of attraction of a mass M at a distance r, with a velocity V. If the masses do not collide, the orbit described will be some form of conic section. In the case of an ellipse, the mass M and m will have, relatively to their common center of gravity, a moment of momentum of orbital motion defined by the equation:

$$M\left(\frac{mr}{M+m}\right)^2 \Omega\sqrt{1-e^2} + m\left(\frac{Mr}{M+m}\right)^2 \Omega\sqrt{1-e^2} = \frac{Mm}{M+m}\,\Omega r^2\sqrt{1-e^2}. \tag{138}$$

where Ω is the mean angular velocity of m about M.

Thus the moment of momentum depends upon the masses, the angular velocity Ω, the radius vector r, and the eccentricity of the orbit in which it is started. If the eccentricity be very small, the moment of momentum is a maximum; if it be very large, *caeteris paribus*, it becomes a minimum. In other words, if the bodies are set in motion nearly in the line joining them, so as to just pass by without a grazing collision, there will be very small moment of momentum given to the new system; while, if the new orbit is wide and round, the moment of momentum will be a maximum.

All these varied conditions arise in actual nature, and what we observe in a group of bodies, such as the solar system, is the outcome of gradual condensation from a nebula; and hence we thus get the mean result for an infinite number of

smaller bodies, most of which have been swallowed up in the central bodies which now govern their motions.

But as the nebulae are very tenuous and vastly expanded masses, they usually collide in falling together, or in passing each other by virtue of difference of proper motions; and the result is not only a definite moment of momentum for the new system thus formed, but also a whirling movement and gradual condensation of the matter which falls together and circulates against resistance, in the resulting motion about the common center. The longer the vortex rotates, the more regular its form will become, and as collision and friction between the parts reduce their velocities, the whole mass gradually settles to smaller and smaller dimensions, until finally it assumes a state of fairly steady motion. If the mass acquires the property of a gas of sufficient density, it may pass into a mechanical state in which a figure of equilibrium is established under the pressure and attraction of its parts. Otherwise the nebula will remain practically free from hydrostatic pressure and each particle pursue an independent path.

§ 45. *How Two Nebulae Coil Up and Settle to Rotation About an Axis.*

The paths of particles which were originally highly eccentric are thus made rounder and rounder, by friction and collision, as the mass condenses, and attains a state of equilibrium. It appears that the nebulae observed in the telescope or revealed by the photographic plate are very rare and still of vast dimensions; and therefore often represent this preliminary movement towards a center, but are not yet in a state of steady motion. Hence collision and friction within these masses may arise which give rise to a feeble light, but the radiation is still further dimmed by the cosmical dust, haze and nebulosity through which it shines.

Any two neighboring swarms of particles, or two passing streams grazing as they pass, would therefore give rise to a spiral nebula; the exact form of the whirl will depend on the initial form and relative movements of the two masses. This is unquestionably the origin of the whirlpool nebulae so abundantly observed in the immensity of space, and heretofore so mysterious to astronomers. Two opposite branches of the spirals thus originate by the meeting of separate streams or by the settling of one stream toward the center, so that the branches coil up as they condense. By studying the forms of the spiral convolutions, their number and regularity, we may form some estimate of the age and state of condensation of the nebula. In some cases many streams approach a single point, and the result is a cosmical vortex of very complex structure, which may sometime develop into a cluster or other group of many bodies.

§ 46. *Analytical Expressions for the Moments of Momentum About the Axes.*

An analytical way of looking at the approach of two nebular masses is to consider the system they form at the initial instant t, and let m_i be the mass of any particle whose coördinates are x_i, y_i, z_i. If we denote the mass of a body whose motion we are considering by m_i and the forces acting severally in the directions of the axes of x, y, z, by X_i, Y_i, Z_i, we may write the differential equations of motion thus:

$$m_i \frac{d^2x_i}{dt^2} = X_i \quad ; \quad m_i \frac{d^2y_i}{dt^2} = Y_i \quad ; \quad m_i \frac{d^2z_i}{dt^2} = Z_i \,. \tag{139}$$

If these equations be multiplied by the multipliers δx, δy, δz, which are arbitrary except that the value of infinity is excluded, and the sum of the products formed, we shall get for all the material points of the system the general equation of dynamics. This expresses D'ALEMBERT's Principle, as employed by LAGRANGE in the *Mécanique Analytique*, and is as follows:

$$\sum_{i=0}^{i=i} \left\{ \left(m_i \frac{d^2x_i}{dt^2} - X_i\right)\delta x + \left(m_i \frac{d^2y_i}{dt^2} - Y_i\right)\delta y + \left(m_i \frac{d^2z_i}{dt^2} - Z_i\right)\delta z \right\} = 0 \,. \tag{140}$$

If we take the sum of the forces in (139) for the entire system of particles, we shall have for the sums of the forces acting along the axes of coördinates:

$$\sum_{i=0}^{i=i} m_i \frac{d^2x_i}{dt^2} = \sum_{i=0}^{i=i} X_i \quad ; \quad \sum_{i=0}^{i=i} m_i \frac{d^2y_i}{dt^2} = \sum_{i=0}^{i=i} Y_i \quad ; \quad \sum_{i=0}^{i=i} m_i \frac{d^2z_i}{dt^2} = \sum_{i=0}^{i=i} Z_i \,. \tag{141}$$

And for the total moments of the forces around the coördinate axes we must take the sums of the products of these several forces by the respective arms on which they act, thus:

$$\left.\begin{aligned}
\sum_{i=0}^{i=i} \left(x_iY_i - y_iX_i\right) &= \sum_{i=0}^{i=i} m_i\left(x_i\frac{d^2y_i}{dt^2} - y_i\frac{d^2x_i}{dt^2}\right) = N \,, \\
\sum_{i=0}^{i=i} \left(y_iZ_i - z_iY_i\right) &= \sum_{i=0}^{i=i} m_i\left(y_i\frac{d^2z_i}{dt^2} - z_i\frac{d^2y_i}{dt^2}\right) = L \,, \\
\sum_{i=0}^{i=i} \left(z_iX_i - x_iZ_i\right) &= \sum_{i=0}^{i=i} m_i\left(z_i\frac{d^2x_i}{dt^2} - x_i\frac{d^2z_i}{dt^2}\right) = M \,,
\end{aligned}\right\} \tag{142}$$

where L, M, N, are the moments of the resultant applied forces around the coördinate axes. These integrals for the whole system represent the algebraic sum of the product of each force by the length of the arm on which it acts.

If p_i be the perpendicular from any point (x_i, y_i, z_i) on the axis of x, and θ_i the angle which p_i makes with the axis of y, then we shall have $y_i = p_i \cos\theta_i$; $z_i = p_i \sin\theta_i$; and when we put $\omega_i = \frac{d\theta_i}{dt}$ = angular velocity about the axis of x, we shall have for the rotation about the x-axis

$$L = \sum_{i=0}^{i=i} m_i\left(y_i\frac{d^2z_i}{dt^2} - z_i\frac{d^2y_i}{dt^2}\right) = \sum_{i=0}^{i=i} m\frac{d}{dt}\left(y_i\frac{dz_i}{dt} - z_i\frac{dy_i}{dt}\right) = \sum_{i=0}^{i=i} m_i\frac{d}{dt}(p_i^2\omega_i)\,. \tag{143}$$

When p_i for each point is invariable, or the mass is rotating as a rigid body, we obtain the following expression:

$$L = \frac{d\omega_i}{dt}\sum m_i p_i^2\,. \tag{144}$$

Now in any system, the integrals L, M, N, are constant, however the system may interact upon itself; and hence if the perpendicular distance p_i decreases under the mutual action of the parts of the system, from such causes as collisions and resistance, then it follows that as p_i^2 decreases, $\frac{d\omega_i}{dt}$ = velocity of axial rotation, must correspondingly increase, and $\frac{d^2\omega_i}{dt^2}$ will be the *acceleration* of the average angular velocity of the nebula around the axis of x, which we may take to coincide with the axis of rotation.

§ 47. *The Winding Up of Spiral Nebulae.*

By these simple principles of the attraction and collision of two approaching streams of cosmical dust every possible form of double spiral may be explained. For the shape of the spiral depends on the circumstances of the approach of the two streams which coalesce to form the nebula, namely: the velocity, the manner of collision, the mass and consequent curvature of the paths of the two swarms as they meet; the resistance developed in the continued winding up of the spirals when they settle into one united whole, as in *Messier* 51, which is one of the grandest and most typical whirlpool nebulae observed in the heavens.

If the shape of the masses and the circumstances of their approach, impact and resistance were known, we could calculate, with some degree of approximation, the form of the resulting spiral nebula, but in default of knowledge on these subjects we must content ourselves with a theory which is perfectly natural and simple, and capable of explaining all the facts disclosed by observation.

When Professor Keeler recognized the great preponderance of spiral nebulae, thus confirming and generalizing the earlier work of Lord Rosse, Lassell and Roberts, he was so surprised at the result and so occupied with the extension

of these explorations that he did not notice how simple an explanation of these phenomena lay close at hand, and his unexpected and premature death unfortunately cut short the future development of this work. The present Director of the Lick, however, has at length printed in Volume VIII of the Publications of that great observatory, the most important of KEELER's photographs. And whilst these are numerous enough to be invaluable as a contribution to our knowledge, one cannot but feel that this work should be extended into a more general survey of the nebulae of the heavens. PROFESSOR MAX WOLF of Heidelberg has done much valuable work along similar lines, and it is hoped that his plans for a general survey of the nebulae may yet be carried to completion. Until this is done in a manner comparable to the much more elaborate *Durchmusterung* of the stars, most investigators will feel that the nebulae have been comparatively neglected, and that our explorations of the heavens have given us only half of a great truth. In the future the nebulae will be entitled to even more consideration than the stars; for they represent systems in process of formation, and thus reveal to us the mighty process of cosmical evolution at various stages of its development.

§ 48. *How Vortices Arise in the Condensation of Nebulae.*

The formation of vortices, through the movement of matter towards a center but with the resultant for all the particles deflected somewhat to one side, is a familiar way of producing spiral movement. And fortunately this is illustrated in our daily experiments with fluids, as when water is being withdrawn from a basin. It is met with also in the movement of cyclones, which revolve in opposite directions in the two terrestrial hemispheres. This is owing to the combination of the rotation of the earth with the movement of the air near the surface. In the northern hemisphere the wind from the south rushes towards the center with an eastward tendency, owing to the greater velocity of the earth's rotation in lower latitude; while that from the north has a corresponding westward tendency, so that a counter-clockwise rotation arises around the center, and the result is a cyclone, or aerial vortex, often of great extent.

In the southern hemisphere where the rotation is around the other pole, the direction of rotation naturally is opposite, or clockwise. All these inferences from theory are confirmed by observation. The change in the direction of the whirling movement in the two terrestrial hemispheres is due to the action of the underlying earth. The air currents have a tendency to preserve whatever direction they take at any instant, and the earth turns under them as in FOUCAULT's

pendulum experiment. Hence the opposite rotations in the two hemispheres is inevitable.

Among the nebulae of infinite space there probably is no common motion of rotation to determine the direction of the vorticose movement; it is all a matter of chance. If the movement in each case could be determined by observation, we should probably find in the grand total as many clockwise as counter-clockwise rotations.

This subject of the condensation and rotation of nebulae has been but little studied heretofore, probably because the nature of these masses has been comparatively obscure. And although it is easy to point out the order of thought on the subject, it is difficult to establish any well defined continuity in the historical development of the theory of cosmical rotation. Among the modern writers one of the earliest to perceive that a multitude of floculent nebulous masses falling towards a center would assume the spiral form, was the philosopher HERBERT SPENCER, who indicated this result in his well known article on the nebular hypothesis in the *Westminster Review* for July, 1858. This was some thirteen years after LORD ROSSE's discovery of the spiral nebula in *Canes Venatici*, in 1845, but only fourteen of these objects were then known; and it is difficult to say how SPENCER reached his conclusions. Nevertheless this probably is the first statement of the observed tendency in direct application to the nebulae.

In a somewhat different way the rotation of the sun and stars, recognized by astronomers since the time of GALILEO, may be said to bear on the question; for the rotation of the earth, planets, sun and stars, which had been surmised since the age of the Greeks, but only proved after the invention of the telescope, implied some mechanical impulse to establish the movement around a center.

As we shall see in the historical part of this work, LEUCIPPUS, DEMOCRITUS and ANAXAGORAS ascribed the supposed effect to unequal velocities of the atoms falling in space, which gave rise to relative motion and therefore spiral movements. In the statement of the nebular hypothesis given in the *Systéme du Monde*, 1796, LAPLACE says in a foot note that in the work *De Motibus Stellae Martis*, KEPLER explains the movement of all the planets in the same direction by means of immaterial spirits emanating from the surface of the sun, retaining the movement of rotation which they had at the surface and communicating this movement to the planets. He had thus concluded that the sun rotates on an axis in a time less than the revolution of *Mercury*, which GALILEO soon afterwards recognized by observation. The hypothesis of KEPLER, remarks LAPLACE, is without doubt inadmissible, but it is remarkable that he has made the identity of the planetary movements to follow from that of the sun, so that this tendency seems natural.

§ 49. *The Observed Spiral Nebulae Represent Mainly the Initial Stages of Condensation.*

Even since the nebular hypothesis was formulated by LAPLACE in 1796, it has been recognized that in the earliest stages of its development the solar system had a whirling movement, which gave the system its moment of momentum about an axis of rotation. In no other way could the orbital momentum of the planets and the axial rotation of the sun have originated. Thus the resultant direction of motion of the matter which came together to form the solar system did not pass through the center of gravity of the system, but through a point at considerable distance from the sun's center. Undoubtedly the matter of our solar system in its earlier stages was widely diffused in the form of a whirlpool or spiral nebula; yet as the orbits at present are so perfectly circular as to have excited the wonder of the greatest geometers in all ages, it is clear that the planetary system was not left in the form which it no doubt assumed in the early spiral stage, but was *gradually transformed under some influence which gave roundness to the orbits.* This was nothing else than the resisting medium formerly pervading the system. The continued action of this medium during long ages has greatly reduced the size and eccentricity of the orbits.

When the spiral mass had thus attained smaller dimensions and the movement had become nearly circular, the nebulosity was gradually absorbed in the sun and planets; so that the planets and satellites were left with nearly circular orbits. In no other way could the orbits of so many planets and satellites have become so nearly perfectly circular. We conclude, therefore, that the solar system itself gives evidence of an original spiral character in the solar nebula. The endowment of the nebula with rotation has been universally recognized by astronomers since the days of LAPLACE; but it has not been shown heretofore that it was spiral in character and slowly decreasing in size, and at the same time growing rounder and rounder under the influence of resistance.

Now when we examine the nebulae in the depths of space, we should expect to find many of them in this early stage of developemnt. The observed spiral nebulae therefore are widely expanded and represent this stage; and have scarcely reached the state of stability in which the nuclei grow into considerable bodies and form systems, such as our solar system. Several distinct grounds may be assigned for this belief.

(1). The spiral nebulae observed in the telescope or found on the photographic plate are of large angular dimensions, and at the smallest admissible distance this would correspond to very large size, in some cases thousands of times

larger than our whole solar system. If such masses are ever to form stellar systems, even of the largest dimensions, they would have to be greatly reduced in size before orbits of the observed size could be produced.

(2). The observed transparency of the nebulae is consistent with this highly tenuous state; and this indicates excessive dispersion of the nebulous matter and corresponds to an early stage of development.

(3). The nebulae as a class are probably so far away that our telescopes disclose to us chiefly the large masses, many of the smaller ones being either unseen, or appearing so faint that they are taken for small stars; yet the recognition that there are at least 120,000 nebulae, and the possibility that the number may be 100 times greater, mostly of spiral structure, corresponding to the earliest and most expanded state of cosmical development, indicates that many more close ones exist beyond the reach of our instruments. In fact it shows that most of the single stars probably have planets revolving about them, and were once spiral nebulae; the nebulosity having gone into the stars except that which survives as planets. Such tiny bodies are however too small to be seen at the great distance of the fixed stars.

(4). The observed closeness of double and multiple stars accords with this view, and shows that the stage of separation, into distinct bodies of comparable mass, under conditions of hydrostatic equilibrium, if it occurs at all, is comparatively late in the stages of nebular development. The radius of the largest spiral nebula may be 1000″, that of the average double star orbit 1″ or less; so that the volumes of the spheres in the two cases are as a billion to one, and the densities in the inverse of this ratio.

(5). The feebleness of the light of the nebulae and their obvious transparency corresponds to luminescence in space, and enables us to see that on the average they are much rarer than the vacuum of an air pump. Such a mass must be at a low temperature, but little above the absolute zero of space. If such a tenuous swarm were heated, the heat would be radiated away in an instant, owing to the great transparency of the cloud.

(6). If the process of double star formation depended on the entanglement of stars of entirely separate origin, rather than on the division of particles by capture within single nebulae, the resulting orbits would be quite as eccentric as those of the comets, which is out of harmony with what we observe among the double and multiple stars. For here the spectroscopic binaries, with orbits on the average much smaller than that of the planet *Mercury*, have an average eccentricity of about 0.17; while the visual binaries with orbits of the average size of the major planets have an eccentricity of about 0.51, three times that of the

spectroscopic binaries. Thus, as the writer has pointed out in the *Monthly Notices* of the Royal Astronomical Society for December, 1907, small eccentricity is observed to be associated with small mean distance, and *vice versa*.

(7). This smaller eccentricity of the smaller orbits might be ascribed either to Tidal Friction, which usually increases the eccentricity along with the mean distance; or to a Resisting Medium, which would decrease the mean distance and eccentricity, and thus render the smaller orbits also the rounder; or to both causes combined. In the actual universe the two antagonistic causes are at work together.

(8). Triple systems, so far as we yet know them, show motion near a common plane, which evidently is that in which the original spiral nebulae rotated. The nebulae and our solar system alike show a striking tendency to develop in their several planes of movement. This could not arise, unless the motion were well ordered before formation began; and thus the spiral tendency evidently precedes that leading to the separation of masses under conditions of equilibrium, if the latter occurs at all.

(9). After the condensing mass has attained approximately a state of equilibrium, under the mutual action of the pressure and attraction of its parts, fission into separate masses of comparatively large size may begin, but this always requires great rotational moment of momentum. If *fluid fission* occurs at all it is by means of figures of equilibrium, which correspond fairly well with those calculated by DARWIN and POINCARÉ.

(10). *The new theory of nebular fission will be developed in Chapter X.*

§ 50. *The Significance of the Double Branches Often Seen Issuing from the Nuclei of Spiral Nebulae; Origin of Curved or Cometary Nebulae.*

While the spiral nebulae have nearly all forms as respects degree of curvature, and some have many branches, it has been remarked with surprise that many of them are made up of double branches, issuing from the opposite sides of the nucleus. What is the cause of this singular arrangement? This is a question which we shall now endeavor to answer. It is well known that the heavens are full of streams; for there are streams of stars drifting in various directions, and of nebulae also, but not yet fully investigated. Now suppose two nebulous streams collide while going in opposite directions, or that one overtakes the other, owing to difference in their velocities, when moving in the same general direction. As the two masses approach, their figures will be distorted and curved, so that the streams will tend to become entangled. A whirling moment will follow, each stream wrapping and

coiling up on the other. Both streams will remain more or less continuous, and the brightest point will be the place of collision, where the two streams meet. From this point the two streams will wind off, and finally have their tails projected in opposite directions, showing how they approached before they coiled up about each other. This explanation is simple and direct and follows from well known mechanical laws. The winding about the origin usually would not be perfectly symmetrical; but in general such spirals would be fundamentally double, made up of inter-wound streams issuing in opposite directions from the nucleus, which is the principal point of contact. If, instead of two separate streams meeting under differences of proper motion, we imagine a single stream of nebulosity which is not perfectly straight left to its own gravitation — this is about equivalent to separate streams already in collision — it is clear that in time it will coil up on itself, and gradually develop into a spiral nebula. Thus it makes little difference whether there be one or more streams, united or distinct, the final result is the same, the development of a whirlpool nebula.

If the secondary body be a dark sun, or other invisible body, the coiling up of the nebulosity about it, or the curving of the stream of nebulosity as it passes by, will give only one branch, and it will often be curved like the tail of a comet.

On the other hand it may happen that the center of gravity of a nebula is near one side. In this case the streams setting towards it will not present a symmetrical outline, especially as seen in projection on the background of the celestial sphere. At a certain stage of the condensation such a nebula would offer more or less of a cometary aspect. The same effect might be produced by the meeting and coiling up of two streams of unequal length; or by the mere gravitational settling of a stream of unsymmetrical figure and heterogeneous distribution of density. In this way a great number of nebulae of irregular figure arise; and the theory of such unsymmetrical condensation is abundantly illustrated throughout Nature.

Modern photography has led us to classify most of the curved nebulae as spiral; others are irregular and bent in all manner of ways, the appearances being due partly to distortion in projection, and partly to actual curvature of the streams of nebulosity in space. The older observers, such as the HERSCHELS and LORD ROSSE, adopted the cometary form in their classification of the figures of the nebulae; but this designation is less used by modern investigators, owing doubtless to the preponderance of the spiral form. It will be seen hereafter that under certain conditions, the streams ordinarily giving rise to spirals may produce all types of curved figures, from the imperfect cometary form to the symmetrical arrangement exhibited by the ring nebulae. In this latter case the streams

miss each other in drifting towards the center, and by whirling about it finally close in and form a continuous girdle of elliptical outline, such as we see in the ring nebula in *Lyra*. The photographs of the nebulae are treated at greater length hereafter, and the reader must be referred to that discussion for a more detailed analysis of their forms; but it seemed desirable to touch upon the subject in explaining the rotations of these masses by means of the known laws of Dynamics.

§ 51. *On the Probability of Rotation Developing as Nebulous Matter Falls Towards a Center.*

The simplest case of this kind is when two bodies approach each other, and both have the form of spheres, so that we may regard them as acting as if collected at their centers of gravity. In the general case of nature there will be n masses, where n is indefinitely great, and their figures may vary from that of a sphere to a train of cosmical dust of any possible form. Curved and irregular wisps of nebulosity will be especially abundant. Under the circumstances the approach of two spheres will be the least favorable to rotation, and we shall first consider this simple case.

Each mass may by hypothesis be regarded as collected at its center of gravity, and this center will describe a right line with uniform velocity, except as modified by the attraction of the other mass. If one moves past the other, under slight attraction, the path traced out will generally be an hyperbola of small curvature, or large major axis. The probability of collision will increase directly with the square of the radius, because the solid angles subtended by the masses as seen from each other depend on this function.

If r be the radius of the two equal spheres, and R the distance between their centers; then the value of the solid angle which would give rise to a collision becomes

$$\omega = 2\pi R \,.\, R\left[1 - \cos\left(\frac{2r}{R}\right)\right] = 4\pi\left\{\left(\frac{r}{R}\right)^2 - \frac{1}{3}\left(\frac{r}{R}\right)^4 + \frac{2}{45}\left(\frac{r}{R}\right)^6 - \;.\;.\;.\;.\;\right\}.$$

At any fixed distance this series is nearly proportional to the square of the radius, since for moderate distances the series converges rapidly. It will be seen from this expression that when the radius is large and the distance small, the chances are most favorable for a collision.

We shall now assume that a collision takes place, and investigate the chances of developing a corresponding rotation. For short distances and small masses which exert a feeble attraction, the hyperbola may be regarded as nearly a straight line. If the motion is nearly rectilinear we see that after collision, rotation will necessarily occur, except when the two centers of gravity move exactly towards the

same point, and (unless moving in the same straight line) with velocities such as to reach it at the same instant. The chances of two centers of gravity moving exactly towards the same point is exactly the same as that of two lines passing through a given point. If we pass movable lines through the centers of gravity of the two masses and revolve them over the plane in which the two bodies are first supposed to move, we see that the probabilities are infinity to one that neither of the lines will pass through the fixed point. They will, however, have to pass through some common point, unless exactly parallel. The probability that any line will miss a given point is ∞ to 1, yet they will certainly pass through one common point; but the chances that both bodies would not be there at the same time is ∞^2 to 1. This, of course, supposes the first movements restricted to a common plane.

In the more general case of cubical space, the two lines in which the bodies move would not lie in a plane, but one would have to be shifted parallel to itself to bring it into intersection with the other, so as to determine a plane. The chance that this latter will not be combined with the two former conditions is ∞^3 to 1. Hence in general the chances that the bodies will not move in the same plane and towards the same point so as to meet there at the same instant is ∞^3 to 1.

We may look at this result in a slightly different way as follows. To reach the point of collision from two unequal distances at the same time, the two velocities would have to be restricted, each to but one out of an infinite number of possibilities, all equally probable. It is therefore ∞^2 to 1 that these particular velocities will not occur, and ∞ to 1 that the lines of motion will not intersect, and hence ∞^3 to 1 that the centers of gravity of the two spheres will not meet there.

We have then a chance of ∞^3 to 1 that the paths of the bodies will not meet in a point and the bodies arrive there at the same instant. *Accordingly as the probability is ∞^3 to 1 that the centers of gravity will not meet in the same point, at the same instant, that is also the probability that rotation will develop.* If they do not meet in the same point, and yet collide, a rotation is sure to ensue. Hence the probability of a rotation developing from *two colliding* spheres moving with all possible velocities in any direction is ∞^3 to 1. And this reasoning holds good for physical bodies, as well as mathematical spheres and points and lines, if we include every grade of rotation from the slowest conceivable up to the swiftest possible. If we neglect all the small or insignificant rotations, it is clear that there would still remain a probability of at least ∞ to 1 that pronounced rotation would develop. We have here considered the simple case of the two spheres, which is the least favorable figure for collisions. Now if all forms of nebulous clouds and wisps and streams be admitted, it is evident that the probability of

rotation would be appreciably higher than ∞ to 1. Hence, in any condensing system of particles, rotation is absolutely inevitable.

Not only may we calculate that rotations ought to arise from the collisions of masses, but we may also see from the study of the heavens that the theory is verified practically by a great multitude of phenomena witnessed in the immensity of space. The rotations seen in the solar system have all arisen in this way. It is shown by observation that at least one-fifth of the fixed stars are double and multiple, the rest having systems of planets too small to be visible to the inhabitants of the terrestrial globe. This indicates that rotations exist everywhere of considerable magnitude, and that accretion in the convolutions of a spiral nebula gradually take place, and the resulting bodies are developed in a Resisting Medium.

Moreover the hazy spectral lines of vast numbers of the stars are ascribed to their rotations on axes so situated that the whirling confuses the light transmitted to the earth from the two limbs of their globes. Finally, the spiral nebulae give visible indications of the universal tendency to whirlpool movements, even in the early stages of condensation, before these systems have settled to a state of steady motion. This tendency to rotation is universal throughout nature.

Consequently we may conclude from these considerations as well as from equations (142) that rotation in condensation towards a center is a universal law of the physical universe. Probably not a satellite, planet, star or nebula is wholly free from such movement, though in some cases the effect may be too small for us to perceive it, or the rotation may be destroyed by special causes such as tidal friction. The number of known stars in the universe probably does not exceed 200 million. But if, with POINCARÉ, we take the number of light and dark bodies five times larger, yet the probability of rotation in condensing masses is still ∞ to 1, and therefore much greater than 1000 million. Accordingly it seems certain that all the bodies in the universe are endowed with some kind of motion of rotation, except where it has been destroyed by other causes, and chiefly by tidal friction.

As all the stars have motion of rotation on their axes they have evidently been formed in vortices, and therefore will have developed planets and satellites or larger companions in the process of condensation. Accordingly it follows that all the stars are attended by systems of planets and satellites, unless double or multiple; and even then these double and multiple systems may have small bodies revolving in regions of stability close to each component star, or far from both of them. The sidereal universe therefore is certainly full of planetary systems, for we see that they have necessarily arisen in the formation of the stars themselves.

Εἰ δ' ἕτερόν τί ἐστι σῶμα τὸ φερόμενον κύκλῳ παρὰ φύσιν, ἔσται τις αὐτοῦ ἄλλη κίνησις κατὰ φύσιν. τοῦτο δ' ἀδύνατον· εἰ μὲν γὰρ ἡ ἄνω, πῦρ ἔσται ἢ ἀήρ, εἰ δ' ἡ κάτω, ὕδωρ ἢ γῆ. ἀλλὰ μὴν καὶ πρώτην γε ἀναγκαῖον εἶναι τὴν τοιαύτην φοράν. τὸ γὰρ τέλειον προτερον τῇ φύσει τοῦ ἀτελοῦς, ὁ δὲ κύκλος τῶν τελείων, εὐθεῖα δὲ γραμμὴ οὐδεμιά · . . . .

Ἀλλὰ μὴν τῶν ἀπὸ τοῦ αὐτοῦ ἐπὶ τὸ αὐτὸ ἐλαχίστη ἐστὶν ἡ τοῦ κύκλου γραμμή · κατὰ δὲ τὴν ἐλαχίστην ταχίστη ἡ κίνησις· ὥστ' εἰ ὁ οὐρανὸς κύκλῳ φέρεται καὶ τάχιστα κινεῖται, σφαιροειδῆ αὐτὸν ἀνάγκη εἶναι. λάβοι δ' ἄν τις καὶ ἐκ τῶν περὶ τὸ μέσον ἱδρυμένων σωμάτων ταύτην τὴν.πίστιν. εἰ γὰρ τὸ μὲν ὕδωρ ἐστὶ περὶ τὴν γῆν, ὁ δ' ἀὴρ περὶ τὸ ὕδωρ, τὸ δὲ πῦρ περὶ τὸν ἀέρα, ...

If an element have another path which is naturally circular, it should have some other natural motion. This, however, is impossible, because fire and air are borne upward, while water and earth are carried downward. But the primitive motion must necessarily continue to follow the same path. For that first completed is by nature of an indefinite extent, and the circle one of the perfect paths, while the straight line never is. . . . . . .

But of these (paths for the motion of the heavens) the shortest one is the path of the circle, and therefore the swiftest motion gives the least path; so that the heavens revolve in a circle and with the swiftest movement and must itself necessarily be of spherical figure. One might infer this also from the stable situation of the settled bodies with respect to the center. For the water is above the earth, and the air above the water, and the fire above the air. — Aristotle, De Coelo, A, 2, 269a, 15–21; B, 4, 287b, 27–33; edition of Prantl, 1881.

CHAPTER V.

Distribution of the Nebulae with Respect to the Milky Way, the Size of the Larger Spiral Nebulae, and Other Considerations Which Invalidate the Theories of These Nebulae Heretofore Advanced.

§ 52. *On the Distribution of the Nebulae with Respect to the Milky Way.*

The nebulae are divided into two classes — the so-called white nebulae and the green; the latter are but a negligible fraction of the whole, perhaps two or three per cent., have bright-line spectra, and so far as known are situated near the plane of the Milky Way. The white nebulae on the other hand give continuous spectra and crowd the heavens in vast numbers. The concentration of the body of the nebulae near the poles of the Galaxy has been known for fully half a century. This was pointed out by Herbert Spencer in 1858, and has been represented graphically by Proctor and Sidney Waters (1869–1873) in the accompanying maps, which are taken from the *Old and New Astronomy* of Proctor and Ranyard, (pp. 727, 728, 729). Nothing could be more impressive than this polar distribution of the nebulae, as far away from the stars and the clusters as possible.

These maps are founded on Sir John Hercshel's *General Catalogue of Nebulae and Clusters* (Phil. Trans., Roy. Soc., 1864). And although the data would be somewhat modified by Dreyer's later revision of this work, the distribution given in the *New General Catalogue of Nebulae*, 1888 (*Memoirs of Royal Astron. Society*, Vol. XLIX, Part I), would no doubt be essentially the same as that indicated in these charts. Even if all the new nebulae recently discovered by Barnard, Swift, Max Wolf, Roberts, Keeler, and others were included, and we had a complete photographic survey of all the nebulae of the heavens, there is no reason to think that this well known antipathy between the location of the stars and of the nebulae would be sensibly changed. The modern surveys simply find more nebulae in all parts of the sky, and consequently the distribution is not materially altered, although the crowding of these objects towards the poles of

the Galaxy may be relatively a little less conspicuous. Such, at least, is Dr. Max Wolf's conclusion based on extensive surveys of these objects, showing that the exhaustive study of nebulae has hardly yet begun.

Just as this work is going through the press, an important paper on the results of the photographic study of the globular clusters has been published by Perrine in *Lick Observatory Bulletin*, No. 155. He confirms the close relationship of these clusters to the Milky Way, and gives a chart which he thinks may safely be said to apply to all clusters classed as globular. Part of Perrine's discussion is as follows: "This distribution brings out a close relationship of some kind to the Milky Way structures. Group *A* coincides very closely with the brightest and most extensive region of the Milky Way proper — the *Sagittarius-Aquila-Ophiuchus* region. The center of this widened region of the Milky Way is practically that of Group *A* of the clusters. The outlying clusters (which are the brightest of these objects) are beyond the limits of the Milky Way. Groups *B* and *C* are found to occupy the regions of the two Nebeculae, respectively. The northernmost region of the Milky Way appears to be almost, if not entirely, devoid of globular clusters. All of these clusters are completely resolved into stars. No trace of nebulosity has been found in any of them."

§ 53. *The Physical Interpretation of the Apparent Antipathy Between the Stars and Nebulae.*

That this extraordinary distribution of the nebulae as far as possible from the stars of the Milky Way has some deep physical significance can scarcely be doubted. What then is the meaning of the observed distribution? To answer this question in the simplest and most unbiased manner, we may ask ourselves how Democritus, Anaxagoras, Aristotle, or any of the Greek natural philosophers would have answered such a question, had they known that cosmical dust is constantly expelled from the stars by electric forces and by the radiation-pressure of their light and driven away from the Milky Way, which they also knew to be composed of small stars too dense to be seen individually? Can any one doubt that the Athenian sages would have said that the nebulae are formed of cosmical dust expelled from the stars, and are therefore located as far away from the Milky Way as possible, being collected principally in its poles?

It is clear that the Physical Universe is governed by two antagonistic principles, both incessantly at work: one being the secular condensation of matter under gravity to form stars and systems, to be followed eventually by another of the opposite character; namely, an expulsion of cosmical dust, under electric forces and the radiation-pressure of the light of the stars, to form nebulae.

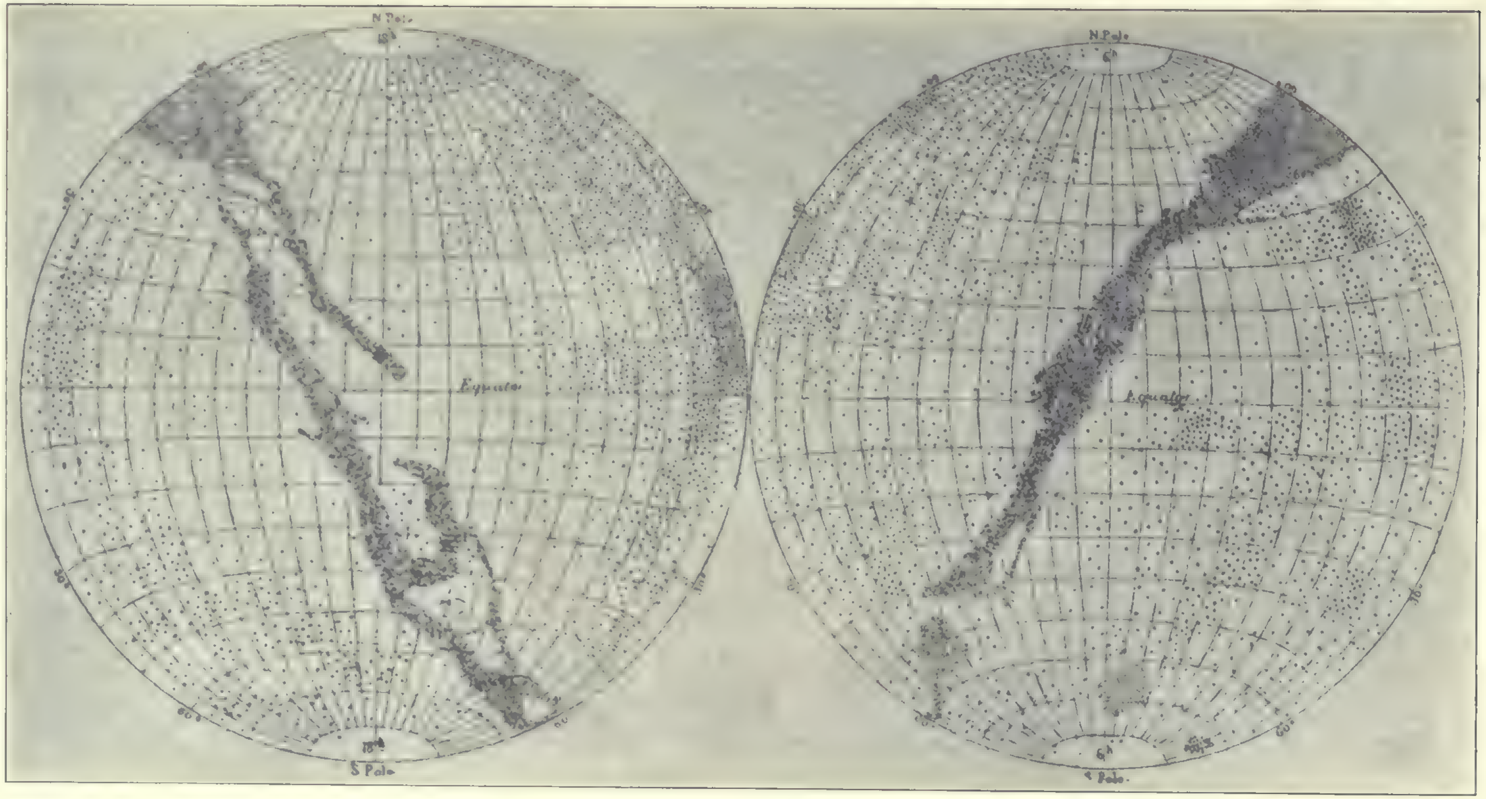

Plate I. Distribution of the Nebulae — Isographic Projection, Showing the Zone of Few Nebulae, from Mr. R. A. Proctor's Paper in the *Monthly Notices* for October, 1869.

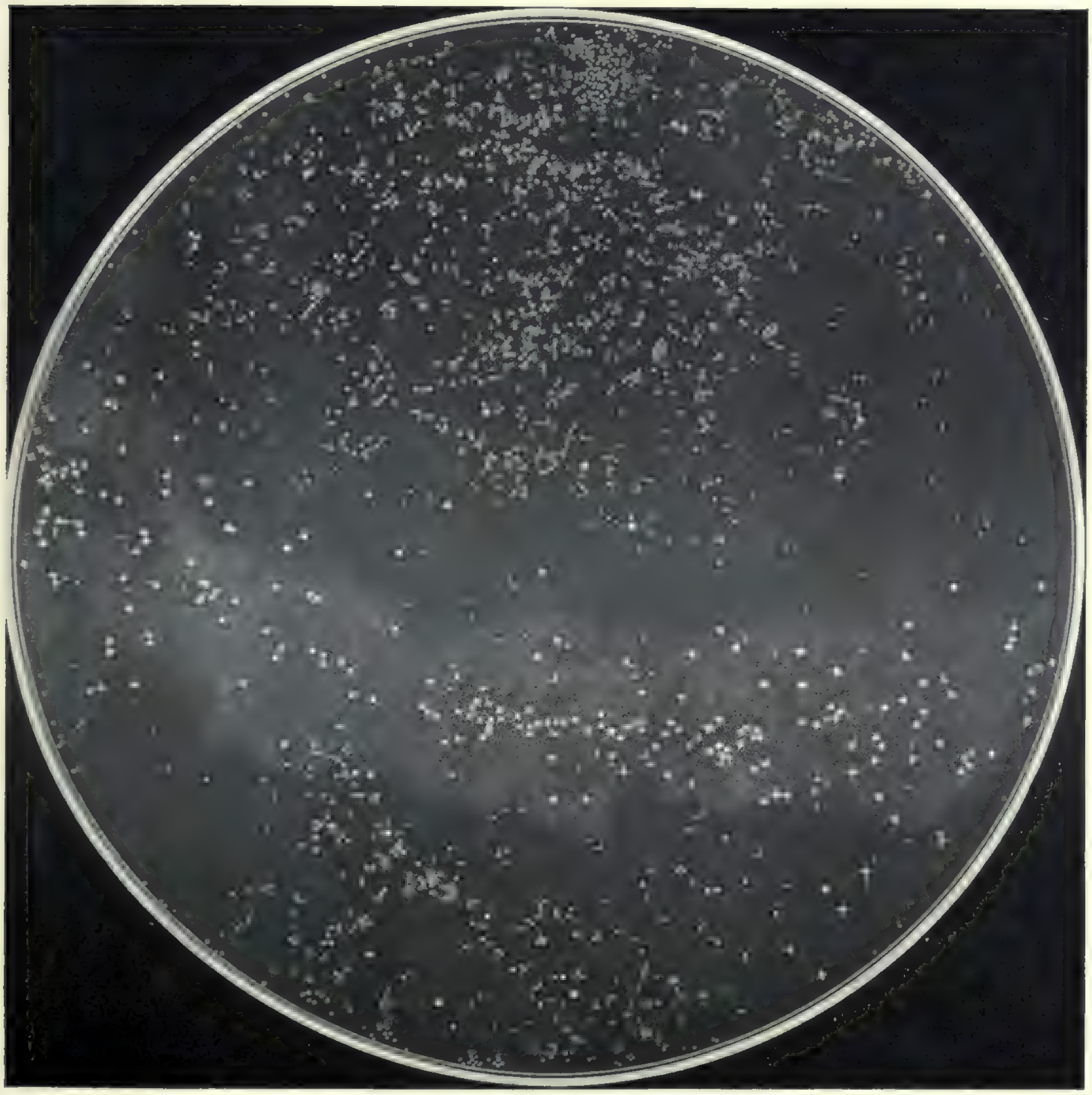

Plate II. The Nebulae and Clusters in the Northern Hemisphere, Plotted on an Equal Surface Projection by Mr. Sidney Waters, from Sir John Herschel's Catalogue. (*The Nebulae are represented by dots, the Clusters by crosses.*)

Plate III. The Nebulae and Clusters in the Southern Hemisphere, Plotted by Mr. Sidney Waters. (*The Nebulae are represented by dots, the Clusters by crosses.*)

Now as the stars are formed mainly in the stratum of the Milky Way, which is spread out over an immense extent but not relatively of great thickness, does it not necessarily follow that the nebulae would be most numerous in the poles of that stratum? This seems to be the true interpretation of the most conspicuous and fundamental division of the universe. There is a constant interchange of matter between the regions of the stars and of the nebulae. One tendency operates to form new stars by the gradual condensation of diffused matter, the other to form new nebulae by the dispersion and subsequent collection of the cosmical dust expelled from stars already far advanced in development.

When the matter is first expelled from the stars, it is driven away indifferently in all directions, but its *permanent path of least resistance* obviously is towards the region of fewest stars; and it therefore collects more or less in various places, but principally in the poles of the Milky Way, and thus gives us the great canopy of nebulae disclosed by observation.*

This, at least, is the way the Greeks would reason in regard to this matter, and the present writer considers it safe to adhere to their simple and direct way of interpreting the most obvious phenomena of the physical world. A similar view would, no doubt, be taken by such modern physicists as ARRHENIUS and others who have studied the theory of the radiation-pressure in the corona of the sun, and the dispersion of cosmical dust from the stars, by the radiation-pressure of their light and by other repulsive forces.

The green nebulae, though comparatively few in number, seem to be most abundant near the plane of the galaxy. And we may account for this most easily by supposing that the matter in these masses is recently expelled from the stars, and some of it is still responsive to neighboring radiations. This may make certain elements glow with inherent light, and hence the bright lines in the spectra of some of these nebulae.

In discussing the distribution of the nebulae, the late MR. A. C. RANYARD remarks that the researches of PROCTOR and WATERS "showed that streams and clusters of nebulae which had been ranked as irresolvable by SIR JOHN HERSCHEL were followed by, and associated with, streams and clusters of nebulae which had been ranked as resolvable, in a manner which rendered it probable that they are associated together, forming distinct systems from, but intimately associated with, the distribution of the lucid or brighter stars, while the large and irregular gaseous nebulae, which are frequently associated with star clusters, are grouped along the Milky Way, and seem to be intimately associated with it. The aggregation

* Greater absorption of light due to augmentation of nebulosity towards the poles of the Galaxy may account for part of the observed contrast between the stars and nebulae.

of star clusters upon the Milky Way, especially along the central region, is also very striking" (*Old and New Astronomy*, p. 730).

§ 54. *The Inadequacy of the Theories Heretofore Proposed for Explaining the Spiral Nebulae.*

Neither Lord Rosse's discovery of spiral nebulae in 1845, nor Lassell's verification of his inferences about 1855 led to any definite theory of these masses.

Lord Rosse's account of *Messier* 51, in the memoir entitled "Observations on Nebulae," presented to the Royal Society, June 20, 1850, and published in the *Philosophical Transactions*, 1850, Part II, contains the following suggestive passage: "We thus observe, that with each successive increase of optical power, the structure has become more complicated and more unlike anything which we could picture to ourselves as the result of any form of dynamical law, of which we find a counterpart in our system. The connection of the companion with the greater nebula, of which there is not the least doubt, and in the way represented in the sketch, adds, as it appears to me, if possible, to the difficulty of forming any conceivable hypothesis. That such a system should exist, without internal movement, seems to be in the highest degree improbable: we may possibly aid our conceptions by coupling with the idea of motion that of a resisting medium; but we cannot regard such a system in any way as a case of mere statical equilibrium. Measurements, therefore, are of the highest interest, but unfortunately they are attended with great difficulties." Such were the problems encountered by the older observers, before the days of Astronomical Photography!

The later work of Isaac Roberts, in 1887–1900, and especially of Keeler, who made a more adequate survey with the Crossley reflector at the Lick Observatory, in 1898–1900, showed clearly that a theory of the spiral nebulae would eventually become necessary for our interpretation of celestial phenomena.

In 1904 an attempt was made by Chamberlin and Moulton of Chicago, to utilize the observed spirality of the nebulae in their continued efforts to solve certain problems relative to the origin of the solar system. The theory developed by Moulton has been published in the *Astrophysical Journal* (Vol. XI, No. 2, March, 1900, and Vol. XXII, No. 3, Oct., 1905). We shall now briefly examine the character of this theory.

The essential point of the Chamberlin–Moulton theory is that if one star passes near another in the course of chance movement due to proper motions, the tidal disturbances thereby arising would cause matter to be ejected from the

stars; and the magnified prominences thus shot forth would form spirals about each mass. The condensation of the ejected material, it is held, would form systems of bodies analogous to our planets.

In considering this theory it should be remembered that it grew out of previous work by CHAMBERLIN, afterwards incorporated in a paper, "On a Possible Function of Disruptive Approach in the Formation of Meteorites, Comets and Nebulae" (*Astrophysical Journal*, Vol. XIV, 17–40, 1901). Thus MOULTON's theory was admittedly an extension of a previous theory, and not the independent outcome of a general examination of the question on its merits. The very title of CHAMBERLIN's paper "On a Possible Function of Disruptive Approach," etc., shows that it was offered merely as *one possible explanation*, and that the $n + 1$ *other possible explanations* had not been fully considered. In view of these facts, it will not be surprising if, when we examine all the phenomena carefully, we find the theory of CHAMBERLIN and MOULTON quite devoid of foundation. It has been remarked with regret that MOULTON, who is by profession an astronomer and mathematician, has also fallen into the habit of offering *possible explanations*, without inquiring *how many other possible explanations are overlooked*. Thus in his paper on the Evolution of the Solar System in *Astrophysical Journal*, October, 1905, Sec. 3 is on a "Possible Origin of Spiral Nebulae." This carelessness is the less excusable in a writer trained in mathematical methods, because it is a fundamental principle in the most exact of all sciences, that the validity of reasoning is assured only when it is supported by necessary and sufficient conditions; that is, it must be shown that the explanation offered is not only sufficient to account for the phenomenon in question, but also that no other possible explanation will account for it. Such reasoning alone is worthy of the traditions of Celestial Mechanics, for NEWTON used this very criterion in establishing the law of universal gravitation. KEPLER having shown by observation that the planets and comets move in conic sections, NEWTON proved that the law of the inverse squares, and no other possible law of attraction, would explain the phenomena. Thus the law of universal gravitation was established forever.

It may, perhaps, occur to the reader that it is not always possible to introduce these strict criteria, and that much of our reasoning in the physical sciences is therefore never very certain; yet if great uncertainty still exists it should always be plainly and distinctly stated. In the case of the speculation indulged in by CHAMBERLIN and MOULTON the course adopted has been misleading in its effects on scientific thought. The publication of such theories was not justifiable, because it was plainly contradicted by the most obvious of celestial phenomena. It can easily be shown to be theoretically unsound, and it is emphatically contra-

dicted by the observed distribution of the spiral nebulae, as we shall see by the following considerations.

§ 55. *Fatal Objections to the Theory of* CHAMBERLIN *and* MOULTON.

We shall now examine this theory of CHAMBERLIN and MOULTON somewhat more critically. It may be admitted that, if the supposed initial approach of two stars be sufficiently close, some ejections of prominences might take place, and the ejected particles might pursue spiral paths. But even if we concede this ejection, would the matter be spread out to such great distances as are observed in our planets, and to the still vaster distances observed in the nebulae? And could the large size, regularity and circularity of the planetary orbits be thus accounted for? Moreover are we justified in postulating such close approaches? The reader will see from these questions how serious the objections are to the *possible explanations* which have been offered. In order to throw as clear a light on this problem as possible, we shall examine the questions in detail, and we first consider:

Some Reasons Why the CHAMBERLIN–MOULTON *Theory of the Origin of Spiral Nebulae is Untenable.* If this theory that the spiral nebulae are formed by the process of disruptive approach, as one star passes another, were true, we should obviously have the following consequences:

(1) Spiral nebulae should be abundant where the stars are densest, as in the clusters and the Milky Way; because there close approaches would on the average be most frequent.

(2) Spiral nebulae would be especially abundant in the densest masses of stars, such as the globular clusters, because close approaches would there attain maximum frequency.

(3) The spiral nebulae should nearly always occur in pairs, for the disruption of one of two passing stars would generally imply the disruption of the other also.

Extensive study of the heavens shows that nature is directly contradictory to these inferences; for the facts are found to be as follows:

(*a*) The spiral nebulae are not abundant in the clusters and Milky Way, the vast majority of them occurring in the region of the poles of the Galaxy.

(*b*) No well defined nebulae are known in the clusters, and dense globular clusters seem to be practically devoid of nebulosity of any kind.* This seems

* Since this was written it has been amply confirmed by an important investigation by PERRINE in *Lick Observatory Bulletin,* No. 155. Two of his conclusions are:

"3. The globular clusters are devoid of true nebulosity.

"4. There is no direct relationship between the globular clusters and the numerous small nebulae. The distribution of the small nebulae appears to be entirely independent of the globular clusters, neither affinity nor avoidance being disclosed."

to indicate the operation of the clustering power observed by the elder HERSCHEL, which is thus gradually clearing the clusters of star-forming material; and directly contradicts the view that disruptive approaches are in progress.

(*c*) So far as known spiral nebulae never occur in pairs; therefore it is evident that two stars do not disrupt each other in passing, nor do they in fact pass very near each other, except under the most extraordinary circumstances.

(*d*) For it may be shown by the theory of probability that approaches of stars sufficiently close to cause tidal outbursts would occur so rarely as to leave no trace in the appearance of the visible universe. PROFESSOR CRAWFORD of the University of California, has justly remarked: "It might in a long time occur once or twice in a universe composed of 100 million stars, but not oftener."

(*e*) The spiral nebulae are numbered by hundreds of thousands, if not by millions, and therefore could not possibly have arisen in this way.

In view of these obvious contradictions, it is clear, therefore, that the CHAMBERLIN–MOULTON theory of the origin of the spiral nebulae is wholly untenable. Indeed one eminent astronomer has expressly remarked on the difficulty of understanding how a theory so completely devoid of real foundation came to be seriously promulgated in a scientific journal. We can only explain the publication of such a theory on the supposition that the whole subject has been very obscure, and accurate reasoning therefore not required.

It is unnecessary to thrice slay the slain, but it may be remarked that the late MISS CLERKE, in her valuable work entitled *Problems in Astrophysics*, p. 445, has pointed out another very obvious weakness in this theory. After describing the theory of CHAMBERLIN and MOULTON, she adds: "The events contemplated in it are on a small scale by comparison with the grandiose dimensions which we must ascribe to spiral nebulae." This point is one which admits of precise examination, and therefore is worthy of further consideration.

§ 56. *What are the Probable Sizes of the Larger Spiral Nebulae?*

A number of the spiral nebulae are found to have apparent radii of many minutes of arc, and they are therefore masses of vast extent. If we take the apparent radius of the larger spiral masses at $12'.5$, the size can not be considered excessive. This is about $\frac{5}{6}$ of the moon's radius, and there might be in the entire heavens some 300,000 such objects without wholly covering the face of the sky.

Now at the distance of α *Centauri* the radius of the earth's orbit would subtend an angle $0''.75$; and a radius of $12'\ 30'' = 750''$, at that distance, would correspond to about 1000 times that of the earth's orbit. But on the average the nebulae cannot be nearer to us than some 30 times the distance of α *Centauri*, and hence

their average diameters of 25′ correspond to bodies 1000 times larger than the orbit of *Neptune!*

Thus the great space covered by the spiral nebulae indicates that many of the larger of these objects are thousands of times larger than our whole solar system. Under the circumstances, the rarity of these masses is not surprising; nor may we expect any appreciable evidence of rotatory motion in periods less than centuries. Accordingly we must squarely face the difficulty presented by the immense dimensions of the spiral nebulae. And although objects of much smaller dimensions undoubtedly exist, any theory proposed could not be considered very satisfactory till it was shown to be adequate to explain the larger masses as well as the smaller ones.

But Our Own Solar System is Already of Vast Extent, and the Dispersion of Matter Ejected from the Sun Over Such a Space would be Very Difficult to Accomplish by Any Indirect Process such as Tidal Disruption. In the attempted explanation of spiral nebulae by prominences ejected from bodies like our sun, a very close approach has to be postulated, in order to make the tidal forces sufficiently powerful to produce the assumed disruption of explosive jets. For the tidal forces vary inversely as the cube of the distances, and the forces therefore become very powerful only when the distance is small. In the present state of our sun, it is safe to say that no very great ejections would take place unless the passing sun came within the orbit of *Mercury*. If the passing star were larger than our sun, or if our sun were less dense and more expanded in volume than it is at present, it might produce an equal effect at greater distances; but the stars on the average are not enough larger than our sun to justify us in admitting a disrupting passage remoter than the orbits of the inner planets, say two astronomical units.

Now just what would happen in such a passage of a large star near our sun, necessarily is more or less obscure; but it seems fairly certain that the change in local gravity accompanied by deformation of the figures of the stars could not produce ejections to extend much, if any, beyond the point of nearest approach of the two bodies. This inference seems to be justified by what is observed among the double stars at their periastron passage in very eccentric orbits.

§ 57. *No Indications of Tidal Disruption Furnished by Double Stars at Close Periastron Passage.*

It is well known that γ *Virginis* has an eccentricity of about 0.9, and yet no outbursts or great changes of brilliancy were observed by W. STRUVE during the passage of 1836. In like manner the double star Σ2525 has an eccentricity of about 0.95; and yet it too passed periastron in 1888 without any violent

disruptions or spiral ejections conspicuous enough to make the outburst visible to such skillful observers as HALL, SCHIAPARELLI, H. STRUVE and BURNHAM.

The fact that eccentric binaries are not converted into nebulae during periastron passage tells very powerfully against such supposed detachment of huge prominences. It is probable that the periastron passage of double stars, in at least some cases, is as close as the perihelia of the orbits of our interior planets to the sun. And if whirlpools of ejected matter were left behind each time such passage occurred, these double stars would probably present a nebulous aspect, which is not confirmed by observation. The observational criterion supplied by the orbital movement of double stars therefore appears to indicate that no great dispersion of nebulous matter to distances such as that of *Neptune* could be produced by the passage of a neighboring star near a body like our sun.

Accordingly it is clear that in such passages of disturbing bodies matter is not ejected to any appreciable extent, owing probably to the intensity of gravity in well developed stars; and also that, even if it were ejected, it could not be spread out to very great distances corresponding to the orbits of the outer planets. Moreover, if so spread out, the orbits of the particles so ejected would nearly all pass near the sun, and be so eccentric that they could never condense into bodies moving in orbits with small eccentricities but large major axes, like the major planets, *Jupiter, Saturn, Uranus* and *Neptune.*

Lastly it is found by observation that spectroscopic binaries on the average have eccentricities not larger than one-third of those found among the wider visual binaries (cf. Paper by the author in *Monthly Notices* of the Royal Astronomical Society for December, 1907), so that the larger orbits have the higher eccentricities among the stars, but not in our solar system. Accordingly we may dismiss the suggestion of this diffusion of matter over vast distances, like those represented by our major planets, as quite devoid of foundation, and wholly inadequate to account for the roundness of the planetary orbits. With regard to outbursts which would give rise to spiral nebulae thousands of times the dimensions of the solar system, the mere suggestion of such a thing shows it to be so improbable that it requires no further notice.

§ 58. *On the Development of Vortices in the Condensation of the Stars from Nebulae, and on the Nature of Philosophic Truth.*

One criticism of the greatest weight against the theory of CHAMBERLIN and MOULTON is based on the obvious inconsistency of the underlying conceptions, and on the arbitrary and gratuitous character of the adopted hypotheses. These

writers assume apparently that the stars were formed originally by the falling together of nebulous matter, though this point is nowhere made clear. If this is the view adopted or tacitly implied, they evidently suppose all the nebulosity to have gone straight to these centres of condensation; and the best plan they can devise, for getting some of it out of the stars to form cosmical systems about them, is to assume arbitrarily that the stars occasionally pass close together, and through tidal disruptions eject some of it to form spiral nebulae (of small size). They do not seem to be aware of the fact that it would be absolutely impossible for diffused nebulous matter to reach these centres without whirling around them, in the process of descent; and that vortices and cosmical systems are thus the inevitable outcome of ordinary gravitational condensation.

The round-about and arbitrary character of their process must strike every reader with wonder and astonishment; and few will believe that even the authors of this theory seriously entertain it. Though MOULTON gives his exposition of the theory in categorical terms, he in one place hints that it is "open to question at every point" (*Astrophysical Journal*, Vol. XXII, No. 3. October, 1905). Of what earthly use is such a theory?

An impartial study of these writings will strongly suggest the possibility that they endeavored to harmonize the irreconcilable by somewhat doubtful methods. They hoped to awaken popular interest by adapting the old nebular hypothesis to the photographic results of ROBERTS and KEELER, respecting the spiral nebulae; but set about it in a way that aroused distrust in the minds of all competent judges. To the serious student of this subject we recommend, as a parallel course of reading, WHEWELL'S account of the cosmical theories put forth by DESCARTES (*History of the Inductive Sciences*, Vol. II, Book VII, Chap. I, pp. 131–140).

We need make no severer criticism of these writers than WHEWELL has made on the cause of the failure of the physical philosophy of the Greeks; namely, that although they observed diligently and exhausted their powers of ingenuity in classification, *it was arbitrary and their ideas therefore not appropriate to the facts;* and the outcome as regards physical science a failure!

In his account of DESCARTES, WHEWELL says: "He always kept up an active correspondence with his friend MERSENNE, who was called, by some of the Parisians, the 'Resident of DESCARTES at Paris,' and who informed him of all that was done in the world of Science. It is said that he at first sent to MERSENNE an account of a system of the universe which he had devised, which went on the assumption of a vacuum; MERSENNE informed him that the *vacuum* was no longer the fashion at Paris; upon which he proceeded to remodel his system and to re-establish it on the principle of a *plenum*."

This account may do the illustrious DESCARTES an injustice, for WHEWELL is treating of the improvements introduced by the Newtonian philosophy; but it is an accurate description of the usual fate of obsequious persons who aim at popularity rather than truth. Like the obliging teacher of Geography, they are always prepared to expound either the round or the flat theory of the Earth, according to the demand. In such accommodating philosophy the Eternal Truth naturally is the last thing to be thought of, for Her Kingdom is not of this Earth.

§ 59. *The Regularity and Circularity of the Planetary Orbits and Their Large Size Inconsistent with the Theory of Tidal Disruption.*

On this point, after what has been said above, it is sufficient to remark that this phenomenon cannot be accounted for without assigning a nearly circular form to the paths of the planets when they were first formed, or else postulating the secular action of a resisting medium which has gradually reduced their major axes and eccentricities. A resisting medium is the only cause known to work effectively in reducing the eccentricity of an orbit; but as the major axis is also greatly reduced by the same cause, such an hypothesis carries with it the implied premise that the planetary orbits were originally much larger than at present, and this embarrasses the spiral hypothesis still more than before.

If it was already incapable of accounting for the spreading out of matter over such vast spaces, it would seem that we dare not stretch the theory to explain an assumed distribution of matter over a space several times that now occupied by the solar system.

In the translation of LAPLACE's Theory, given in the historical part of this volume, we see that it was shown by the immortal author of the *Mécanique Céleste*, over a century ago, that planets formed of matter carried away from the sun would at each revolution return very near to this star and have a small perihelion distance. If the ejection was due to tidal disruptions arising during the passage of a disturbing body, the case would be slightly different from that considered by LAPLACE, in the celebrated hypothesis of BUFFON, that a comet by striking the sun had carried away a torrent of matter which condensed into the planets; and there might be circumstances under which some of the matter would revolve in orbits with considerable perihelion distances; but most of it assuredly would return in paths passing near the surface of the sun. And as large masses like the major planets could not be accounted for without assuming that their primordial orbits were originally very much larger than they are now, not even a resisting

medium would explain how this matter could be gathered into these large masses moving in almost circular orbits at such great distances. An explanation of the origin of the planets by such a strained and artificial hypothesis is therefore wholly impossible.

Under the circumstances it is obvious that these speculations of CHAMBERLIN and MOULTON are not only invalid, but also highly misleading. And now, as if to make matters still worse, MOULTON has finally suggested (Publication No. 107 of the Carnegie Institution, 1909, p. 160) that matter is disintegrating by the release of enormous sub-atomic energies, under the extremes of temperature and pressure existing in the stars. Another superfluous hypothesis, without the least observational foundation!

§ 60. *The Significance of* BODE'S *Law of Planetary Distances.*

In his Presidential Address on Cosmical Evolution, quoted hereafter at the beginning of the account of the *Modern Theories of Cosmogony*, PROFESSOR SIR G. H. DARWIN attaches considerable significance to BODE'S law, and seems to think that the regions where the planets developed were regions of stability in the solar nebula. Within certain limits there is much to commend in this view, and yet it must not be unreservedly accepted. Let us examine the question a little more closely. BODE'S law is stated thus: Write a series of 4's, and to the second add 3; to the third add 3×2; the fourth 3×4, and so on, doubling the multiplier by 3 each time. Thus we have for the distance of any planet $x = 4 + 2^{n-2}.3$, and the values are:

TABLE EXHIBITING BODE'S LAW AS APPLIED TO THE SOLAR SYSTEM.

Planet	BODE'S Hypothetical Distance $x = 4 + 3.2^{n-2}$		Observed Distance	Error of BODE'S Law
Mercury	4	= 4	3.9	+ 0.1
Venus	4+3	= 7	7.2	− 0.2
The Earth	4+2 .3	= 10	10.0	± 0.0
Mars	$4+2^2.3$	= 16	15.2	+ 0.8
The Asteroids	$4+2^3.3$	= 28	26.5	+ 1.5
Jupiter	$4+2^4.3$	= 52	52.0	± 0.0
Saturn	$4+2^5.3$	=100	95.4	+ 4.6
Uranus	$4+2^6.3$	=196	191.9	+ 4.1
Neptune	$4+2^7.3$	=388	300.6	+87.4

It will be seen by this table that the agreement of the actual distances with those given by BODE'S law is not very close, in several instances; while in two cases the failure is complete — namely, in the case of *Neptune*, where the appear-

ance of the system is perfectly regular, and in the case of the Asteroids, which are spread over the whole interval between *Mars* and *Jupiter*. This last position, indeed, near the greatest planet, might be expected to exhibit some irregularity, but no departure from the formula should be expected at *Neptune*, if the law of BODE is a real physical law, and there it is that the failure is most complete.

It follows, therefore, that the law of BODE is neither general nor exact, but a kind of empirical numerical relation having no geometrical or physical significance.

PROFESSOR NEWCOMB has made the following thoughtful criticism of BODE's Law: "It will be seen that before the discovery of *Neptune* the agreement was so close as to suggest the existence of an actual law of the distances. But the discovery of this planet in 1846 completely disproved the supposed law, and there is now no more reason to believe that the proportions of the solar system are the result of any exact and simple law whatever. It is true that many ingenious people employ themselves from time to time in working out numerical relations between the distances of the planets, their masses, their times of rotation, and so on, and will probably continue to do so; because the number of such relations, which can be made to come somewhere near to exact numbers, is very great. This, however, does not indicate any law of nature. If we take forty or fifty numbers of any kind, say the years in which a few persons were born, their ages in years, months and days at some particular event in their lives; the numbers of the houses in which they live, and so on, we shall find as many curious relations among the numbers as have ever been found among those of the planetary system. Indeed such relations among the years of the lives of great actors in the world's history will be remembered by many readers as occurring now and then in the public journals" (*Popular Astronomy*, Edition of 1878, pp. 237–238).

Another effective criticism of BODE's law is based upon the failure in the case of the satellite systems. Since the publication of the author's recent paper in the *Astronomische Nachrichten*, No. 4308, probably no one has thought of the satellites as detached from the planets about which they revolve, or of the planets as detached from the sun; yet on the other hand if BODE's law holds for planets which have been captured, it might be expected to be repeated in some analogous form among the satellites of the great planets. Such, however, is not the case, because the distances of the remote satellites of *Jupiter* and *Saturn* break suddenly and unaccountably with the intervals holding among the satellites in the inner parts of these systems. Moreover, the motion of the outer satellite in each case is retrograde. Thus any analogy with BODE's law, or with any similar law whatever, entirely disappears; and all this antiquated speculation falls to the ground. If such relations of distance should be found in the systems of *Mars* or *Uranus*,

but not in those of *Jupiter* and *Saturn*, the proof will be all the more complete that some apparently harmonious relations are simply the work of chance. For all these bodies were captured and have had their distances adjusted under the action of a Resisting Medium.

It is needless to add that we concur in PROFESSOR NEWCOMB's view that the law of BODE as applied to the planets is mainly a chance outcome, and not a real law of nature. And yet there is this much truth in DARWIN's view that BODE's law represents a certain periodicity corresponding to regions of stability in the solar nebula: as the spiral nebula which formed our system developed, a planet would form or survive in any wide space of our system not already occupied by such a body. But no two planets would form very close together, because under the secular action of the resisting medium, the mean distances have been greatly diminished and the eccentricities decreased, and any two bodies originally developing near the same mean distance would sooner or later have united into one mass.

The planets have thus swept up the rubbish in the regions on either side of where they move; but the clearing up of cosmical dust is restricted to zones of limited width, and where the space is large other planets have been formed, or rather have *survived*, in the general clearing up of the system. Accordingly BODE's law is not a true law of nature, as that term is usually employed, because it rests on no physical basis; yet in any spiral nebula some such periodic relationship is likely to develop, the exact nature of it depending on the initial conditions of the nebula.

Nearly all the matter of the primordial nebula of the planetary system went into the sun, and hence the large planets are confined to the outskirts of the system; while the group of small bodies composing the terrestrial planets are the few surviving remnants of the inner parts of our nebula, all the rest having been swallowed up in the great central mass, which has 746 times as much matter as all the other bodies combined, and thus completely dominates the system. In the formation of the satellites also nearly all the matter has gone into the planets. In his address DARWIN points out how it will happen in the course of the immeasurable ages required for the growth of the planets and satellites that the central body will capture most of the cosmical dust which circulates among the bodies as remnants of the primordial nebula.

§ 61. *Significance of the Spiral Forms Observed Among the Nebulae.*

In view of what has been shown in the preceding sections we can now see that the matter expelled from the fixed stars constituting the stellar universe is wafted about in all manner of streams and finally again collects together under

the power of its own gravitation. Hence, after the electric charges are dissipated and the matter again gathers into masses and streams, it condenses to form stars. The streams drift about till they encounter other streams with which they unite to form spiral nebulae and these develop into planetary systems and double and multiple stars and clusters.

(1). Since the known attractive forces will not give the form of spiral observed, which is nearest the equiangular or perhaps Archimedean spiral (implying forces varying as $\frac{h^2}{r^3}$ and $\frac{h^2}{r^3}\left(\alpha + \frac{\beta}{r^2}\right)$, it follows that the spires of a nebula do not represent paths pursued by particles under central forces, but chance motion of matter drifting under gravity and proper motion and settling towards a center.

(2). And in this coiling up of the nebula the matter has not reached a condition of approximate equilibrium, but is slowly settling to this state. Hence, the spirals are not separating or breaking up, but the whole mass is condensing under friction and resistance. When the tendency to the center becomes checked by the internal resistance of the mass or the paths of the particles become circular, under the action of the resisting medium, the rotation as a whole will become about steady. In some of the nebulae it is observed that the coils of the spirals are nearly circular. Such cases, we may infer, afford examples where the two opposite branches of the spirals must already have been revolving a very long time, or else where they started on very circular paths in the beginning.

(3). At the great distance of the nebulae we see chiefly very expanded nebulae, and hence not the condensed stage of fluid fission, if that ever takes place, but chiefly that of the earliest settlement towards equilibrium.

(4). The spiral nebulae appear to be quite transparent, and they must therefore in general be at low temperature. If they were heated they would soon cool off. Hence planetary bodies formed in condensing spirals are initially at low temperature, and only become highly heated as the central condensation becomes of considerable size.

§ 62. *All the Theories of Spiral Nebulae Heretofore Advanced Disproved, Except Those of* DEMOCRITUS *and the Greeks, who First Conceived the Development of Vortices.*

In view of the above considerations one cannot read the highly artificial theories recently promulgated for explaining the spiral nebulae without being impressed with their absolute inferiority compared even to those formulated 2300 years ago by the Greeks, who first conceived the idea of atoms falling in space with unequal velocities and thus developing vortices. DEMOCRITUS is the principal

author of this theory among the ancients, and it has never been quite lost sight of by the moderns. Of late years, however, it is truly remarkable what fantastic and improbable theories have been current!

It is needless to say that not one of these artificial theories, such as those based on tidal disruption, close approach, and collision of stars, has the slightest foundation. Even NEWTON, when composing the *Principia,* over two centuries ago, remarked in the General Scholium that the Deity had placed the stars at immense distances apart, so that they would not fall upon one another by their mutual gravitation. As modern research shows that close approaches and collisions are practically impossible, or at most would occur with the utmost rarity, it is difficult to account for the circulation of such doctrines by writers of recognized standing. Our age, however, is one of great multiplicity of publications, and consequently of confused, disordered, and superficial thought; and therefore one need not be surprised or disappointed at the output of an abundance of error; yet for that very reason it is all the more necessary to preserve a clearly marked line of distinction in the public mind between the speculations which are worthless and those which are of some value. All speculations which lead to truth are of temporary or permanent value, but those which lead to false conceptions and serve simply to propagate error are pernicious and wholly detrimental to the progress of science.

CHAPTER VI.

On the Movement and Close Approach of Separate Stars.

§ 63. *On the Various Conceivable Ways in which Stellar Systems Might Have Originated.*

If we were asked to name all the conceivable ways in which stellar systems might have originated, we should have to consider the following processes:

(1). The formation of nuclei in a spiral nebula coiling up and whirling about a center under conditions which are essentially devoid of hydrostatic pressure; or of true figures of equilibrium under the pressure and attraction of their parts. The former process appears to be the more general and is illustrated by the evolution of the solar system. The latter corresponds to the rupture of figures of equilibrium as determined mathematically by Poincaré and Darwin, and was first applied to the double stars by the present writer in 1892.

(2). The mutual approach of separate stars which might become physically connected after they were already developed, owing to the action of disturbing forces. If they became entangled in such a way that their connection was permanent, we should have pairs of coupled stars not very unlike those now observed in the heavens.

(3). The extremely close approach of separate stars by which tidal disruptions would become so powerful that systems might be formed out of the material ejected, which is an hypothesis recently promulgated by Chamberlin and Moulton.

(4). The entanglement of stars of independent origin through resistance in collision, which would give rise to the dispersion of some of their matter into a nebula, and might result in the coupling of separate bodies into a system founded on the ruin wrought by the collision. This last hypothesis was put forth by Professor A. W. Bickerton of New Zealand, in 1879,* was viewed as a possibility by Lord Kelvin in 1887; it has since been favorably considered by Dr. Isaac Roberts, in connection with Nebulae and Clusters, and by Arrhenius in his new

* In a paper on Cosmic Evolution, *Philosophical Magazine*, August, 1900, p. 217, Professor Bickerton points out that as far back as 1869 Dr. Johnstone Stoney discussed Grazing Collisions before the Royal Society, and suggested such an origin for new stars and double stars.

work entitled *Das Werden Der Welten*, translated into English under the title, *Worlds in the Making* (Harper & Brothers, 1908).

We shall now examine these several theories with some care, and see if it is possible to arrive at criteria by which all but one of these four possibilities may be excluded. As these processes include all the conceivable modes in which systems may arise, we may thus hope to arrive at the true law of nature. It will be convenient to begin with LORD KELVIN'S account of a collision, and afterwards to consider these several possible explanations in the reverse order to that in which they are stated.

§ 64. *Lord Kelvin's Description of the Effects of a Collision.*

In his *Popular Lectures and Addresses* (Vol. I, pp. 413–417), LORD KELVIN gives the following interesting discussion of the collision of two stars. He seems to think such accidents happen occasionally, and therefore it is well to consider such a possibility as that pictured by this great master of physical science:

"To fix the ideas, think of two cool solid globes, each of the same mean density as the earth and of half the sun's diameter, given at rest, or nearly at rest, at a distance asunder equal to twice the earth's distance from the sun. They will fall together and collide in exactly half a year. The collision will last for about half an hour, in the course of which they will be transformed into a violently agitated incandescent fluid mass flying outward from the line of the motion before the collision and swelling to a bulk several times greater than the sum of the original bulks of the two globes.* How far the fluid mass will fly out all around from the line of collision it is impossible to say. The motion is too complicated to be fully investigated by any known mathematical method; but with sufficient patience a mathematician might be able to calculate it with some fair approximation to the truth. The distance reached by the extreme circular fringe of the fluid mass would probably be much less than the distance fallen by each globe before the collision, because the translational motion of the molecules constituting the heat into which the whole energy of the original fall of the globes becomes transformed in the first collision, takes probably about three-fifths of the whole amount

* "Such incidents seem to happen occasionally in the universe. LAPLACE says: 'Some stars have suddenly appeared, and then disappeared after having shown for several months with the most brilliant splendor. Such was the star observed by TYCHO BRAHE in the year 1572, in the constellation *Cassiopeia*. In a short time it surpassed the most brilliant stars and even *Jupiter* itself. Its light then waned away, and finally disappeared sixteen months after its discovery. Its colour underwent several changes; it was at first of a brilliant white, then of a reddish yellow, and finally of a lead-coloured white, like to *Saturn*.'" (HARTE'S *Translation of* LAPLACE'S *System of the World*, Dublin, 1830).

of that energy. The time of flying out would probably be less than half a year when the fluid mass must begin to fall in again towards the axis. In something less than a year after the first collision the fluid will again be in a state of maximum crowding towards the centre, and this time even more violently agitated than it was immediately after the first collision; and it will again fly outward, but this time axially towards the places whence the two globes fell. It will again fall inwards and after a rapidly subsiding series of quicker and quicker oscillations it will subside, probably in the course of two or three years, into a globular star of about the same mass, heat and brightness as our present sun, but differing from him in this, that it will have no rotation.

"We suppose the two globes to have been at rest when they were let fall from the mutual distance equal to the diameter of the earth's orbit. Suppose, now, that instead of having been at rest they had been moving transversely in opposite directions with a relative velocity of two (more exactly 1.89) metres per second. The moment of momentum of these motions round an axis through the centre of gravity of the two globes perpendicular to their lines of motion, is just equal to the moment of momentum of the sun's rotation round his axis. It is an elementary and easily proved law of dynamics that no mutual action between parts of a group of bodies, or of a single body, rigid, flexible or fluid, can alter the moment of momentum of the whole. The transverse velocity in the case we are now supposing is so small that none of the main features of the collision and of the wild oscillations following it, which we have been considering, or of the magnitude, heat and brightness of the resulting star, will be sensibly altered; but now, instead of being rotationless, it will be revolving once round in twenty-five days and so will be in all respects like to our sun.

"If, instead of being at rest initially, or moving with the small transverse velocities we have been considering, each globe had a transverse velocity of three-quarters (or anything more than .71) of a kilometre per second, they would just escape collision, and would revolve in ellipses round their common centre of inertia in a period of one year, just grazing each other's surface every time they came to the nearest points of their orbits.

"If the initial transverse velocity of each globe be less than, but not much less than .71 of a kilometre per second, there will be a violent grazing collision, and two bright suns, solid bodies bathed in flaming fluid, will come into existence in the course of a few hours and will commence revolving round their common centre of inertia in long elliptic orbits in a period of little less than a year. Tidal interaction between them will diminish the eccentricities of their orbits, and if continued long enough will cause the two to revolve in circular orbits round their

centre of inertia with a distance between their surfaces equal to 6.44 diameters of each" (Lecture on the Sun's Heat, pp. 413–417).

This may be regarded as the most accurate of known pictures of what would be the effect of a collision if it occurred. Whether such events really occur between compact stars separated by great distances is another question which we shall presently discuss. *The passage of stars through nebulae is no doubt an occasional event of some importance, and most of our new stars which suddenly blaze forth with so much splendor may be traced to some such cause.* The nebulae, however, are clouds of cosmical dust often themselves thousands of times larger than our whole solar system. A collision with such an enormous cloud of dust is not at all analogous to collisions between well developed stars, which are all compact globes of very small size. Hence our new stars due to collisions with nebulae* are temporary, not permanent, as would be the case if two stellar globes came into collision. The fact that no permanent star has blazed forth within the historical period since the age of the Greeks therefore shows us that collisions of stars are certainly extremely rare events, if, indeed, they occur at all in intervals less than something like a million years.

While these views seem to be amply justified, on solid mathematical grounds, it is proper to say that they depart from those held by Lord Kelvin, not only in his mature but also in his advanced years. In his well known paper "On the Clustering Power of Gravitational Matter in any Part of the Universe," read to the British Association in 1901, published in the *Philosophical Magazine* for August and reprinted in the *Observatory* for November, 1901, Lord Kelvin gives the following additional discussion of the problems now under consideration:

"Newcomb has given a most interesting speculation regarding the very great velocity of *Groombridge* 1830, which he concludes as follows: 'If, then, the star in question belongs to our stellar system, the masses or extent of that system must be many times greater than telescopic observation and astronomical research indicate. We may place the dilemma in a concise form, as follows:

" 'Either the bodies which compose our universe are vastly more massive and numerous than telescopic examination seems to indicate, or *Groombridge* 1830, is a runaway star, flying on a boundless course through infinite space with such momentum that the attraction of all the bodies of the universe can never stop it.'

"In all these views the chance of passing another star at some small distance such as one or two or three times the Sun's radius has been overlooked; and that this chance is not excessively rare seems proved by the multitude of *Novas* (collisions and their sequels) known in astronomical history. Suppose, for example,

* The fall of a large comet upon a dark or feebly luminous star might also produce a nova.

Groombridge 1830, moving at 370 kilometres per second, to chase a star of twenty-times the sun's mass, moving nearly in the same direction with a velocity of 50 kilometres per second, and to overtake it and pass it as nearly as may be without collision. Its own direction would be nearly reversed and its velocity would be diminished by nearly 100 kilometres per second. By two or three such casualties the greater part of its kinetic energy might be given to much larger bodies previously moving with velocities of less than 100 kilometres per second. By supposing reversed the motions of this ideal history, we see that *Groombridge* 1830 may have had a velocity of less than 100 kilometres per second at some remote past time, and may have had its present great velocity produced by several cases of near approach to other bodies of much larger mass than its own, previously moving in directions nearly opposite to its own, and with velocities of less than 100 kilometres per second. Still it seems to me quite possible that NEWCOMB's brilliant suggestion may be true, and that 1830 *Groombridge* is a roving star which has entered our galaxy, and is destined to travel through it in the course of perhaps two or three million years, and to pass away into space never to return to us" (*Observatory*, November, 1901, pp. 410–411).

From this discussion it will be seen that LORD KELVIN's views underwent no change in his advanced years, and that he always greatly overrated the probability of collisions. POINCARÉ has been more penetrating in his study of this problem, and has given us a most useful criterion by which we may judge the average tendencies arising in the proper motions of the bodies of the firmament.

§ 65. *The Collision Theory Advanced by* BICKERTON *and Since Entertained by Others.*

This theory has been before astronomers for more than a quarter of a century, but apparently was regarded as too improbable for serious consideration till it was given credit by LORD KELVIN's discussion. It has been adopted by DR. ISAAC ROBERTS, in a different form, and given a certain amount of recognition by ARRHENIUS, in his new and suggestive work, *Worlds in the Making*. It has also been entertained by others, and has its defenders at the present time. As was the case with the late LORD KELVIN, so also with PROFESSOR ARRHENIUS; he is an eminent physicist, rather than an astronomer; and some of the difficulties which occur to the astronomer and mathematician did not occur to him. The eminent Swedish physicist was misled into adopting some of the theories of CHAMBERLIN and MOULTON, which seem to be quite devoid of foundation. For these reasons it is necessary to discuss certain theories which otherwise could be passed over without comment. The accompanying diagrams are from the writings of BICKERTON and ARRHENIUS respectively:

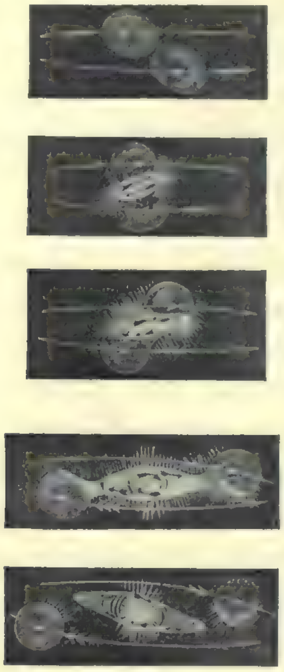

FIG. 14. THE FIRST FIGURE AT THE TOP, REPRESENTS BICKERTON'S THEORY OF A PAIR OF STARS DISTORTED AND COMING INTO IMPACT; THE SECOND FIGURE, A PAIR OF STARS IN IMPACT; THE THIRD FIGURE, STARS PASSING OUT OF IMPACT, AND FORMATION OF THIRD BODY; THE FOURTH FIGURE, SHOWING ENTANGLEMENT OF MATTER IN EACH BODY; THE FIFTH FIGURE, TWO VARIABLES AND A TEMPORARY STAR.

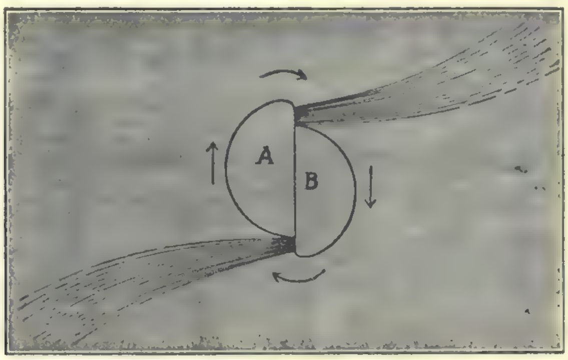

FIG. 15 THE EFFECTS OF A COLLISION, ACCORDING TO ARRHENIUS.

It does not seem necessary to dwell on these diagrams beyond remarking that they might be more or less satisfactory, *if such collisions really occurred; but as it seems certain that such chance impacts seldom or never take place, owing to the immense mutual distances of the stars, and the small size of their globes, they represent events too purely hypothetical to be of any interest.*

To enable one to see this somewhat more clearly, we may remark that the

average space between the stars is a distance of the same order of magnitude as that which separates us from *α Centauri*. This is 275,000 radii of the earth's orbit, which itself is about 219 times the sun's radius. Accordingly, it follows that the solar radius is only 1:60,225,000 of that of the sphere representing the average distance of the fixed stars, here taken to be equal to that of *α Centauri*. At *α Centauri* the sun's angular diameter is only 0″.00387, and about 22 trillion such suns would be required to cover the celestial sphere. If then the spaces separating the fixed stars are to their radii as 60 million to unity, we see that when projected at random with any constant velocity of considerable magnitude,* they cannot come into collision unless the motion is so accurately aimed as to make their radii overlap, the chance of which could not be more than 1 in 22,000,000,000,000. This probability is certainly small enough to be neglected.†

Accordingly while it is possible for a very few collisions to occur in the heavens, in the course of infinite time, it is evident that they are events far too rare to leave any visible impress on the observed aspects of the physical universe. For with the great extent of the Milky Way and the smallness of the proper motions, the body of the stars never come near one another at all; and it is doubtful if, in the ordinary course of nature, one collision between actual stars occurs in a million years, even if we adopt POINCARÉ's estimate of 1000 million suns for the total collection of bodies composing the sidereal universe.

The argument adduced by POINCARÉ, in his address on the Milky Way and the Theory of Gases, which was inspired by LORD KELVIN's paper above quoted, is another way of reaching a similar result. Here POINCARÉ estimates in effect that a star might traverse the Milky Way some 16,000 times without coming nearer another body than the distance of *Neptune* from the sun. This conclusion of POINCARÉ is in full accord with the evidence of the telescope which discloses to the practical astronomer the fact that only an excessively small fraction of the background of the sky is covered with stars, so that there is always and everywhere the most ample space between them. As much enlarged as the luminous stars appear to be, owing to the spurious discs due to the effect of the telescope, it would by chance hardly be possible to aim a projectile so as to come near any of them; and since the real discs are always much below the limit of telescopic vision,

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* Such that the hyperbolic path pursued has no sensible curvature and may be regarded as almost rectilinear.

† If there are 1000 million stars like our Sun and all at the *distance of the nearest*, this calculation shows that only 1 : 22000th of the surface of the celestial sphere would be covered by them. At the actual distance of the stars the space covered by them would be vastly less. If the Sun filled the orbit of *Neptune* the space occupied would be increased by the factor $(6570)^2$; and if the distance be multiplied by 5000, the total result would be $\frac{1}{22000}\left(\frac{6570}{5000}\right)^2 = \frac{1.623}{22000} = \frac{1}{13556}$; which agrees well with POINCARÉ's result given below, and more fully considered in the Chapter on Clusters.

the difficulty of aiming a star so as to produce a collision is vastly greater yet. In fact it is much greater than we can well conceive. We may therefore dismiss the collision theory as so wholly devoid of foundation as to require no further consideration.

§ 66. *On the Close Approach of Separate Stars, and on the Theory that Systems are Formed by Tidal Disruption.*

After what has been said above and more fully set forth in § 55 of Chapter V, it is not necessary to dwell at any great length on the untenability of this hypothesis. We have shown that to be effective the approaches would have to be extremely close, and even then the disruptions would be of small importance and could not give rise to the ejection of prominences large enough to form cosmical systems at all comparable to our solar system in size.

POINCARÉ estimates that if spheres as large as the orbit of *Neptune* be drawn about each of the 1000 million stars assumed to compose the sidereal universe, the apparent diameter of the corresponding circles at average distance of the fixed stars would be about 0″.1, and they would cover only a sixteen-thousandth part of the surface of the celestial sphere. Hence a star started at random with a considerable velocity, might on the average traverse the Milky Way 16,000 times without coming nearer to any star than the distance of *Neptune* from the sun. Now it is true that some regions of the heavens are more crowded with stars than the average, while others are much less so; but, from this estimate of POINCARÉ, it is evident that close approaches of stars would be the rarest of celestial phenomena.

An experienced mathematician who has given some attention to this problem has justly remarked that one or two such close approaches might occur in the universe in the course of a very long time, but not more. It is difficult to imagine how the untenability of this theory could be more impressively shown than by such illustrations as these.

§ 67. *On the Mutual Approach and Possible Physical Entanglement of Separate Stars to Form Cosmical Systems.*

From the above considerations we see how little probability there is of the close approaches of separate stars forming systems by collisions or tidal disruptions, and it only remains to consider the possibility of the physical entanglement of stars passing near one another, and thus giving rise to the permanent connection of separate masses into stellar systems.

It is clear that owing to the great distances which separate them, two stars will almost never pass near each other, and if they should the only known cause by which they could become permanently entangled and united into a physical system is a resisting medium. This latter, of course, is a real cause, and in the form of nebulous matter, or clouds of cosmical dust, is widely diffused throughout the universe. Around some of the stars there might be diffused a considerable amount of cosmical dust, and this diffused nebulosity might so retard the velocity of a star in passing by that the two would become permanently entangled. In some few cases their relative velocities might be so far reduced that they could not afterwards escape the power of their mutual gravitation, and would continue to revolve in very long periods, and in orbits of great eccentricity. It is undeniable that this might occasionally happen, but it is not likely to happen oftener than once in about 1000 close approaches; and since even a single close approach is an exceedingly rare event, it follows that the physical entanglement of passing stars moving with considerable velocities is almost an impossibility. Yet in case of such unexpected capture the resulting orbit would be of vast dimensions, and the eccentricity but little below that of a parabola, say about 0.995.

It is true that in time the resisting medium, if it continued to act, might reduce the major axis and periodic time, and also diminish the eccentricity; but the process of this transformation would be excessively slow, because the resisting medium is rare and would offer but little hindrance to a body having the great orbital momentum of a star like our sun.

Accordingly as the event of close approach itself would be a rare phenomenon, the encountering of sensible resistance much rarer still, and the transformation of the orbit to an ellipse with eccentricity sensibly below 0.99 very slow and difficult, because the original period would be excessively long and the major axis large; it follows that, although a system could just possibly be produced in this way, yet the probability of its actual occurrence is so slight that we may dismiss the hypothesis as requiring no further consideration.

It is not by such extremely improbable and highly artificial processes that the sidereal universe has been strewn with stellar systems about one-fifth as numerous as the stars themselves, and in many cases made up not only of two stars but also of three or four bodies, moving in a common direction with orbits but slightly inclined, and thus constituting multiple stars of imposing grandeur. These stellar systems have arisen from the development of nebulae, which usually take the spiral form and coil up in such a way as give a fundamental plane of motion to each system.

After this general survey of the last three of the four possible ways in which the stellar systems might have originated (indicated by the first section of this chapter), we seem finally to be reduced, by exclusion, to the first process as the true law of nature. We shall presently examine that process with some care. But meanwhile it is advisable to consider the theory of the approach of separate bodies from the point of view adopted by LAPLACE in his theory of the comets.

§ 68. LAPLACE'S *Theory of the Projection of Comets into the Sphere of the Sun's Attraction.*

In the part of this work which deals with the solar system, and also in the translation of the Nebular Hypothesis, as formulated in the *Systéme Du Monde*, 1796, allusion is made to LAPLACE'S Theory that the comets are foreign to our solar system. Though now believed to be incorrect, this theory is important in itself, and because of the light it throws on the problem of the approach of separate stars, and we shall therefore give it in full. Most of this theory was included in an article *Sur Les Cométes*, published by LAPLACE in the *Connaissance Des Temps* for 1816. It has been analyzed and presented in a slightly different form by TODHUNTER in his *History of the Theory of Probability from the Time of* PASCAL *to that of* LAPLACE (pp. 491–494).

LAPLACE remarks that out of the hundred comets which had been observed up to this time not one had been ascertained to move in a hyperbola, and he proposes to show by the theory of probability that this outcome might be expected, the chances being very great that a comet would move either in an ellipse or in a parabola, or in an hyperbola of such large transverse axis that it is not easily distinguished from a parabola in the part of the orbit covered by observations.

Suppose r to be the radius of the sphere of the sun's activity, which may be 100,000 times the radius of the earth's orbit, and denote by V the velocity of the comet at the time when it enters the sphere of the sun's attraction, so that r is the comet's radius vector at that instant. Let a be the semi-axis major of the orbit which the comet is to describe, e the eccentricity, D the perihelion distance, ϖ the angle which the tangent to the orbit makes with the radius vector r drawn to the sun. The velocity V is in the tangent to the orbit, and when the sun's mass is taken as unity and the sun's mean distance as the unit of distance, we have the following well known formulae:

$$\left.\begin{aligned} V^2 &= \frac{2}{r} - \frac{1}{a}, \\ rV\sin\varpi &= \sqrt{a(1-e^2)}\,, \\ D &= a(1-e)\,. \end{aligned}\right\} \qquad (145)$$

By eliminating a and e, from these equations, we get

$$\sin^2 \varpi = \frac{2D - \frac{2D^2}{r} + D^2V^2}{r^2V^2}; \tag{146}$$

and we easily find from this an expression for $1 - \cos \varpi$ as follows:

$$1 - \cos \varpi = 1 - \frac{\sqrt{1 - \frac{D}{r}}}{rV} \sqrt{r^2V^2\left(1 + \frac{D}{r}\right) - 2D}. \tag{147}$$

When all directions which tend inward are taken to be equally probable the chance that the direction will lie between 0 and ϖ is evidently $1 - \cos \varpi$; and the corresponding values of the perihelion distance is 0 and D. LAPLACE supposes all values of D equally probable and determined the probability of the perihelion distance lying between 0 and D. TODHUNTER criticizes his procedure, but reaches the same result, by a slightly different process, as follows:

$$\text{Let} \quad \psi(V) = 1 - \frac{\sqrt{1 - \frac{D}{r}}}{rV} \sqrt{r^2V^2\left(1 + \frac{D}{r}\right) - 2D}. \tag{148}$$

Then when all directions of projection with respect to the radius vector drawn to the sun are considered equally probable, a comet starting with a velocity V has the chance $\psi(V) = 1 - \cos \varpi$ that its perihelion distance will lie between 0 and D.

Suppose we assume as a fact that the perihelion distance does lie between 0 and D, but that we do not know the velocity of projection. It is then required to find the probability that the initial velocity lies between assigned limits, which is a problem in inverse probability. The solution is:

$$P = \frac{\int_{V_1}^{V_2} \psi(V)\, dV}{\int_{V'}^{V''} \psi(V)\, dV}, \tag{149}$$

where the limits V_1 and V_2 are the assigned limits, and V' and V'' are the extreme admissible values of V, corresponding to the lower and upper limits respectively.

In order to find the value of the numerator of (149) LAPLACE puts

$$\sqrt{r^2V^2\left(1+\frac{D}{r}\right)-2D}=rD\sqrt{1+\frac{D}{r}}-z\,, \tag{150}$$

and for the assigned limits of V he takes

$$V_1=\frac{\sqrt{2D}}{r\sqrt{1+\frac{D}{r}}}\,,\quad\text{and}\quad V_2=\frac{i}{\sqrt{r}}\,. \tag{151}$$

Integrating between these limits the expression for $\psi(V)$ in (148) he finds approximately

$$\int_{r\sqrt{1+\frac{D}{r}}}^{\frac{i}{\sqrt{r}}}\left\{1-\frac{\sqrt{1-\frac{D}{r}}}{rV}\sqrt{r^2V^2\left(1+\frac{D}{r}\right)-2D}\right\}dV=\frac{(\pi-2)\sqrt{2D}}{2r}-\frac{D}{ir\sqrt{r}}\,, \tag{152}$$

the terms depending on higher powers of r in the denominator being rejected as insensible, because r is very large. In evaluating $\int_{V'}^{V''}\Psi(V)\,dV$ it is clear that the upper limit of V may be taken to be infinity, so that i will be infinite, and the required chance becomes

$$\left\{\frac{(\pi-2)\sqrt{2D}}{2r}-\frac{D}{ir\sqrt{r}}\right\}\div\frac{(\pi-2)\sqrt{2D}}{2r}=1-\frac{\sqrt{2D}}{i\,(\pi-2)\sqrt{r}}\,. \tag{153}$$

If $i^2=2$, the velocity would be the highest which would permit the orbit to be an ellipse. In the equation $V^2=\frac{2}{r}-\frac{1}{a}$, we may put $a=-100$, and then

$$V^2=\frac{r+200}{100r}\;;\quad\text{thus}\quad i^2=\frac{r+200}{100}\,. \tag{154}$$

And when we use this value of i, we find the chance that the orbit shall be either an ellipse or a parabola or an hyperbola with transverse axis greater than 100 times the radius of the earth's orbit; the chance that the orbit will be an hyperbola with a smaller transverse orbit will be $p=\frac{\sqrt{2D}}{i(\pi-2)\sqrt{r}}$, which agrees with LAPLACE's result. When LAPLACE supposes D equal to 2, r equal to 100,000 and $i^2=\frac{r+200}{100}$, he finds the probability of the event to be

$$p' = \frac{1}{5714}. \tag{155}$$

LAPLACE remarks that although in his analysis he had supposed that all values of D between 0 and 2 were equally probable, yet observation shows that the comets with perihelion distance less than 1 greatly predominate over those with larger perihelion distance, and he proceeds to calculate the effect of this modification of his hypothesis.

This in brief is LAPLACE's theory of the projection of comets into the sphere of the sun's attraction. It is not admitted to-day as the true origin of comets, because most of these bodies are shown to be original members of the solar system, but it is a very instructive example of the treatment of certain problems, and has a close connection with the theory of the approach of separate stars.

§ 69. *In Passing Each Other Two Separate Stars Would Usually Describe Hyperbolas.*

This proposition requires very little treatment. For it is well known that the velocity of a body at any point of an ellipse about the sun is equal to that which would be acquired in falling from the corresponding curve of zero velocity, or the circumference of a circle with radius $2a$, equal to the major axis of the ellipse. This is known as WHEWELL's theorem, after DR. WHEWELL of Cambridge, who first discovered it. Thus it is clear that at great distances the velocity is always small, when the orbit is an ellipse. This also follows from the formula: $V^2 = k^2\left(\frac{2}{r} - \frac{1}{a}\right)$, when a is very large, and r nearly equal to $2a$. At great distances the velocity has to be very small, otherwise the orbit cannot be an ellipse. The parabola is simply the limiting curve when the major axis of the ellipse becomes infinite. Consequently all relative velocities above a very small one will give hyperbolas, and the transverse axes will rapidly increase with the increase of the velocity.

Now it happens that many stars in their proper motions have velocities comparable with that of the earth in its orbit, and very small relative velocities are unusual. Thus if two stars approach near each other in virtue of difference of proper motion, we may be sure they will usually move in hyperbolas, and that the paths will have but very slight curvature.

This is another way of appreciating the difficulty of separate stars being captured by close approach. They could not be captured, except by the rarest combination of circumstances, and it is also certain that a close approach would be the most extraordinary of astronomical events. Practically such close approaches would occur, if at all, only in clusters, where the stars are already dense, and in rapid motion under the attraction of the whole mass of stars.

§ 70. *The Only Other Conceivable Way in Which the Stellar Systems Could Originate —namely, by the Formation of Nuclei in Spiral Nebulae Free from Hydrostatic Pressure, or by the Rupture of Nebulous Masses Rotating in Equilibrium Under the Pressure and Attraction of Their Parts.*

Having considered in the preceding sections three of the four possible methods by which the stellar systems might have originated, and having found all but one of these to be inadmissible, it only remains to examine the first of these methods, to ascertain whether the stellar systems could have originated in this way. That is by the formation of nuclei in a spiral nebula, or by rupture of the figure of equilibrium assumed by a nebulous mass in rotation under the pressure and attraction of its parts.

The first part of this problem is verified by observation and will be carefully considered hereafter; the latter problem is an old one in physical science, and has been carefully investigated by many eminent mathematicians. We shall here consider only the work of DARWIN and POINCARÉ who agree in the conclusion that when a homogeneous mass of incompressible fluid undergoes such acceleration of rotation in contraction that it becomes unstable the mass divides into two fairly equal parts. In fact the process of division is shown to be such that very unequal division is impossible. Thus double and multiple stars might be formed in this way by the breaking down of figures of equilibrium, but not planets and satellites, because in the solar system the attendant bodies are always too small relatively to the large central masses which govern their motions. Such small bodies are formed by the survival of nuclei in spiral nebulae, and are thus all captured or added from without, but have never been detached, because there is little or no hydrostatic pressure.

Now what do we find to be the fact among the double stars? The brightnesses are often nearly equal, and so far as known the masses are always of the same order; for example, no double star at present known has a companion so small as one-tenth of its own mass. Hence we may safely infer that as a general rule the masses are about as nearly equal as the brightnesses of the components. Accordingly the equal and comparable masses observed and inferred generally to exist among the double stars seem to conform closely to the mathematical researches of POINCARÉ and DARWIN on the breaking up of the figures of equilibrium of rotating masses of fluid. This was first pointed out by the writer in 1892, and has been verified by later researches in the various branches of stellar astronomy, especially in work on certain variable stars and in researches on spectroscopic and visual binaries, but the fission is mainly dynamical, not hydrostatic.

The orbits of double stars are not eccentric enough to indicate that these objects have been captured after they were fully developed, unless meanwhile the eccentricity has been reduced by the secular action of a resisting medium. These systems may have originated in one or both of two possible ways, and in no others whatsoever: (1) They may have been formed by the rupture of true figures of equilibrium of rotating masses of fluid, as I long ago inferred from the researches of Poincaré and Darwin. In this event the primordial orbits would have small eccentricities, and in time both the major axis and the eccentricity would be increased by the secular effects of tidal friction; (2) They may have arisen from the survival of nuclei like those seen in the coils of spiral nebulae. In this case the primordial orbits would have been of rather large eccentricity, and would have been decreased in size and rounded up under the secular action of the resisting medium. Thus in one hypothesis the primordial orbits were small and round, and have been gradually enlarged and elongated by the secular effects of tidal friction; in the other, the primordial orbits were originally large and decidedly elongated, and have since been reduced in size and rounded up under the secular effects of a resisting meidum. It is observed that the orbits have almost every degree of eccentricity, except that appropriate to comets — say from about 0.95 to 1.00 — while on the other hand very round orbits are common for close systems, such as the spectroscopic binaries. The fact that the larger orbits have the larger eccentricities does not tell decisively in favor of either theory. This effect might be produced either by tidal friction or by a resisting medium, or by a combination of both causes.

It is unfortunate that the effects of the two causes cannot yet be clearly separated. Both are real causes, and as one is exactly the opposite of the other, there is no entirely certain method of separating them. But we believe that a criterion for separating the two effects may hereafter be developed. For example, it is shown in the theory of the origin of the solar system that the nebulous mass was never in equilibrium under the pressure and attraction of its parts, and that the bodies were not detached but captured, and this circumstance has made the attendant bodies so small. This mode of formation was repeated in our system so many times, always with the same uniform result, that there can be no possible doubt that the absence of hydrostatic pressure in a nebula usually leads to the development of small attendant bodies. It is found by calculation that the probability of small bodies is more than a decillion decillion to unity. This is as all of the grains of fine sand contained in a sphere of the fixed stars, with radius equal to 200,000 radii of the earth's orbit, to one.

Now if such a criterion based on probability is valid, it will follow that such

relatively large masses as the double stars do not arise from the preservation of the small isolated nuclei in a whirlpool nebula devoid of hydrostatic pressure, but must result from the capture of many of these small masses by the larger nuclei. The dynamical process by which this capture takes place is more fully discussed in Chapter X, and need not be dwelt on here. As the nebulae in different cases have all degrees of hydrostatic pressure, from perfect fluidity on the one hand to the entire absence of such pressure on the other, it follows that the two processes of cosmical development, one depending on capture, by a dynamical process, under the action of a resisting medium, the other on the exertion of more or less hydrostatic pressure from the center, must often pass by insensible degrees one into the other; so that in the actual universe both processes are at work together. This is decidedly the most probable result, when all influences are taken into account; and more than this cannot be safely inferred at present.

In the case of immense nebulae the result is similar to that of many separate nebulae brought near together — the formation of star clusters. For owing to the great tenuity of such a gigantic cloud of cosmical dust, it necessarily cannot act as a whole, but each part must develop under its own forces. This explains clusters and groups of stars, in a simple and satisfactory manner. The occurrence of large bodies in such groups is due to the breaking up of the whole system into lesser units, without any true hydrostatic pressure, owing to the immense extent of the whole cloud. Thus in gigantic nebulae large bodies may form, because each part acts independently of the whole. In lesser degree the same principle applies to the development of double stars and of stellar systems generally; for any nucleus once started tends to grow, wherever conditions admit of a supply of material, unless overshadowing disturbing forces intervene to modify the natural order of development.

The study of variable stars also points to the resisting medium as the more dominant cause, but here again the same two causes, namely, Tidal Friction, and the Resisting Medium, are entangled together, though in time it may be possible to separate them by spectroscopic or other criteria yet to be developed.

Τόδε δε μηδείς ποτὲ φοβηθῇ τῶν Ἑλλήνων, ὡ̔ς οὐ χρὴ περι τὰ θεῖα ποτὲ πραγματεύεσθαι θνητοὺς ὄντας · πᾶν δε τούτου διανοηθῆναι τοὐναντίον, ὡ̔ς οὔτε ἄφρον εστι ποτὲ το θεῖον, οὔτε αγνοεῖ που τὴν ἀνθρωπίνην φύσιν · ἀλλ' οἶδεν ὅτι, διδάσκοντος ἀυτοῦ, ξυνακαλουθήσει καὶ μαθήσεται τα διδάσκομενα.

— PLATO, *Epinomis*, p. 988.

Nor should any Greek have any misgiving of this kind; that it is not fitting to inquire narrowly into the operations of superior Powers, such as those by which the motions of the heavenly bodies are produced: but, on the contrary, men should consider that the Divine Powers never act without purpose, and that they know the nature of man: they know that by their guidance and aid, man may follow and comprehend the lessons which are vouchsafed him on such subjects. — WHEWELL, *History of the Inductive Sciences*, Vol. I, p. 108.

CHAPTER VII.

The Secular Effects of the Action of a Resisting Medium Upon the Orbital Motions of the Heavenly Bodies.

§ 71. *Historical Development of the Theory of a Resisting Medium.*

The effect of a resisting medium upon the motions of revolving bodies has long been a subject of investigation. In the *Principia*, 1687, Sir Isaac Newton treated quite fully of the action of resisting media upon the motions of bodies; and gave especial attention to the effects on projectiles and on the oscillations of pendulums. In Lib. II, § 1, Scholium, he remarks: "In medium void of all tenacity, the resistance made to bodies are in the duplicate ratio of the velocities," that is, as the square of the velocities. He also shows that the resistance is proportional to the squares of the diameters of the bodies, or proportional to the surfaces exposed to resistance (Lib. II, § VII, Cor. 3). Hence, if K denote some constant, he shows that the resistance varies according to the formula:

$$F = Kv^2 . \, r^2 . \, \sigma \quad ; \tag{156}$$

where σ is the density of the medium, viewed as made up of hard, discrete particles.

Newton did not treat of the resistance to the motions of the planets, because he supposed the heavenly spaces to be practically devoid of matter, and he alludes to these regions as places "where there is no air to resist their motions" (General Scholium). He refers, however, to the effect of a resistance near the sun upon the motions of comets (Lib. III, Prop. XLII, Probl. XXII, just before the General Scholium to the *Principia*): "The Comet which appeared in the year 1680, was in its perihelion less distant from the Sun than by a sixth part of the Sun's diameter: and because of its extreme velocity in that proximity to the Sun, and some density of the Sun's atmosphere, it must have suffered some resistance and retardation; and therefore, being attracted something nearer to the Sun in every revolution, will at last fall down upon the body of the Sun. Nay, in its aphelion, where it moves the slowest, it may sometimes happen to be yet farther retarded by the attractions of other Comets, and in consequence of this

retardation descend to the Sun. So fixed Stars that have been gradually wasted by the light and vapours emitted from them for a long time, may be recruited by Comets that fall upon them; and from this fresh supply of new fewel, those old Stars, acquiring new splendor, may pass for new Stars. Of this kind are such fixed Stars as appear on a sudden and shine with a wonderful brightness at first, and afterwards vanish by little and little. Such was that Star which appeared in *Cassiopeia's* Chair; which CORNELIUS GEMMA did not see upon the 8th of November, 1572, though he was observing that part of the heavens upon that very night, and the skie was perfectly serene; but the next night (Nov. 9) he saw it shining much brighter than any of the fixed Stars, and scarcely inferiour to *Venus* in splendor. TYCHO BRAHE saw it upon the 11th of the same month when it shone with the greatest lustre; and from that time he observed it to decay by little and little; and in 16 months time it entirely disappeared" (MOTTE's Translation).

It is clear from these and other passages in the *Principia*, that NEWTON had considered the action of a resisting medium in its terrestrial rather than in its celestial aspects; yet his allusion to the action on a comet passing near the sun shows that he was aware that the effect might sometimes become sensible in the motions of the heavenly bodies.

The first writer of importance to depart from the NEWTONIAN doctrine that the heavenly spaces are empty and to announce the conclusion that the celestial regions are not perfectly transparent was CHESEAUX of Geneva, discoverer of the celebrated comet of 1744, which had a great perihelion distance and, it is said, at least five tails. In his treatise on this comet (Traité de la Cométe qui a paru en 1743 et 1744, 8°, LAUSANNE et GENÈVE, 1744, p. 223) CHESEAUX says that the light suffers a certain extinction in consequence of the diffused material spread over the immensely long path which it traverses in coming to the eye of the observer from the great distance of the fixed stars.

The problem of the absorption of light in space was subsequently considered by OLBERS, and at length treated much more fully by the elder STRUVE in his *Études d'Astronomie Stellaire,* Petersbourg, 1847; but the results were not entirely conclusive, except as to the loss of light from distant stars necessarily reducing their brightness, and diminishing the depths to which our telescopes can penetrate. In our time the hypothesis of CHESEAUX, that light is extinguished in space, is fully confirmed by the great extent, and wide diffusion of faint nebulosity over the background of the sky. Such clouds of cosmical dust prevent the background of the heavens from appearing perfectly black,* and conceal from our vision the

* cf. "Remarks on a Brownish Appearance of the Sky Noticed in Certain Constellations of the Southern Hemisphere," by the author, *A.N.* 3618.

most distant regions of the universe; just as the haze in our atmosphere diminishes the distinctness of neighboring objects and finally extinguishes the light of the more distant ones altogether, whether they be terrestrial bodies, as trees, houses, mountains, and clouds, or celestial bodies, such as the moon and stars when they first appear above the horizon. Even in the clear air of California the writer has often seen the Sierra Nevada mountains, at a distance of about 100 miles, reduced to the last extremity of faintness by the absorption of their light.

The first mathematician to treat of the effects of the resisting medium upon the motions of the planets, was the celebrated EULER, in a letter published in the *Philosophical Transactions* of the Royal Society for 1749, pp. 141–142. He showed the earth to be gradually approaching the sun, under the action of a medium which resisted its instantaneous velocity, and allowed the curvature of the orbit to increase; so that the path is no longer a closed curve representing a true Keplerian ellipse, but in reality a spiral with coils very closely wound.

§ 72. EULER'S *Remarks on the Secular Effects of the Resisting Medium upon the Orbital Motion of the Earth, and on the Origin of the Planets at a Great Distance from the Sun — With Brief Notice of the More Modern Theory of a Resisting Medium.*

In view of the results briefly indicated by the writer in *Astronomische Nachrichten*, 4308,* and of the paramount part played by the Resisting Medium in shaping the orbits of the planets and satellites, as well as the paths of the attendant bodies in other cosmical systems observed in the immensity of space, the remarks of the celebrated LEONARD EULER are of so much interest to contemporary astronomers and mathematicians that they should be quoted at greater length. As already pointed out these remarks are included in the *Philosophical Transactions* of the Royal Society for 1749, pp. 141–142, under the title: "Part of a Letter from LEONARD EULER, Professor of Mathematics at Berlin and F.R.S., to the REV. MR. CASPAR WETSTEIN, Chaplain to the Prince of Wales, dated Berlin, June 28, 1749; read Nov. 2, 1749." And this is followed by a similar extract from a second letter to WETSTEIN, dated Berlin, December 20, 1749, read March 1, 1750.

The views of EULER here set forth are very remarkable, not only for the insight they show into the mechanism of the heavenly motions, but also into what is now proved to be the true mode of origin of our solar system. It must be remembered that, in reaching these views on Cosmogony, EULER preceded both KANT (1755), and LAPLACE (1796); and that he was the first mathematician

* cf. Also *A.N.* 4334; and *Proc. Amer. Philos. Soc.*, Vol. XLVII, No. 191, 1909, pp. 119–128.

since NEWTON to consider the secular effects of a Resisting Medium. His views on the origin of the planets are therefore quite free from any kind of prejudice, and the direct outcome of the continued action of small forces which he believed to be operative in the heavenly spaces.

Sir ISAAC NEWTON seems to have held that the spaces where the planets move are essentially as devoid of matter as a vacuum. This is in fact expressly stated in first paragraph of the General Scholium to the *Principia*. Yet he may have believed that some waste matter is diffused in the celestial spaces, for in the paragraph just before the General Scholium, he says: "The vapors which arise from the Sun, the fixed stars, and the tails of the Comets may meet at last with, and fall into, the atmosphere of the planets by their gravity."

We have already remarked that CHESEAUX was the first to express the views that the heavenly spaces are not perfectly transparent, but that light suffers a certain amount of absorption or extinction in passing over great distances (cf. L. de CHESEAUX, Traité de la Comète qui a paru en 1743 et 1744, 8°, LAUSANNE et GENÈVE, 1744, p. 223). This account of CHESEAUX was written five years before the promulgation of EULER's views, and it is uncertain to what extent, if at all, NEWTON and CHESEAUX had influenced EULER in reaching the conclusion that the planets suffer resistance in their motion about the Sun, but the possibility of such suggestion should be borne in mind. EULER felt that the observed propagation of light indicated the presence of æther or other subtile matter in space, and the movements of the heavenly bodies showed that it exerted a sensible influence in the course of ages.

The extracts from EULER's letters are as follows:

(1). *First Letter*: "XXII. MONSIEUR LE MONNIER writes to me that there is, at Leyden, an Arabick manuscript of IBN JOUNIS (if I am not mistaken in the name, for it is not distinctly written in the letter), which contains a history of Astronomical observations. M. LE MONNIER says, that he insisted strongly on publishing a good translation of that book. And as such a work would contribute much to improvement of Astronomy, I should be glad to see it published. I am very impatient to see such a work, which contains observations that are not so old as those recorded by PTOLEMY. For having carefully examined the modern observations of the sun with those of some centuries past, although I have not gone further back than the 15th century, in which I have found WALTHER's observations made at Nüremberg; yet I have observed that the motion of the Sun (or of the Earth) is sensibly accelerated since that time; so that the years are shorter at present than formerly; the reason of which is very natural, for if the earth, in its

motion, suffers some little resistance (which cannot be doubted, since the space through which the planets move, is necessarily full of some subtile matter, were it no other than that of light), the effect of this resistance will gradually bring the planets nearer and nearer the sun; and as their orbits thereby become less, their periodical times will also be diminished. Thus in time the earth ought to come within the region of *Venus*, and in fine into that of *Mercury*, where it would necessarily be burnt. Hence it is manifest that the system of the planets cannot last forever in its (present) state. It also incontestably follows that this system must have had a beginning; for whoever denies it must grant me, that there was a time, when the earth was at the distance of *Saturn* and even farther, and consequently that no living creature could subsist there. Nay there must have been a time when the planets were nearer to some fixt stars than to the Sun; and in this case they could never come into the solar system. This then is a proof, purely physical, that the world in its present state, must have had a beginning, and must have an end. In order to improve this notion, and to find with exactitude how much the years become shorter in each century, I am in hopes that a great number of older observations will afford me the necessary succours."

(2). *Second Letter*: "XXIII. I am still thoroughly convinced of the truth of what I advanced that the orbs of the planets continue to be contracted, and consequently their periodic times grow less. . . . . The late DR. HALLEY has also remarked that the revolutions of the moon are quicker at present than they were in the time of the ancient CHALDEANS, who have left us some observations of Eclipses." EULER then discusses the difficulty of finding the number of days since the time of PTOLEMY, and thinks the uncertainty may be a day or two; also raises the question whether the length of the day is constant: "At present we measure the length of the day by the number of oscillations which a pendulum of given length makes in this space of time; but the ancients were not acquainted with those experiments, whereby we might have been informed, whether a pendulum of the same length made as many vibrations in a day as now. But even though the Ancients had actually made such experiments, we could draw no inferences from them, without supposing, that gravity, on which the time of an oscillation depends, has always been of the same force; but who will ever be in a condition to prove this invariability in gravity?" He finally concludes that both the lengths of the year and of the day are diminishing, "so that the same number of days will answer nearly to a year."

The views of EULER here set forth, that the earth and other planets were at one time farther from the Sun than at present, are so remarkable that

it is scarcely necessary to do more than bring them to the attention of astronomers.

The theory of the Resisting Medium was discussed in a prize paper in 1762 by BOSSUT, who remarks that the effect would become much more sensible in the motion of the moon than in that of the planets, and suggests this physical cause to account for the secular acceleration of the moon, first alluded to by HALLEY in 1693 and more fully confirmed by DUNTHORNE in 1749, the cause of which was then unknown.

The action of a Resisting Medium was afterwards treated by LAPLACE in a very general manner in the *Mécanique Céleste* (Lib. X, Chap. VII, § 18). With characteristic penetration he showed that when the density of the medium increases towards the center the major axis and eccentricity of the orbit will incessantly decrease; yet it did not occur to him to apply this result to explain the roundness of the orbits of the planets and satellites. We shall treat of LAPLACE'S conclusions hereafter, and show their immediate connection with the formation of the planetary system.

In 1819 ENCKE found that the comet which bears his name but had been observed as long ago as 1795 did not seem to return in a period which was exactly constant, but appeared to be shortened each time by from two to three hours; and explained it by the resistance arising from a medium pervading the heavenly spaces. This hypothesis soon became very celebrated, and has since been further discussed by many eminent mathematicians, including PLANA (1825), MOSSOTTI (1826), ENCKE (1831), HANSEN (1835), MÖLLER (1872), VON ASTEN (1878), OPPOLZER (1880), and BÄCKLUND (1895). The motions of the short-period comets have been further discussed by BÄCKLUND, VON HAERDTL, BOHLIN and others, but no very decided result has been obtained, some of the indications pointing one way and some the other; yet the general effect has been to diminish the importance of the part supposed to be played by a Resisting Medium in the present state of our system. A brief outline of the theory of a Resisting Medium will now be given, in order to make secure the line of argument adopted in this work.

§ 73. *Elements of the Theory of Resistance.*

In a discontinuous medium, as was long ago remarked by NEWTON, we may take the resistance to be proportional to the surfaces of the section of the moving planet made by a plane perpendicular to the direction of the velocity; proportional also to the density of the medium σ, and to some function of the velocity $\Psi(V)$. Finally the resistance refers to unit mass, and therefore we should introduce the

reciprocal of the mass of the planet; so that if we denote by K some constant, we shall have for the force exerted by the Resisting Medium the expression

$$F = \frac{KS\,\sigma\,\Psi(V)}{m}. \tag{157}$$

As there are m units of mass in the planet, the total resistance is mF.

If now we suppose that the resistance depends only on the distance from the sun, we have

$$F = H\,\Psi(V)\,\Phi\left(\frac{1}{r}\right) \quad ; \quad \sigma = \Phi\left(\frac{1}{r}\right) \quad ; \quad H = \frac{KS}{m}. \tag{158}$$

With this expression for the forces involved, it remains to find the effect upon the motion of a planet. For this purpose we shall resume the formulae for the variation of the elements, considered as arbitrary constants, introduced by the integration of the differential equations of motion. As the disturbing force F, due to the Resisting Medium, is always directed in the plane of the orbit, we may put the orthogonal component $W = 0$, and therefore also $\frac{di}{dt} = 0$, $\frac{d\Omega}{dt} = 0$, so that the inclination and node remain fixed, and are not subject to change by the action of the Resisting Medium.

If we denote by ψ_0 the angle which the tangent to the orbit makes with the prolongation of the radius vector, and by R and S the components of the disturbing force resolved in the direction of the radius vector and of the perpendicular to the radius vector respectively, we have the following expressions for the variations of the velocity (cf. LAPLACE's *Mécanique Céleste,* Liv. X, Chap. VII, § 18; TISSERAND's *Mécanique Céleste,* Tome I, p.433, Eq. A; Tome IV, p. 218; WATSON's *Theoretical Astronomy,* pp. 552–555; or similar works on Celestial Mechanics):

$$\frac{dV}{dt} = R\cos\psi_0 + S\sin\psi_0. \tag{159}$$

It is well known, however, that the components of the velocity are

$$\left.\begin{aligned} V\cos\psi_0 &= \frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p}}\,e\sin v\,, \\ V\sin\psi_0 &= r\left(\frac{dv}{dt}\right) = \frac{k\sqrt{p(1+m)}}{r}\,, \\ V &= \frac{ds}{dt} = \sqrt{dr^2 + r^2dv^2} = r^2dv\sqrt{\left(\frac{dr}{r^2dv}\right)^2 + \frac{1}{r^2}}. \end{aligned}\right\} \tag{160}$$

Accordingly we get

$$V^2 = \frac{k^2(1+m)}{p}\left[\frac{p^2}{r^2} + e^2\sin^2 v\right] = \frac{k^2(1+m)}{p}\left[(1+e\cos v)^2 + e^2\sin^2 v\right]$$
$$= \frac{k^2(1+m)}{p}\left[1 + e^2 + 2e\cos v\right];$$

or

$$V = \frac{k\sqrt{1+m}}{\sqrt{p}}\left(1 + e^2 + 2e\cos v\right)^{\frac{1}{2}}. \tag{161}$$

But the resistance is proportional to the square of the velocity, $\left(\frac{ds}{dt}\right)^2$; and hence using the expressions (160) for the components parallel to and perpendicular to the radius vector we get

$$R = -K\Phi\left(\frac{1}{r}\right)\frac{k\sqrt{1+m}}{\sqrt{p}}e\sin v\frac{ds}{dt} \quad ; \quad S = -K\Phi\left(\frac{1}{r}\right)\frac{k\sqrt{p(1+m)}}{r}\frac{ds}{dt}. \tag{162}$$

§ 74. *The Perturbation of the Longitude in the Orbit, Due to the Action of Resisting Medium, is Periodic.*

If we denote by χ the longitude in the orbit, the formula for the variation of this element is

$$\frac{d\chi}{dt} = \frac{1}{k\sqrt{p(1+m)}}\frac{1}{e}\left[-p\cos v \,.\, R + (p+r)\sin v \,.\, S\right]. \tag{163}$$

Substituting from (162) the value of R and S, we get

$$e\frac{d\chi}{dt} = \frac{-K\Phi\left(\frac{1}{r}\right)}{k\sqrt{p(1+m}}\left[-p\cos v\frac{k\sqrt{1+m}}{\sqrt{p}}e\sin v + (p+r)\frac{k\sqrt{p(1+m)}}{r}\sin v\right]\frac{ds}{dt},$$
$$= -K\Phi\left(\frac{1}{r}\right)\left[-e\cos v\sin v + (1 + e\cos v + 1)\sin v\right]\frac{ds}{dt};$$

or

$$e\frac{d\chi}{dt} = -2K\Phi\left(\frac{1}{r}\right)\sin v\frac{ds}{dt}. \tag{164}$$

But by (161) $\frac{ds}{dt} = V = \frac{k\sqrt{1+m}}{\sqrt{p}}\left(1 + e^2 + 2e\cos v\right)^{\frac{1}{2}}$, and $dt = \frac{r^2 dv}{k\sqrt{p(1+m)}}$, and hence

$$Vdt = \frac{r^2}{p}\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}}dv \quad ; \quad \text{therefore}$$

$$ed\chi = -\frac{2K\Phi\left(\frac{1}{r}\right)r^2}{p}\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}}\sin v dv. \tag{165}$$

The function $K\Phi\left(\frac{1}{r}\right)\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}} r^2$ is always positive, and we may therefore expand it in a series, proceeding according to the cosines of v and its multiples.

Assume therefore

$$K\Phi\left(\frac{1}{r}\right)r^2\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}} = A + B\cos v + C\cos 2v + D\cos 3v + E\cos 4v + \ldots, \quad (166)$$

in which the coefficients A, B, C, D, are positive numbers and functions of e Hence we have

$$ed\chi = -\frac{2}{p}\left(A + B\cos v + C\cos 2v + D\cos 3v + E\cos 4v + \ldots\right)\sin v dv.$$

When this expression is integrated with respect to v we find

$$e\delta\chi = \frac{2}{p}\left(A\cos v + \frac{1}{4}\cos 2v + \ldots\right). \quad (167)$$

Accordingly we conclude that χ is subject to periodic perturbations only, and that a resisting medium produces no permanent change of the longitude in the orbit.

By a similar course of reasoning, applied to the formula for the variation of the mean anomaly,

$$\frac{dM}{dt} = -\cos\phi\frac{d\chi}{dt} - \frac{2r\cos\phi \,.\, R}{k\sqrt{p(1+m)}} + \int\frac{d\mu}{dt}dt, \quad (168)$$

where $e = \sin\phi$, it may be made clear that only periodic changes arise from the first and second terms of the right member. And it is easily shown that the secular change results wholly from the third term, by virtue of the change in the mean daily motion μ. This really depends, however, on the change in the mean distance, which we shall now consider.

§ 75. *Secular Change in the Mean Distance.*

The general formula for the variation of the semi-axis major is

$$\frac{da}{dt} = \frac{2a^2}{k\sqrt{p(1+m)}}\left(e\sin vR + \frac{p}{r}S\right). \quad (169)$$

If we introduce for R and S their values, and put $dt = \frac{r^2dv}{k\sqrt{a(1-e^2)(1+m)}}$ we shall get

$$da = -\frac{2a^2r^2}{k\sqrt{p(1+m)}}\left\{e\sin v K\Phi\left(\frac{1}{r}\right)\frac{k\sqrt{1+m}}{\sqrt{p}}e\sin v\frac{k}{\sqrt{p(1+m)}}\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(1+e\cos v\right)K\Phi\left(\frac{1}{r}\right)\frac{k\sqrt{p(1+m)}}{r}\frac{k}{\sqrt{p(1+m)}}\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right\}\frac{dv}{k\sqrt{p(1+m)}}$$

$$= -\frac{2a^2K\Phi\left(\frac{1}{r}\right)r^2}{p^2(1+m)}\left[e^2\sin^2 v\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}+\left(1+e\cos v\right)\frac{p}{r}\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right]dv\,.$$

$$da = -\frac{2a^2K\Phi\left(\frac{1}{r}\right)r^2}{p^2(1+m)}\left[e^2\sin^2 v\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}+\left(1+e\cos v\right)^2\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right]dv\,,$$

$$= -\frac{2a^2K\Phi\left(\frac{1}{r}\right)r^2}{p^2(1+m)}\left[\left\{e^2\sin^2 v+(1+e\cos v)^2\right\}\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right]dv\,,$$

$$= -\frac{2a^2K\Phi\left(\frac{1}{r}\right)r^2}{p^2(1+m)}\left[\left(1+2e\cos v+e^2\right)\left(1+2e\cos v+e^2\right)^{\frac{1}{2}}\right]dv\,;$$

or

$$da = -\frac{2a^2K\Phi\left(\frac{1}{r}\right)r^2}{p^2(1+m)}\left[\left(1+2e\cos v+e^2\right)^{\frac{3}{2}}\right]dv\,. \tag{170}$$

This has the same general form as equation (166), except that the expression $(1 + 2\,e\cos v + e^2)^{\frac{1}{2}}$ is now raised to the power $\frac{3}{2}$. And as

$$K\Phi\left(\frac{1}{r}\right)r^2\left(1+2e\cos v+e^2\right)^{\frac{1}{2}} = A + B\cos v + C\cos 2v + D\cos 3v + E\cos 4v + \ldots\ldots;$$

it is evident that we may multiply both members by $(1 + 2e\cos v + e^2)$ and write the result in the form:

$$K\Phi\left(\frac{1}{r}\right)r^2\left(1+2e\cos v+e^2\right)^{\frac{3}{2}} = A' + B'\cos v + C'\cos 2v + D'\cos 3v + E'\cos 4v + \ldots\ldots \tag{171}$$

Substituting this series for the equivalent expression in (170), and integrating, we get

$$\delta a = -\frac{2a^2}{p^2(1+m)}\left[A'v + \text{periodic terms}\right]. \tag{172}$$

§ 76. *The Secular Change of the Eccentricity.*

The perturbations of the eccentricity are given by the expression

$$\frac{de}{dt} = \frac{1}{k\sqrt{p(1+m)}}\left\{p\sin v\,.\,R + \frac{p}{e}\left(\frac{p}{r}-\frac{r}{a}\right)S\right\} \tag{173}$$

Introducing the values of R and S from equations (162), we get

$$\frac{de}{dt} = -\frac{K\Phi\left(\frac{1}{r}\right)}{k\sqrt{p\,(1+m)}}\left\{ p \sin v \left(\frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v \frac{ds}{dt}\right) + \frac{p}{e}\left(\frac{p}{r} - \frac{r}{a}\right)\left(\frac{k\sqrt{p\,(1+m)}}{r}\frac{ds}{dt}\right)\right\},$$

$$= -K\Phi\left(\frac{1}{r}\right)\left\{ e \sin^2 v + \frac{p}{e}\left(\frac{p}{r} - \frac{r}{a}\right)\frac{1}{r}\right\}\frac{ds}{dt}. \tag{174}$$

Now the expression $\frac{ds}{dt} = V = \frac{k\sqrt{1+m}}{\sqrt{p}}\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}}$; and

$$\frac{p}{e}\left(\frac{p}{r} - \frac{r}{a}\right)\frac{1}{r} = \frac{a\,(1-e^2)}{e}\left[\frac{(1 + e\cos v)^2 - 1 + e^2}{1 + e\cos v}\right]\frac{1 + e\cos v}{a\,(1 - e^2)} = \frac{2e\cos v + e^2\cos^2 v + e^2}{e}$$

$$= 2\cos v + e\cos^2 v + e\,.$$

The whole brace in (174) therefore gives

$$2\cos v + e\,(\sin^2 v + \cos^2 v) + e = 2\,(\cos v + e)\,.$$

Accordingly we have

$$\frac{de}{dt} = -2K\Phi\left(\frac{1}{r}\right)\left(\cos v + e\right)\frac{k\sqrt{1+m}}{\sqrt{p}}\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}}. \tag{175}$$

And since $dt = \frac{r^2 dv}{k\sqrt{p\,(1+m)}}$, we get

$$de = -\frac{2K\Phi\left(\frac{1}{r}\right)r^2}{p}\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}}\left(e + \cos v\right)dv\,. \tag{176}$$

But by (166) the expression

$$K\Phi\left(\frac{1}{r}\right)r^2\left(1 + 2e\cos v + e^2\right)^{\frac{1}{2}} = A + B\cos v + C\cos 2v + \;.\;.\;.\;.\;;$$

and hence the product

$$(A + B\cos v + C\cos 2v + \ldots)(e + \cos v) = Ae + (A + eB)\cos v + (eC + C\cos v)\cos 2v + \ldots \tag{177}$$

Introducing these values into (176) and integrating, we have

$$\delta e = -\frac{2}{p}\left\{ Aev + \text{periodic terms}\right\}. \tag{178}$$

Laplace derives the equivalent of this expression (*Mécanique Céleste*, Liv. X, Ch. VII, § 18) and remarks that the major axis always decreases, because δa in (172) always is negative. And when $\Phi\left(\frac{1}{r}\right)$ increases as the distance from the

sun decreases, the eccentricity also steadily decreases. "Therefore at the same time that the planet approaches towards the sun, by the effect of the resistance of the medium, the orbit will become more circular." For the secular changes of the semi-axis major and eccentricity LAPLACE put his expressions in the form

$$da = -Kdv \left\{ 2a^2\varphi\left(\frac{1}{a}\right) \right\}, \tag{179}$$

$$\frac{de}{e} = -Kdv \left\{ a\varphi\left(\frac{1}{a}\right) + \varphi'\left(\frac{1}{a}\right) \right\}, \tag{180}$$

where $\varphi'\left(\frac{1}{a}\right)$ is the differential of $\varphi\left(\frac{1}{a}\right)$ divided by the differential of $\frac{1}{a}\cdot$ If we multiply the equation (180) by $2da$ and divide the product by the equation (179), we shall get

$$\frac{2de}{e} = \frac{da}{a} + \frac{da}{a^2}\,\frac{\varphi'\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right)} = \frac{da}{a} - d\left\{\frac{1}{a}\,\frac{\varphi'\left(\frac{1}{a}\right)}{\varphi\left(\frac{1}{a}\right)}\right\} = \frac{da}{a} - \frac{d\left\{\varphi\left(\frac{1}{a}\right)\right\}}{\varphi\left(\frac{1}{a}\right)}. \tag{181}$$

The integral of this expression is

$$\log e^2 = \log a - \log\varphi\left(\frac{1}{a}\right) + \log q^2, \tag{182}$$

where q is an arbitrary constant. Passing to numbers, we have

$$e^2 = \frac{aq^2}{\varphi\left(\frac{1}{a}\right)}, \quad \text{or} \quad e = q\sqrt{\frac{a}{\varphi\left(\frac{1}{a}\right)}}\cdot \tag{183}$$

After deriving this expression, LAPLACE concludes his discussion with the remark: "Hence we easily perceive that, while a decreases, and $\varphi\left(\frac{1}{a}\right)$ increases, the value of the eccentricity e will incessantly decrease" (cf. *Mécanique Céleste*, Liv. X, Ch. VII, § 18, BOWDITCH Translation).

In the treatment of this physical problem, LAPLACE showed his characteristic insight into the great laws of nature; yet it is remarkable that it did not occur to him that the roundness of the orbits of the planets and satellites could be explained by this cause quite as simply as by the theory of a rotation which would gently detach these masses and set them revolving in orbits nearly circular. A veteran astronomer, PROFESSOR GEORGE DAVIDSON of San Francisco, who was among the first to whom the writer unfolded the present theory, justly remarked, in regard to the resisting medium which LAPLACE had demonstrated to have formerly acted

against the motions of *Jupiter's* satellites, that "LAPLACE *had the true cause in sight but he did not carry it far enough to discover the real process by which the solar system was formed.*"

Apparently it simply did not occur to the illustrious author of the *Mécanique Céleste* to apply this idea to the roundness of the orbits of the planets and satellites. At the same time it must be remembered that the difficulty since encountered in connection with the Laplacian formulation of the nebular hypothesis, when we imagine the sun and planets expanded till they filled the orbits of the planets and satellites, and seek to calculate their theoretical times of rotation, which are thus found to be entirely inconsistent with a velocity of rotation that would produce separation, as BABINET very impressively showed in 1861 (*Comptes Rendus* LII, p. 481), had not arisen in LAPLACE's time. And as the great geometer had devoted much attention to the figures of equilibrium of rotating masses of fluid, following the precedent set by NEWTON in the *Principia*, it was natural for him to ascribe the detachment of the planets and satellites to the breaking down of figures of equilibrium.

Even POINCARÉ and DARWIN within the past twenty years have still sought to test the nebular hypothesis by the same process, and STRATTON's work on Planetary Inversion in 1906 proceeded on the same general lines. And although they did not succeed in throwing light on the genesis of planets and satellites they reached results of considerable dynamical importance. This early work of POINCARÉ and DARWIN gave us a first approximation to the theory of the evolution of double and multiple stars, as was long ago shown by the writer in his *Inaugural Dissertation* at the University of Berlin (Die Entwickelung der Doppelstern-Systeme, Berlin, 1892).

§ 77. TISSERAND's *Investigation of the Secular Effects of a Resisting Medium, in Which the Action Varies as Any Power of the Velocity and Any Inverse Power of the Distance.*

If we suppose the resistance to vary as some power of the velocity, $\psi(V) = V^p$; and the density as any function of the distance from the sun's center $\Phi\left(\frac{1}{r}\right) = \frac{1}{r^q}$; then it is obvious that we shall have

$$F = H\psi(V)\,\Phi\left(\frac{1}{r}\right) = H\frac{V^p}{r^q}. \tag{184}$$

Consequently we get

$$\left.\begin{aligned}\frac{1}{a}\frac{da}{dt} &= -\frac{2H}{1-e^2}\Big(1+2e\cos v+e^2\Big)\frac{V^{p-1}}{r^q},\\ \frac{de}{dt} &= -2H\Big(e+\cos v\Big)\frac{V^{p-1}}{r^q};\end{aligned}\right\}\tag{185}$$

and therefore

$$\left.\begin{aligned}\frac{1}{a}\frac{da}{dt} &= -\frac{2H'}{1-e^2}\Big(1+2e\cos v+e^2\Big)^{\frac{p+1}{2}}\Big(1+e\cos v\Big)^{q-2},\\ \frac{de}{dt} &= -2H'\Big(e+\cos v\Big)\Big(1+2e\cos v+e^2\Big)^{\frac{p-1}{2}}\Big(1+e\cos v\Big)^{q-2},\\ H' &= H\frac{k^{p-2}}{\Big[a\,(1-e^2)\Big]^{\frac{p}{2}+q-2}}.\end{aligned}\right\}\tag{186}$$

On purely mathematical grounds it is well known that we may always have the following convergent development

$$\Big(1+2e\cos v+e^2\Big)^{\frac{p-1}{2}}\Big(1+e\cos v\Big)^{q-2} = A_0+A_1\cos v+A_2\cos 2v+\ldots\tag{187}$$

where the coefficients A_0 and A_1 may be represented by the expressions

$$\left.\begin{aligned}A_0 &= \frac{1}{\pi}\int_0^\pi\Big(1+2e\cos v+e^2\Big)^{\frac{p-1}{2}}\Big(1+e\cos v\Big)^{q-2}.\,dv,\\ A_1 &= \frac{2}{\pi}\int_0^\pi\Big(1+2e\cos v+e^2\Big)^{\frac{p-1}{2}}\Big(1+e\cos v\Big)^{q-2}.\,\cos v dv.\end{aligned}\right\}\tag{188}$$

The formulae (186), by means of these values, become

$$\left.\begin{aligned}\frac{da}{dv} &= -\frac{2aH'}{1-e^2}\Big\{A_0(1+e^2)+A_1e+[A_1(1+e^2)+2eA_0+eA_2]\cos v+\ldots\Big\},\\ \frac{de}{dv} &= -2H'\Big\{A_0e+\tfrac{1}{2}A_1+(A_0+eA_1+\tfrac{1}{2}A_1)\cos v+\ldots\Big\}.\end{aligned}\right\}\tag{189}$$

Integrating these expressions we have

$$\left.\begin{aligned}\delta a &= -\frac{2aH'}{1-e^2}\Big[A_0(1+e^2)+A_1e\Big]v+\Big\{-\frac{2aH'}{1-e^2}\Big[A_1(1+e^2)+2eA_0+eA_2\Big]\sin v+\ldots\Big\},\\ \delta e &= -2H'(A_0e+\tfrac{1}{2}A_1)v+\{-2H'(A_0+eA_1+\tfrac{1}{2}A_2)\sin v+\ldots\}.\end{aligned}\right\}\tag{190}$$

The second parts of the right members of these expressions are periodic, and may

be neglected; the first parts alone give the secular inequalities, and in these we may therefore substitute nt for v, so that we have finally

$$\left.\begin{aligned}\frac{\delta a}{a} &= -\frac{2H'}{1-e^2}\Big[A_0(1+e^2)+A_1e\Big]nt\,,\\ \delta e &= -2H'(A_0e+\tfrac{1}{2}A_1)\,nt\,.\end{aligned}\right\}\qquad(191)$$

In considering the coefficients A_0, and A_1 we may observe that the first of the equations (188) shows that A_0 is essentially positive; while the expression for A_1 gives, on integrating by parts,

$$A_1 = \frac{2e}{\pi}\int_0^\pi\Big(1+2e\cos v+e^2\Big)^{\frac{p-3}{2}}\Big(1+e\cos v\Big)^{q-3}\Big[(p-1)(1+e\cos v)$$

$$+(q-2)(1+2e\cos v+e^2)\Big]\sin^2 v\,dv\,.$$

This expression is essentially positive if we have simultaneously $p\geqq 1$, $q\geqq 2$; hence it follows that δe is essentially negative, and the eccentricity incessantly diminishes (cf. TISSERAND'S *Mécanique Céleste*, Tome IV, Chap. XIII, § 93, pp. 219–221). This is a general result of the same purport as that already derived on a more restricted hypothesis relative to the laws of the resisting medium and its assumed mode of action.

§ 78. *Particular Case in Which the Resistance Varies as the Square of the Velocity and Inversely as the Square of the Distance — Hypothesis of* ENCKE.

We shall now consider the case in which $F = H\frac{V^2}{r^2}\cdot$ This is the hypothesis adopted by ENCKE in the investigation of the comet which bears his name, and which was long believed to give evidence of the action of a resisting medium, by an observed acceleration of between two and three hours in the predicted times of return to perihelion. The preceding expressions, in terms of the true and eccentric anomalies respectively, in ENCKE'S hypothesis, become

$$\left.\begin{aligned}A_0(1+e^2)+A_1e &= \frac{1}{\pi}\int_0^\pi\Big(1+2e\cos v+e^2\Big)^{\frac{3}{2}}dv = \frac{1-e^2}{\pi}\int_0^\pi\frac{(1+e\cos u)^4\,du}{(1-e\cos u)^{5/2}}\,,\\ A_1+2A_0e &= \frac{2}{\pi}\int_0^\pi\Big(1+2e\cos v+e^2\Big)^{\frac{1}{2}}\Big(e+\cos v\Big)dv = \frac{2(1-e^2)^2}{\pi}\int_0^\pi\frac{\cos u\,(1+e\cos u)^3\,du}{(1-e^2\cos^2 u)^{5/2}}\,,\end{aligned}\right\}\quad(192)$$

where u denotes the eccentric anomaly.

We need to consider only the even powers of $\cos u$, and consequently may replace u by $90° - u$; thus we get

$$\left.\begin{aligned}
A_0(1+e^2) + A_1 e &= \frac{2(1-e^2)^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{(1 + 6e^2\sin^2 u + e^4 \sin^4 u)\,du}{\triangle^5}, \\
A_1 + 2A_0 e &= \frac{4(1-e^2)^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{3e\sin^2 u + e^3\sin^4 u}{\triangle^5}\,du, \\
\triangle^2 &= 1 - e^2\sin^2 u.
\end{aligned}\right\} \tag{193}$$

Substituting for $e^2 \sin^2 u$ its value $1 - \triangle^2$, these expressions become

$$\left.\begin{aligned}
A_0(1+e^2) + A_1 e &= \frac{2(1-e^2)^2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{8}{\triangle^5} - \frac{8}{\triangle^3} + \frac{1}{\triangle}\right) du, \\
A_1 + 2A_0 e &= \frac{4(1-e^2)^2}{\pi e}\int_0^{\frac{\pi}{2}} \left(\frac{4}{\triangle^5} - \frac{5}{\triangle^3} + \frac{1}{\triangle}\right) du.
\end{aligned}\right\} \tag{194}$$

By the general formula

$$(2p+1)(1-e^2)\int_0^{\frac{\pi}{2}} \frac{du}{\triangle^{2p+3}} - 2p(2-e^2)\int_0^{\frac{\pi}{2}} \frac{du}{\triangle^{2p+1}} + (2p-1)\int_0^{\frac{\pi}{2}} \frac{du}{\triangle^{2p-1}} = 0; \tag{195}$$

and this gives

$$\int_0^{\frac{\pi}{2}} \frac{du}{\triangle^3} = \frac{1}{1-e^2}\int_0^{\frac{\pi}{2}} \triangle du; \quad \int_0^{\frac{\pi}{2}} \frac{du}{\triangle^5} = \frac{2}{3}\frac{2-e^2}{(1-e^2)^2}\int_0^{\frac{\pi}{2}} \triangle du - \frac{1}{3(1-e^2)}\int_0^{\frac{\pi}{2}} \frac{du}{\triangle}. \tag{196}$$

Designating the complete integrals of the first and second species by F_1, E_1

$$F_1 = \int_0^{\frac{\pi}{2}} \frac{du}{\triangle}; \quad E_1 = \int_0^{\frac{\pi}{2}} \triangle\, du; \tag{197}$$

it is easily shown that

$$\left.\begin{aligned} A_0(1+e^2)+A_1e &= 2\frac{1-e^2}{3\pi}\left[8\frac{1+e^2}{1-e^2}E_1-(5+3e^2)F_1\right], \\ A_1+2A_0e &= 4\frac{1-e^2}{3\pi e}\left[\frac{1+7e^2}{1-e^2}E_1-(1+3e^2)F_1\right]; \end{aligned}\right\} \tag{198}$$

$$H_{2,2}=\frac{A_0(1+e^2)+A_1e}{2A_0e+A_1}=1+\frac{A(1-e)-A_1}{A_1+2A_0e}(1-e)=\frac{e}{2}\frac{8(1+e^2)E_1-(1-e^2)(5+3e^2)F_1}{(1+7e^2)E_1-(1-e^2)(1+3e^2)F_1} \tag{199}$$

Putting $e = \sin\varphi$, in (191), we may calculate the following simple expressions for the secular variations:

$$\left.\begin{aligned} \frac{\delta n}{n} &= \frac{3H'}{\cos^2\varphi}\left[A_0(1+e^2)+A_1e\right]nt, \\ \delta\varphi &= \frac{-H'}{\cos\varphi}(2A_0e+A_1)\,nt, \end{aligned}\right\} \tag{200}$$

$$\frac{\delta n}{n} = -\frac{3\delta\varphi}{\cos\varphi}H_{2,2}; \tag{201}$$

where $H_{2.2}$ is as defined above.

These elliptic integrals are easily evaluated by means of the Tables of LEGENDRE, if we take out the tabular values corresponding to the value of e for the planet or comet in question. In the case of ENCKE's Comet, $e = 0.85$, $F_1 = 2.10995$, $E_1 = 1.22810$, $H_{2,2} = 0.97$ (cf. TISSERAND's *Mécanique Céleste*, Tome IV, pp. 221–223). The formulae (200) and (201) or their equivalents have been used by ENCKE, PLANA, BACKLUND, TISSERAND and others, in the investigation of the motions of actual bodies of the solar system.

For a long time ENCKE's investigations were believed to show the effect of a true resisting medium upon the motion of his comet; but the inquiry has since been extended to other comets also, and the whole subject re-examined with greater care by VON ASTEN, VON HAERDTL, BOHLIN and BACKLUND, with less conclusive results. According to these investigators the secular changes in the motions of ENCKE's Comet are found after some thirty-five revolutions to be somewhat arbitrary, and not the same at successive periods. DR. G. W. HILL writes to the author (April 15, 1909) that the late PROFESSOR ASAPH HALL was of the opinion that ENCKE had not established the existence of a resisting medium from his discussion of the observations of his comet, and that HALL's view seems to be confirmed by BACKLUND's investigations.

DR. HILL himself is very skeptical about the effectiveness of the resisting medium in the present condition of the solar system. "I certainly think," he adds, "that the observations of *Jupiter* and *Saturn* might go on for 10,000 years without our detection of any effects of a resisting medium." The views of the

illustrious author of the *New Theory of Jupiter and Saturn* are always entitled to great weight, and he may be right regarding inefficiency of the resisting medium at present pervading our system; but it seems to the author that despite their admitted excellence the observations of the last 150 years are not sufficiently accurate to justify us in extending our exterpolation over a period sixty-six times as great. In view of the unexpected sources of error brought to light from time to time in the finest modern observations, such extreme exterpolation would be most hazardous, and could not be justified by past experience.

No doubt the effect of the resisting medium is slight for short periods of time. VON HAERDTL concluded that THEODORE VON OPPOLZER'S claim that WINNECKE'S Comet was accelerated like ENCKE'S is devoid of foundation.

BACKLUND'S investigations of ENCKE'S Comet are by far the most elaborate that have ever been made; and his conclusion, that the action is variable and somewhat arbitrary, shows that the resistance cannot well be that of an ethereal fluid, such as ENCKE imagined 90 years ago, but must be due to swarms of cosmical dust encountered in the movement of the comet about the sun.

Whatever doubt may arise as to the effectiveness of the resisting medium in the present state of the solar system, there can be no possible doubt as to its power in our system at the epoch when the planets were formed. The observed roundness of the orbits of the planets and satellites is an everlasting witness to the presence of a resisting medium against which these bodies revolved for immeasurable ages. There is no other admissible explanation of this phenomenon, and as the resisting medium is a vera causa, on the secular effects of which all mathematicians are agreed, we may hold that it has as surely rounded up these orbits as if we had witnessed the transformation within the short period of human history covered by exact observations.

§ 79. *Probable Law of Density in the Nebulae.*

In spite of the immense distances of the nebulae and the great inherent difficulty which confronts us in any attempt to subject them to careful investigation, it may be affirmed that in these objects *as a class* the density increases towards the center, so that a nucleus and well defined star often indicate the progress of condensation. Although there are a few exceptions to this general rule, they are so rare that it has been a recognized fact of observation since the time of the elder HERSCHEL. And as all stages of development exist in these primitive masses, with correspondingly great variations in the distribution of density, it might be supposed that the nebulae are entirely chaotic and not even subject to approximate laws of internal arrangement. Within certain limits there is undoubtedly great

and arbitrary variation in the external form and internal distribution of the density of the nebulae, but this becomes less and less pronounced as the development proceeds; and in those nebulae which are condensing and which have already taken the spiral form, the centripetal forces gradually establish order and uniformity with a tendency towards definite arrangements of density, though the law of distribution varies with the progress of the condensation.

This is why the nebulae were often noticed to have stellar nuclei by the elder HERSCHEL, whose observations have been amply confirmed by modern photography. Moreover, this conspicuous natural phenomenon was correctly interpreted by SIR JOHN HERSCHEL, in the *Philosophical Transactions* for 1833, as indicating progressive increase of density towards the center. But it has never been determined what the approximate laws of density are, and the subject is therefore deserving of some attention in this chapter.

There are only a few laws to which we shall refer, but these will probably indicate the general tendency in nature.

(1). The law of density for a gaseous mass in convective equilibrium, when the molecules are reduced to the state of single atoms. This arrangement for a monatomic gas, with $k = 1\frac{2}{3}$, has been found (*A.N.* 4053) to give the law of density indicated in the figure. The central density is exactly six times the mean density of the entire mass.

(2). The law of density for common gases, with $k = 1\frac{2}{5}$, as found by LANE, RITTER, LORD KELVIN, and other investigators (cf. paper by the author "On the Temperature of the Sun and on the Relative Ages of the Stars and Nebulae," *Transactions of the Academy of Sciences of St. Louis*, Vol. X, No. 1, p. 29). These investigations have been shown (cf. *Astronomische Nachrichten*, 4053, p. 326), to agree in fixing the central density of such a mass of gas at twenty-three times the mean density. The curve corresponding to this distribution of density is shown in the diagram.

Both this law for biatomic gases, and the above law for monatomic gases, suppose the gaseous mass to be in convective equilibrium under the temperature, pressure and attraction of its parts; but as a nebula is not strictly a mass in equilibrium, but rather a swarm of dust undergoing gradual transformation, these results cannot be expected to hold accurately true for the nebulae as a class; yet they may give a good indication of the tendency in nature.

(3). The law of density $\Phi\left(\frac{1}{r^2}\right)$, which was the hypothesis used by ENCKE, in his investigation of the resistance suffered by the comet which bears his name.

The curve for this law of density is also shown in the diagram. It is very steep towards the center, and correspondingly flat towards the periphery.

(4). The law of density $\Phi\left(\frac{1}{r}\right)$, which, in the simplest form, gives a rectangular hyperbola referred to its asymptotes. This curve has a somewhat more gradual slope than that depending on ENCKE's hypothesis.

Law of Density.	Radius of Nebula, with Corresponding Density as Illustrated in the Diagram.										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Monatomic Gas $k = 1\frac{2}{3}$	10.	9.671	8.744	7.379	5.792	4.197	2.768	1.613	0.770	0.235	0.0000
Common Gas $k = 1\frac{2}{5}$	10.	8.8	6.4	3.9	2.0	1.0	0.4	0.15	0.038	0.0054	0.0000
$\Phi\left(\frac{1}{r^2}\right)$	∞	10.0	2.5	1.11111...	0.625	0.400	0.27777...	0.204081...	0.1625	0.12345	0.10000
$\Phi\left(\frac{1}{r}\right)$	∞	2.0	1.0	0.6666...	0.5000	0.4000	0.3333...	0.2857...	0.2500	0.2222	0.20000

None of these laws of simple inverse powers of the distance, however, is to be regarded as applicable to the conditions arising in nature. They all give too great a density towards the center; yet they may convey to the mind some approximation to the tendency in gravitational condensation. In some nebulae one law will apply, in others a very different law would have to be sought. If we were to seek the law giving least central condensation, as probably most nearly representing average conditions, in a fairly well-ordered nebula, we should no doubt have to choose the monatomic law. When, however, a break occurs in the continuity of the nebula, by the development of a stellar nucleus, all these laws obviously would fail.

Now it is important to recall that, however the density of the nebular resisting medium increases towards the center, the eccentricity of the orbit diminishes. As we have seen above, this result was reached by LAPLACE *in* 1805. *The significance of such a law can hardly be overestimated. Only a homogeneous nebula, with curve of density represented in the above diagram by a horizontal straight line, leaves the eccentricity unchanged.*

To produce an *increase of the eccentricity* the density must *increase towards the periphery of the nebula.* This corresponds to a curve in the above diagram rising upward on the right and passing downward on the left. But this condition is dynamically so unstable as not to require our detailed consideration. It would

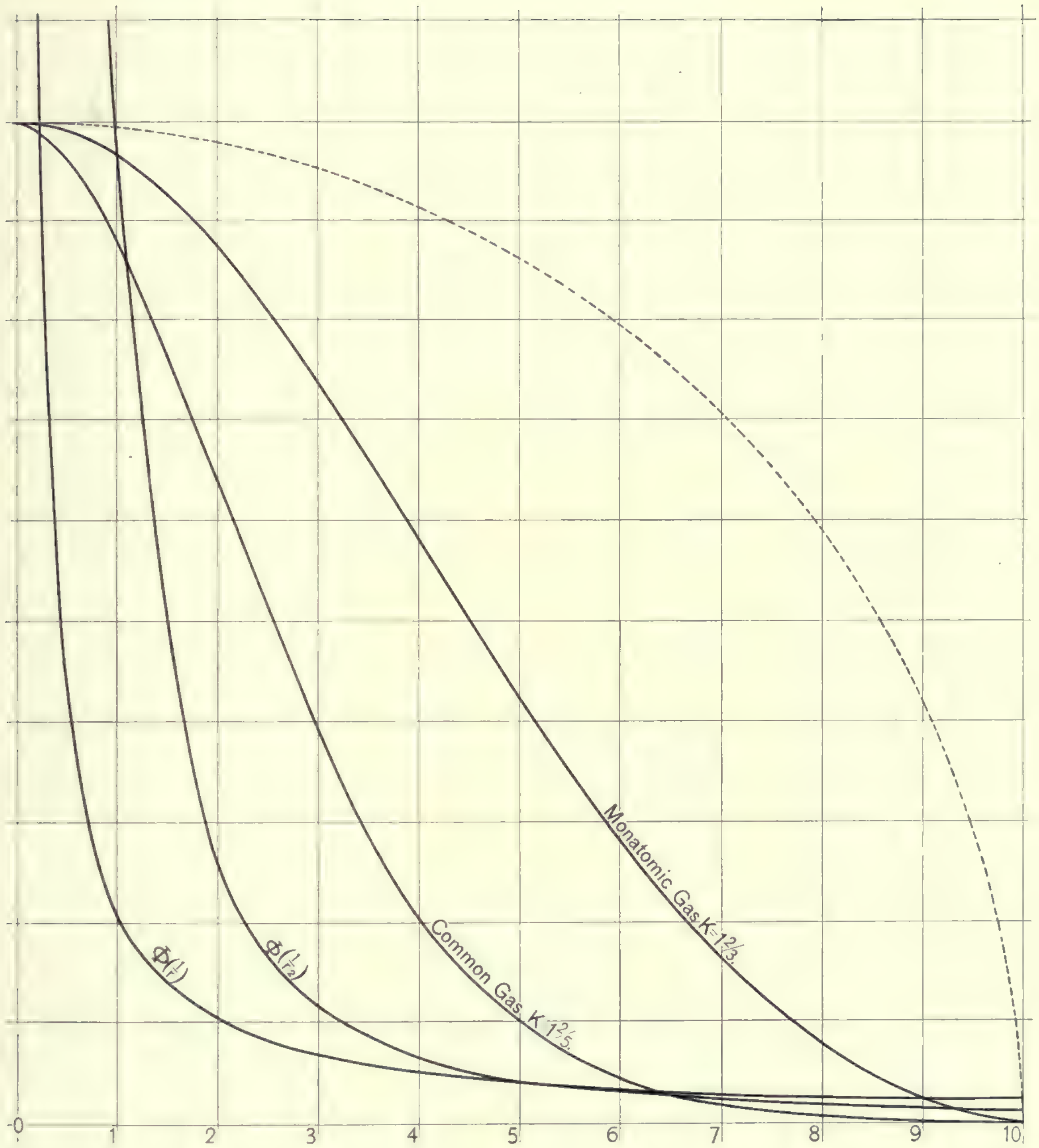

PLATE IV. DIAGRAM ILLUSTRATING SEVERAL LAWS OF DENSITY IN A NEBULA.

at best prove to be of very short duration, and would leave but little trace of its influence in the appearances of the heavenly bodies.

If a nebula should coil up in such a way as to be of the annular form, which is found by observation to be an exceedingly rare occurrence, a body revolving about the center of gravity of the whole ring system might have the eccentricity of its orbit increased. As ring nebulae or spherical nebulae with density increasing toward the periphery are excessively rare, it does not seem necessary to dwell longer on these unusual results. No case of a secular increase of the eccentricity, owing to this peculiar distribution of density, is yet known; and it is doubtful if such a theoretical phenomenon can ever be verified by observation.

§ 80. *The General Theorem that the Eccentricity Diminishes When the Density of the Resisting Medium Increases Towards the Center, According to Any Law, Announced by* LAPLACE *in* 1805.*

Owing to the important part played by the nebular Resisting Medium in reducing the size and rounding up of the orbits of the planets and satellites during the remote ages of the past, the question of the discovery of the possibility of such a remarkable secular effect is of considerable interest. During the early work on the theory of the Resisting Medium interest was centered mainly in the decrease of the major axis and period of revolution; and but little attention was given to the changes of the eccentricity. For it was not supposed that in the present state of the solar system the alterations of the eccentricity would become sensible within the brief time covered by exact observations; and it did not occur to investigators that these small changes might be important in throwing light upon the mode of formation of our system.

Under the circumstances it seemed advisable to make an attempt to look up the records of the early work, in order to ascertain, if possible, who was the discoverer of the important law that the eccentricity usually decreases. The original papers of the earliest investigators of the secular effects of a Resisting Medium could not be found in California, and recourse had to be had to the older libraries of Europe. Mr. W. H. WESLEY, Assistant Secretary of the Royal Astronomical Society in London, and M. P. PUISEUX, Astronomer at the Paris Observatory, have been good enough to lend their aid in this examination of the original papers of early writers. As some of these scattered works of the eighteenth century probably are not to be had in America and are scarce even in Europe, it will readily be appreciated how indispensable the coöperation of these two gentlemen has been in the attainment of the results indicated below. Though these

* c.f. *A.N.* 4351

results are meager and perhaps incomplete, they are the best yet available to the modern student, and, in fact, his only guide in dealing with this highly important subject. We therefore publish the conclusions reached by WESLEY and PUISEUX, in the hope that if any further information is to be had anywhere this will operate to call it out and bring it to the attention of astronomers.

The comprehensive investigation given by LAPLACE, in the *Mécanique Céleste* (Liv. X, Chap. VII, § 18), has long been familiar to astronomers, and is presumably the earliest work of this kind; but the question arose whether any analogous result had been obtained by earlier writers, and especially by EULER and LAGRANGE, who had exhaustively treated so many of the great problems in Celestial Mechanics. When MR. WESLEY came to examine the subject, he did not find the rare works in question in the Library of the Royal Astronomical Society; but was able to obtain most of them from the older and more complete library of the Royal Society. M. PUISEUX, on the other hand, had at his disposal the very complete library of the National Observatory at Paris.

The result of MR. WESLEY's search was substantially as follows, the authors being named in chronological order:

(1). EULER has treated of the effects of a Resisting Medium in his prize paper, *Nouvelles Recherches sur le vrai mouvement de la Lune*: *Théorie de la Lune*. (Recueil des pièces qui ont remporté les prix de l'Académie Roy. des Sciences, Tome IX.) The author considers that he has entirely developed and fixed all inequalities, without exception, which can influence the moon's place by more than ten seconds; he has shown that none of these inequalities can produce a secular equation, so there can be no doubt that the secular equation observed must be due to the resistance of the medium in which the planets move. He does not pursue the matter further, but ends his "Théorie de la Lune" by the statement that he has absolutely proved that the secular equation in the motion of the moon cannot be produced by the forces of attraction alone.

MR. WESLEY also found another discussion in EULER's "Opuscula Varii Argumenti" (Berlin, 1746), the first volume of which contains a little treatise of 32 pages entitled "De Perturbatione Motus Planetarum, a resistentia aetheris orta." This is what it is called on the title page, but the chapter itself is headed "De relaxatione motus planetarum." The author discusses the question of a medium filling space, which is required for the transmission of light, and the effect of such a medium on planetary movements. It would not cause any change in the position of perihelion, and comparison with ancient observations does not appear to show any sensible change in the periodic times. (He does not mention the secular acceleration of the moon). He concludes that a resisting medium

would diminish the periodic times of bodies moving through it, *in proportion to the eccentricities of their orbits*: "Quia autem pro planetis vidimus, eorum tempora periodica eo magis ceteris paribus diminui, quo eorum orbitae magis sunt excentricae." So that, as the planets have small eccentricities, they would be little affected, but the comets more so. He instances the comet which, according to HALLEY, appeared in 1531, 1607, and 1682, showing a diminution of period of from 76 to 75 years.

It will be seen by this that EULER recognized that the changes of eccentricity would be more considerable in an elongated than in a very round orbit, but it would seem that he did not investigate the effect of the resistance for definite laws of density of the medium.

(2). BOSSUT (CHARLES). Recherches sur les alterations que la résistance de l'éther peut produire dans le mouvement moyen des planètes. 4to. Charleville, 1766.

The author says this is the work which was crowned by the Acad. Roy. de Sc., in 1762, and which will be printed by the Academy. He decides that the resistance of the ether makes no change in the position of Perihelion or Aphelion, but tends to constantly diminish the major axis of orbit, and time of revolution. He concludes that resistance of ether is shown by secular acceleration of the moon's motion, and that a secular acceleration of the earth's motion appears indicated by the observations — while for *Jupiter* and *Saturn* it cannot be sensible till after a very long period. In his conclusions he makes no mention of the eccentricity, but on page 46 he says: "Je ne parle pas ici de la petite variation qu'elle produirait dans l'excentricité des orbites planétaires et cométaires: cette variation n'est pas suffisante pour établir ou pour infirmer l'existence de la cause dont il s'agit."

(3). LAGRANGE has three memoirs in *Recueil*, Tome IX, on the Libration of the Moon, on the Inequalities of *Jupiter's Satellites*, and on the Problem of Three Bodies, respectively. He treats these subjects entirely from the point of view of attraction — as purely gravitational — and without the least mention of a resisting medium.

The general result of M. PUISEUX's investigation was similar to that of MR. WESLEY. He writes;

Le Tome VII du *Recueil des pièces qui ont remporté les prix de l'Académie Royale des Sciences* ne contient pas de mémoire de BOSSUT sur la question du milieu résistant. Mais le Tome VIII de la même collection contient deux mémoires, l'un de BUSSUT qui obtenu le prix, l'autre de J. A. EULER (le fils), qui a ramporté l'accessit. Des conclusions de deux auteurs sont les mêmes. Ils examinent

successivement le cas d'une excentricité très petite et d'une excentricité notable, mais moindre que 1. Ils trouvent que la résistance, supposée proportionelle à la vitesse ou au carré de la vitesse, doit produire sur l'excentricité une diminution, mais que cette diminution est trop faible pour être constatée par l'observation. Le seul effet mesurable, d'après eux doit être une accélération du moyen mouvement. . . . .

Dans les deux mémoires d'EULER (Tome IX du *Recueil*), il est déclaré que l'accélération séculaire du mouvement de la Lune ne peut l'expliquer que par l'influence d'un milieu résistant. . . . .

D'autres recherches que j'ai faites m'ont donné un résultat negatif. Il parait donc probable que LAPLACE est le primier à avoir signalé une diminution notable de l'excentricité comme effet possible d'un milieu résistant. Le titre du mémoire de J. A. EULER contenu dans le Tome VIII est: *Mémoire dans lequel on examine si les planètes se meuvent dans un milieu dont la résistance produise quelque effet sensible sur leur mouvement,* par J. A. EULER (1762).

From this independent examination of the principal authorities of the eighteenth century by MESSRS WESLEY and PUISEUX, we may conclude with the latter that LAPLACE was the first to establish the important law that the eccentricity decreases when the density of the resisting medium increases toward the center. This beautiful result may be called LAPLACE's Theorem. The secular action of this physical cause has left a profound impress upon the size and shape of the orbits of the planets and satellites; and it seems appropriate that the effect on the eccentricity should have been first investigated by the great mathematician to whom we owe the nebular hypothesis respecting the origin of the solar system. Though that hypothesis as originally given was incorrect, the general idea was essentially sound, and it has served us for one hundred years, and now finally led us to the true theory of the evolution of the planets and satellites.

CHAPTER VIII.

The Restricted Problem of Three Bodies, or Theory of the Motion of a Sun and Planet Revolving in Circles About Their Common Center of Gravity and Acting Upon a Particle or Satellite of Infinitesimal Mass.

§ 81. *General Remarks on the Problem of Three Bodies.*

We have already pointed out in Chapter III that the general problem of three bodies is insoluble; yet some light may be thrown upon the nature of the movement in many cases, and especially where the influence of the second and third bodies is slight. A system made up of two equal stars, or two unequal bodies such as the Sun and *Jupiter*, gives rise to a partition of all space into three distinct regions — one around each mass separately, and one around both together— within which the control is vested in these two masses separately or conjointly. There are thus two subordinate spheres of influence within which each mass dominates, if the third body is once within it and moving with a velocity so small that it cannot cross the border and escape; and a larger sphere of influence, enclosing both bodies, within which the attraction of the two masses conjointly is predominant.

If a particle therefore enters the region of such a system of two bodies, it may move around both, in a path which is not re-entrant. If it passes into either of the smaller spheres of influence, with appropriate velocity, it may remain there for a time or quickly pass out again, according to the initial velocity, direction, and other conditions of the motion. Finally, if it comes in with considerable velocity, such as we usually describe as hyperbolic, it may immediately pass out again and return to infinity. But in passing through the system its path may be greatly transformed, and it is in this way that many of the comets have been captured by planets such as *Jupiter*. All the satellites, likewise, have been captured and have had their orbits reduced in size and rounded up under the secular action of the nebular resisting medium formerly pervading our system.

Owing to the great importance of the theory of such movements, both when freely traced in empty space, and when resisted by a nebular medium, as in the

physical universe, we shall give it with some care, so as to throw all the light possible upon the obscure process by which not only the periodic comets and asteroids but also the satellites have been captured, and their orbits transformed to a state of comparatively great stability.

The theory of the capture of comets has been placed on a firm basis by the older researches of BURKHARDT and LAPLACE, and the more modern researches of ADAMS, LEVERRIER, SCHIAPARELLI, H. A. NEWTON, TISSERAND, CALLANDREAU, POOR, and other astronomers and mathematicians; but heretofore little or nothing has been published on the capture of satellites, and naturally this new theory has to be developed in considerable detail. It has always been held that the satellites originally were detached or evolved from the planets, in the *annular form*, as imagined by LAPLACE, or in some analogous way (cf. Paper on Planetary Inversion, by MR. F. J. M. STRATTON, of Cambridge, England, in the *Monthly Notices* of the Royal Astronomical Society, April, 1906).

The discussion in this and the following chapters will necessarily be incomplete, because the subject of the restricted problem of three bodies has not yet been fully developed by mathematicians; and moreover, the very brief account here given has to be limited to a condensation of the more general results obtained in recent researches on periodic orbits and related subjects. But if there is little room for new mathematical theory in so brief a sketch, perhaps the application of the results already obtained, when combined with the principles of a Resisting Medium, may prove valuable in throwing light upon the origin of the satellites of the solar system and of other similar systems existing in space.

Results of pure analytical theory, unless applied to the actual heavens, are likely to be considered too ideal to interest the astronomer and practical mathematician. Neglect of these ideal results, on the other hand, retards the progress of the physical sciences, because many phenomena remain unexplained, which might be made clear by a deeper knowledge of pure and applied Dynamics. The difficulty of connecting the analysis of the mathematician with the actual phenomena of nature has always been considerable; and of course there will be no exception in this case. But even if our first attempts are only partially successful, the improvement in our grasp of physical problems may quite repay the effort put forth, and suggest lines of thought along which the inquiry may be extended.

It is scarcely necessary to point out that the results here condensed are the outgrowth of the methods of JACOBI, HILL, POINCARÉ and DARWIN. I have made use especially of PROFESSOR SIR G. H. DARWIN's celebrated memoir on "Periodic Orbits" (*Acta Mathematica*, Vol. XXI), which was itself a development of previous conceptions due to HILL and POINCARÉ. The great advance in the Lunar Theory

made by the illustrious American geometer in 1877 rests partly on the previous work of EULER, and LAGRANGE. But it was the development of the theory resulting from the integral obtained by JACOBI, in 1836, that gave DR. HILL his most powerful methods of attacking certain problems relative to the motion of three bodies. DARWIN'S paper has the great merit resulting from the actual numerical calculation of a large number of periodic and non-periodic orbits, with a general classification of the results, so as to enable one to make out the nature of the path in many cases. And fortunately these paths are sufficiently numerous and varied to enable one to apply the results to the problems connected with the origin of the satellites, with firm confidence that the paths given are real approximations to what has occurred millions of times in the actual history of the solar system.

§ 82. *Differential Equations of Motion for a System Made Up of a Sun and Planet Revolving in Circles About Their Common Center of Gravity and of a Particle Subject to Their Attraction.*

Before proceeding we shall consider the system of special units of Mass, Space and Time originally introduced by DR. G. W. HILL in his "Researches in the Lunar Theory" (*Collected Mathematical Works*, Vol. I. p. 293), for reducing the differential equations to their simplest forms. Whatever be the relative magnitude of the sun and planet, the unit of mass may be so chosen that the sum of the two masses is unity:

$$M + m = 1 \quad ; \tag{202}$$

and the individual masses become

$$M = 1 - m \quad \text{and} \quad m \; ; \quad \text{or} \quad 1 - \mu \quad \text{and} \quad \mu , \quad \text{where} \quad \mu \leqq \tfrac{1}{2} .$$

And whatever be the interval of space separating the sun and planet, the unit of distance may be so chosen that the constant distance between the centers of the bodies $1 - \mu$ and μ shall be unity, or

$$\triangle_{1,2} = \sqrt{\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2} = 1 . \tag{203}$$

And the unit of time may be so chosen that the constant of attraction

$$k^2 = \frac{4\pi^2}{\tau^2} \frac{a^3}{M + m} = n^2 a^3 = 1 , \tag{204}$$

where τ is the periodic time, and n the mean motion of the planet about the sun (cf. GAUSS, *Theoria Motus*, Lib. I, § 1).

If with these simplifications we take the origin of coördinates at the center of gravity* of the two masses $1-\mu$ and μ, and make the plane of the coördinate axes $\xi\eta$ coincide with the plane of motion; then it is evident that the coördinates of the three bodies will become $\xi_1, \eta_1, 0_1$; $\xi_2, \eta_2, 0_2$; and ξ, η, ζ, respectively. Accordingly, we have for the radii vectores ρ_1, ρ_2 of the particle referred to the two bodies $1-\mu$, and μ:

$$\rho_1 = \sqrt{(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2} \quad ; \quad \rho_2 = \sqrt{(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2}\,. \tag{205}$$

And the differential equations of motion for the particle become

$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2} + (1-\mu)\frac{(\xi-\xi_1)}{\rho_1^3} + \mu\frac{(\xi-\xi_2)}{\rho^3} = 0\,, \\
&\frac{d^2\eta}{dt^2} + (1-\mu)\frac{(\eta-\eta_1)}{\rho_1^3} + \mu\frac{(\eta-\eta_2)}{\rho_2^3} = 0\,, \\
&\frac{d^2\zeta}{dt^2} + (1-\mu)\frac{\zeta}{\rho_1^3} + \mu\frac{\zeta}{\rho_2^3} = 0\,.
\end{aligned}\right\} \tag{206}$$

Now since by the special system of units, $M+m$, a, and k^2 are each unity, it follows that the mean angular motion also becomes unity; for we have

$$n = k\frac{\sqrt{M+m}}{a^{3/2}} = 1\,. \tag{207}$$

Let us now refer the motion of these bodies to a new system of coördinates (x, y, z). The origin remains unchanged, but the axes rotate in the plane of motion $\xi\eta$, in which the bodies $1-\mu$ and μ revolve, with uniform angular velocity $nt = t$, since by the special system of units $n = 1$ (cf. Poincaré, *Mécanique Céleste*, Tome I, p. 81). Then we have for the coördinates of the particle and of the sun referred to the rotating axes,

$$\left.\begin{aligned}
\xi &= x\cos t - y\sin t\,, \\
\eta &= x\sin t + y\cos t\,, \\
\zeta &= z\,;
\end{aligned}\right\} \tag{208}$$

$$\left.\begin{aligned}
\xi_1 &= x_1\cos t - y_1\sin t\,, \\
\eta_1 &= x_1\sin t + y_1\cos t\,, \\
\zeta_1 &= 0\,.
\end{aligned}\right\} \tag{209}$$

The equations for the coördinates of the planet are similar to those for the sun, except that the subscript 1 is changed to 2, according to the notation previously adopted.

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* In his celebrated memoir on *Periodic Orbits*, p. 102, Darwin takes the origin at the center of the Sun. This arrangement has decided advantages, but in our general examination of the question it seems best to place the origin at the center of gravity.

If now we take the second differentials of (208) and substitute these functions and derivatives in (206), we shall get:

$$
\left.\begin{aligned}
&\left\{\frac{d^2x}{dt^2} - \frac{2dy}{dt} - x\right\}\cos t - \left\{\frac{d^2y}{dt^2} + \frac{2dx}{dt} - y\right\}\sin t \\
&+ \left\{(1-\mu)\frac{(x-x_1)}{\varrho_1^{\,3}} + \mu\frac{(x-x_2)}{\varrho_2^{\,3}}\right\}\cos t - \left\{(1-\mu)\frac{(y-y_1)}{\varrho_1^{\,3}} + \mu\frac{(y-y_2)}{\varrho_2^{\,3}}\right\}\sin t = 0\,, \\
&\left\{\frac{d^2x}{dt^2} - \frac{2dy}{dt} - x\right\}\sin t + \left\{\frac{d^2y}{dt^2} + \frac{2dx}{dt} - y\right\}\cos t \\
&+ \left\{(1-\mu)\frac{(x-x_1)}{\varrho_1^{\,3}} + \mu\frac{(x-x_2)}{\varrho_2^{\,3}}\right\}\sin t + \left\{(1-\mu)\frac{(y-y_1)}{\varrho_1^{\,3}} + \mu\frac{(y-y_2)}{\varrho_2^{\,3}}\right\}\cos t = 0\,, \\
&\frac{d^2z}{dt^2} + (1-\mu)\frac{z}{\varrho_1^{\,3}} + \mu\frac{z}{\varrho_2^{\,3}} = 0\,.
\end{aligned}\right\} \tag{210}
$$

When we transform the first two of these expressions by multiplying them successively by $\cos t$ and $\sin t$, and $-\sin t$ and $\cos t$, respectively, and adding the results, we obtain:

$$
\left.\begin{aligned}
&\frac{d^2x}{dt^2} - \frac{2dy}{dt} - x + (1-\mu)\frac{(x-x_1)}{\varrho_1^{\,3}} + \mu\frac{(x-x_2)}{\varrho_2^{\,3}} = 0\,, \\
&\frac{d^2y}{dt^2} - \frac{2dx}{dt} - y + (1-\mu)\frac{(y-y_1)}{\varrho_1^{\,3}} + \mu\frac{(y-y_2)}{\varrho^{\,3}} = 0\,, \\
&\frac{d^2z}{dt^2} + (1-\mu)\frac{z}{\varrho_1^{\,3}} + \mu\frac{z}{\varrho_2^{\,3}} = 0\,.
\end{aligned}\right\} \tag{211}
$$

These equations may be still further simplified by taking the position of the x-axis at the origin of time so that it coincides with, and therefore ever afterwards passes through, the centers of the two bodies $1-\mu$ and μ, which makes $y_1 = 0$, $y_2 = 0$. Accordingly, we have finally

$$
\left.\begin{aligned}
&\frac{d^2x}{dt^2} - \frac{2dy}{dt} - x + (1-\mu)\frac{(x-x_1)}{\varrho_1^{\,3}} + \mu\frac{(x-x_2)}{\varrho_2^{\,3}} = 0\,, \\
&\frac{d^2y}{dt^2} + \frac{2dx}{dt} - y + (1-\mu)\frac{y}{\varrho_1^{\,3}} + \mu\frac{y}{\varrho_2^{\,3}} = 0\,, \\
&\frac{d^2z}{dt^2} + (1-\mu)\frac{z}{\varrho_1^{\,3}} + \mu\frac{z}{\varrho_2^{\,3}} = 0\,;
\end{aligned}\right\} \tag{212}
$$

which are the differential equations of the motion of a particle referred to rectangular axes rotating with uniform angular velocity, corresponding to the gravitational attraction of the sun and planet, and having the x-axis always coinciding with the line joining the two principal masses. This system of coördinates has been so arranged as to involve as variables only the coördinates of the particle; for the coördinates of the two larger bodies are constant, owing to the manner in which the axes have been adjusted and set in rotation at the initial epoch $t = 0$.

It will be seen from equations (212) that the solution of these differential equations is of the sixth order, but we may reduce the solution two orders lower by restricting the motion of the particle to the plane of the principal bodies, as is done in DARWIN's memoir on *Periodic Orbits*. In this case the equation in z vanishes, and only those in x and y remain; and we have simply

$$\left.\begin{aligned}\frac{d^2x}{dt^2}-\frac{2dy}{dt}-x+(1-\mu)\frac{(x-x_1)}{\sqrt{(x-x_1)^2+y^2}}+\frac{\mu(x-x_2)}{\sqrt{(x-x_2)^2+y^2}}=0\,,\\ \frac{d^2y}{dt^2}+\frac{2dx}{dt}-y+(1-\mu)\frac{y}{\sqrt{(x-x_1)^2+y^2}}+\frac{\mu y}{\sqrt{(x-x_2)^2+y^2}}=0\,;\end{aligned}\right\} \tag{213}$$

the solution of which is of the fourth order.

§ 83. JACOBI's *Integral of the Differential Equations for the Motion of the Particle Referred to the Rotating Axis.*

In the *Comptes Rendus de l'Académie des Sciences de Paris*, Tome III, p. 59, the celebrated mathematician K. G. J. JACOBI has given an integral of equations (212), which is of high importance in the restricted problem of three bodies. It has been further discussed by PROFESSOR G. W. HILL, in his famous papers on the Lunar Theory, entitled: "On the Part of the Motion of the Lunar Perigee which is a function of the Mean Motions of the Sun and Moon," and "Researches in the Lunar Theory." The first of these papers was printed privately at Cambridge, Mass., in 1877; and has since been reprinted in *Acta Mathematica*, Vol. VIII, pp. 1–36, 1886; also in the *American Journal of Mathematics*, Vol. I. p. 18; and in HILL's Collected Mathematical Works, Vol. I, pp. 243–270. The "Researches in the Lunar Theory" were communicated to the National Academy of Sciences in April, 1877, and printed in Vol. I of the *American Journal of Mathematics*, 1878; and recently reprinted in HILL's Collected Works, Vol. I, pp. 284–334. The subject has been treated also by DARWIN in his well known paper on "Periodic Orbits," in *Acta Mathematica*, Vol. XXI, p. 102, 1897, and by others (cf. WHITTAKER's *Analytical Dynamics*, p. 342).

If we put for brevity

$$2\Omega = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2}, \tag{214}$$

it will be found that equations (212) may be reduced to the form

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \frac{\partial\Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \frac{\partial\Omega}{\partial y}, \\ \frac{d^2z}{dt^2} \qquad\quad &= \frac{\partial\Omega}{\partial z}. \end{aligned}\right\} \tag{215}$$

As Ω is a function of x, y, z only, these equations may be integrated by multiplying them successively by $2dx$, $2dy$, $2dz$, respectively, and adding the products. This makes the sum of the second members an exact differential, and on multiplying by dt and integrating we get

$$2\int\left\{\frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} + \frac{\partial\Omega}{\partial z}\frac{dz}{dt}\right\}dt = 2\Omega - C\ .$$

Therefore for the integral of (215) we have

$$\left.\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= V^2 = 2\Omega - C \\ &= x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C\ , \quad \text{(by 214)}. \end{aligned}\right\} \tag{216}$$

This is the integral obtained by JACOBI in 1836, and therefore called by DR. HILL the Jacobian Integral (cf. HILL'S *Collected Mathematical Works*, Vol. I, p. 244). But as the three differential equations (215) are of the second order, five more integrals would be required to give a complete solution of the problem.

In case the motion of the particle were restricted to the xy-plane, the differential equations depending on z would be excluded, and three others besides this integral of JACOBI would suffice. If the first of these three integrals were obtained, the last two could be found by means of JACOBI'S *Last Multiplier*, as explained in the celebrated *Vorlesungen über Dynamik* (cf. also WHITTAKER'S *Analytical Dynamics*, pp. 271–272). But this first integral has not been obtained, and even the solution in a plane remains incomplete. In the *Acta Mathematica*, Vol. XI, BRUNS has proved that no new algebraic integrals exist; and in *Les Methodes Nouvelles de la Mécanique Céleste*, Tome I, Chap. V, POINCARÉ has shown that there are no new uniform transcendental integrals, even when one of the two large bodies, as μ, is very small compared with the other $1 - \mu$ (cf. WHITTAKER'S *Analytical Dynamics*, Chap. XIV, pp. 346–373).

Now the integral $V^2 = 2\Omega - C$, above given represents the velocity of the particle under the attraction of the two bodies, and has therefore been called by LORD KELVIN (cf. *Philosophical Magazine*, Vol. 34, p. 34) and afterwards by

DARWIN, the equation of *relative energy*. And since $V^2 = 2\Omega - C = 0$ will represent a zero velocity, it is clear that $2\Omega = C$ gives the critical values of the family of curves of the more general tpye $2\Omega = C + V_0^2$, where V_0 is any constant velocity. The curves $2\Omega = C$ therefore define the regions of space where the velocity is zero, and the particle moves for an instant as if rigidly connected with the rest of the system. The particle cannot cross the boundaries thus established but must turn back toward the general direction from whence it came. We shall now examine into the nature of these portions of relative space.

§ 84. *The Equation of Relative Energy and the Surfaces of Zero Relative Velocity.*

Equation (216) gives us a relation between the square of the velocity in tridimensional space, and the coördinates of the particle referred to the rotating axes. If therefore the constant of integration C can be determined numerically, by the initial conditions of the motion or otherwise, this integral determines the velocity of the particle at all points of the rotating space. On the other hand, when we have the velocity given for some one point, as for example by observation in the case of actual heavenly bodies, this integral of JACOBI gives the locus of all those points of the relative space where the particle may move. Finally if V is made zero the equation $2\Omega = C$ or

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} = 0\,, \tag{217}$$

will define the surfaces at which the relative velocity will be zero, or the three bodies may move for an instant as parts of a single rigid body (cf. DARWIN'S *Periodic Orbits*, p. 106). By varying the parameter C within sufficiently wide limits, we get the entire family of surfaces covering all space throughout the universe.

On one side of these surfaces the velocity will be real, on the other side imaginary. The equation of the surfaces of relative zero velocity given by JACOBI'S integral is therefore

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2 + z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2 + z^2}} = C\,. \tag{218}$$

These surfaces were first discussed by DR. G. W. HILL, in his famous "Researches in the Lunar Theory" (HILL'S *Collected Works*, Vol. I, pp. 294–304), and are well known; but must here be treated with sufficient fullness to render the course of the subsequent reasoning clear and intelligible. For it is on the use of HILL'S closed surfaces about the planets that the capture theory of satellites is shown

to depend. In fact the capture of satellites necessarily results from the shelter afforded stray bodies by these closed surfaces, as soon as we introduce the secular action of the resisting medium. These surfaces operate to gather small bodies into these sequestered regions and fix them there, as inevitably as water runs down hill and collects in low places, or a wheel on the edge of a rut settles to the bottom before it finally attains a position of stability. The downward path for the whole system, however, is always one of *Least Action*, as will be more fully established in §§ 88–90.

If z is made zero we obtain the equations for the intersection of the surfaces with the xy-plane, thus:

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C\,. \tag{219}$$

In the equation (218) y and z occur only in the second power, and therefore the surfaces are symmetrical with respect to the xy and xz-planes. And if $\mu = \frac{1}{2}$ the surfaces are symmetrical with respect to the yz-plane also. But in the more general case of $\mu \leqq \frac{1}{2}$, the surfaces are deformations of the symmetrical ones arising in the special case $\mu = \frac{1}{2}$. As z occurs symmetrically and only in the second power, it is evident that a line parallel to the z-axis which pierces the surface at all will do so in two real points. When the corresponding roots of the equation are imaginary, the line does not pierce the surface. For, as DR. HILL remarks, these surfaces are contained within a cylinder made by revolving about the z-axis a line parallel to it at the distance $\sqrt{C}$. This is the radius of the cylinders which become asymptotic to the surfaces at $z^2 = \infty$. For when z^2 increases indefinitely, equation (218) approaches the limit $x^2 + y^2 = C$, which makes $\sqrt{C}$ the radius of the asymptotic cylinders. As already remarked the properties of these surfaces were first outlined by DR. G. W. HILL in his celebrated "Researches in the Lunar Theory," 1877 (cf. HILL's *Collected Works*, Vol. I, pp. 300–303). It seems advisable to discuss them somewhat carefully in the following sections, and to illustrate the equations by figures such as have been calculated by HILL, DARWIN, and others.

§ 85. *Nature of the Surfaces of Constant Relative Energy on the xy-Plane.*

If now we examine equation (219) we see that for large values of x and y, which satisfy the equation, the last two terms of the left member become small, and we may write the expression in the form

$$x^2 + y^2 = C - \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} - \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C - \epsilon\,. \tag{220}$$

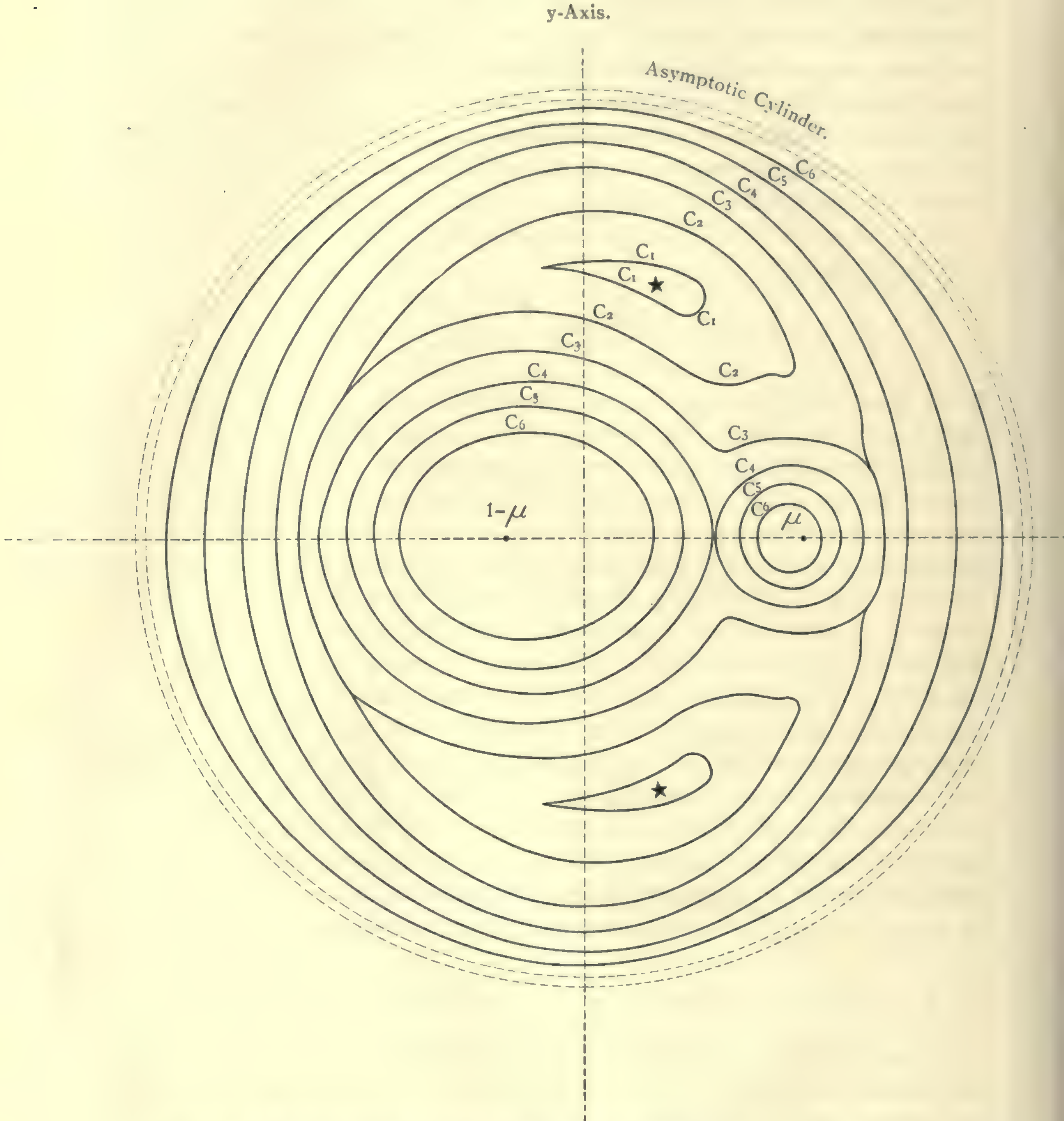


FIG. 16. THE SURFACES OF CONSTANT RELATIVE ENERGY AS TRACED ON THE xy-PLANE, THE RATIO OF THE MASSES BEING ABOUT AS 4 TO 1.

where ϵ is a quantity which may be taken as small as we please. But this is obviously the equation to a circle whose radius is $\sqrt{C - \epsilon}$. Accordingly, one branch of the curve in the xy-plane approaches a circle within the asymptotic cylinder. And the larger the constant C is, the larger are the resulting values of x and y; and consequently the smaller the value of ϵ, and the more nearly circular the curve becomes as it approaches the asymptotic cylinder.

On the other hand, very small values of x and y satisfying (219) make the first two terms unimportant, and the resulting expression may be reduced to the form

$$\frac{1-\mu}{\sqrt{(x-x_1)^2+y^2}} + \frac{\mu}{\sqrt{(x-x_2)^2+y^2}} = \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} = \frac{C}{2} - \frac{x^2+y^2}{2} = \frac{C}{2} - \epsilon\,. \tag{221}$$

(cf. HILL's "Researches in the Lunar Theory," *Collected Mathematical Works*, Vol. I, pp. 297–304; DARWIN's memoir on "Periodic Orbits," *Acta Mathematica*, Vol. XXI, p. 106).

If we examine this simplified expression, we shall find that it is the expression for the *equipotential curves* for the two centers of force, $1 - \mu$ and μ. When the

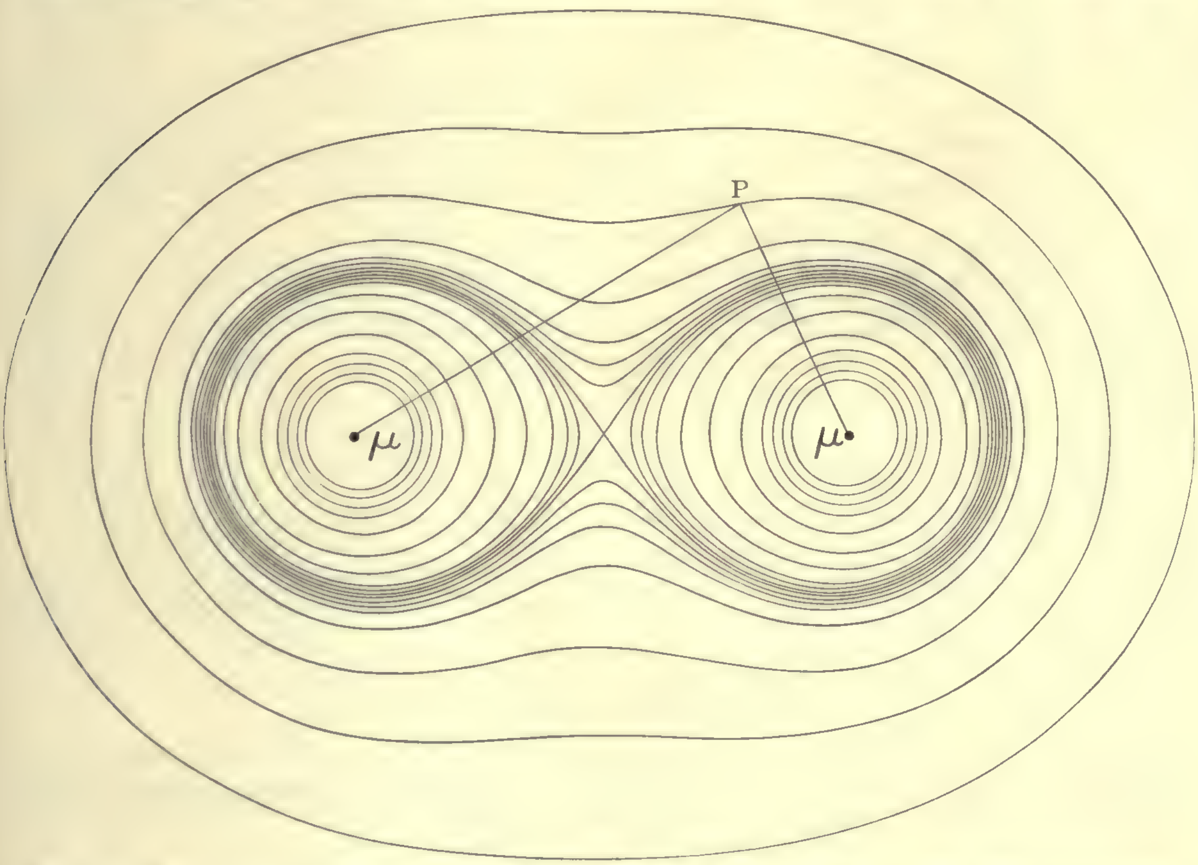


FIG. 17. EQUIPOTENTIAL SURFACES ABOUT TWO EQUAL MASSES, SUCH AS WE OFTEN FIND IN A TYPICAL DOUBLE STAR.

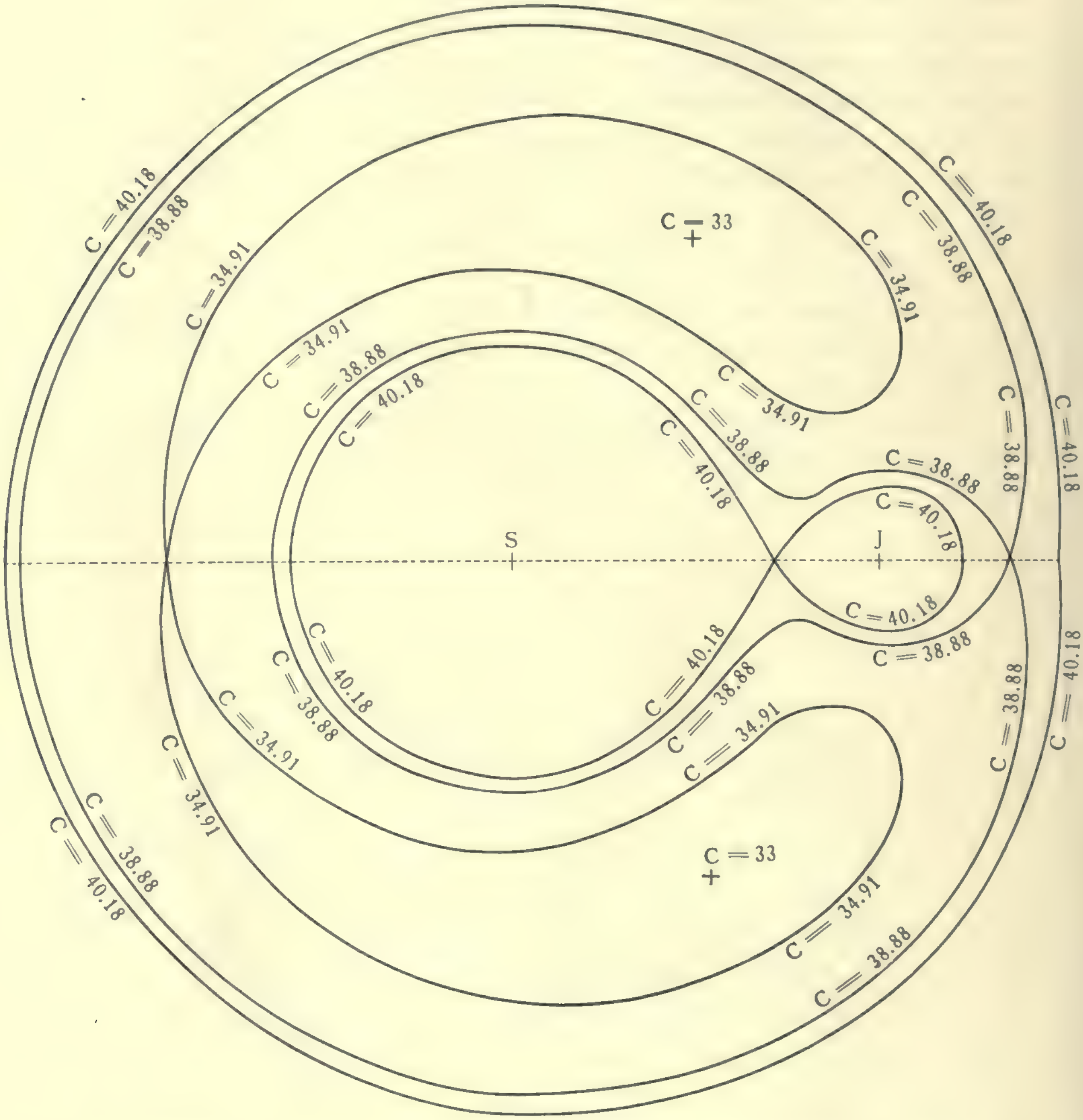


FIG. 18. DARWIN'S DIAGRAM OF THE CURVES OF CONSTANT RELATIVE ENERGY IN THE xy-PLANE, THE RATIO OF THE MASSES BEING AS 10 TO 1.

values of C are large these curves consist of closed ovals around each of the bodies $1-\mu$ and μ; but as this constant decreases, the surfaces recede from the two centers and at length unite between the bodies, giving an hour-glass shaped figure, with unequal bulbs. As C decreases still further, the curve becomes an oval enclosing both bodies, which passes by degrees from a figure resembling an ellipse to that of a circle at a considerable distance from the origin. The accompanying illustration exhibits the arrangement of such equipotential curves about two equal masses μ and μ, such as a typical double star. If these curves were revolved about the line joining the two bodies the curves would generate the corresponding equipotential surfaces (cf. THOMSON and TAIT's *Natural Philosophy*, Vol. I, Part II, § 508), which, however, would correspond to a static rather than to a kinetic system, for when the bodies revolve like a double star the surfaces are distorted by the effects of centrifugal force, and become of greater extent in the direction parallel to the plane of the orbit.

The next illustration exhibits the curves on the xy-plane in the typical case $1-\mu = 10\mu$, calculated by PROFESSOR SIR G. H. DARWIN in his celebrated paper on "Periodic Orbits" in *Acta Mathematica*, Vol. XXI. The curves here traced by DARWIN are of the highest interest, because they show the sub-divisions of space accurately calculated and drawn to scale.

The method of calculating these curves is outlined in HILL's "Researches in the Lunar Theory" (*Collected Works*, pp. 284–334), and in DARWIN's memoir on "Periodic Orbits," pp. 106–118. It is explained also in the memoirs of GYLDÈN and in a paper of BOHLIN in the *Acta Mathematica*, Tome X, p. 109. A brief account of these surfaces may be found also in MOULTON's "Introduction to Celestial Mechanics," pp. 193–196.

§ 86. *The Curves of Constant Energy on the xz and yz-Planes.*

If we wish to find the forms of the curves of intersection made by the surfaces on the xz-plane, we put y equal to zero in equation (218) and trace the curves given by the expression

$$x^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2+z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2+z^2}} = C. \qquad (222)$$

If the values of x and z satisfying this equation are large, nothing but the first term remains important; and hence we may put

$$x^2 = C - \epsilon \quad , \quad \text{or} \quad x = \sqrt{C-\epsilon}\,.$$

which gives a symmetrical pair of straight lines parallel to the z-axis.

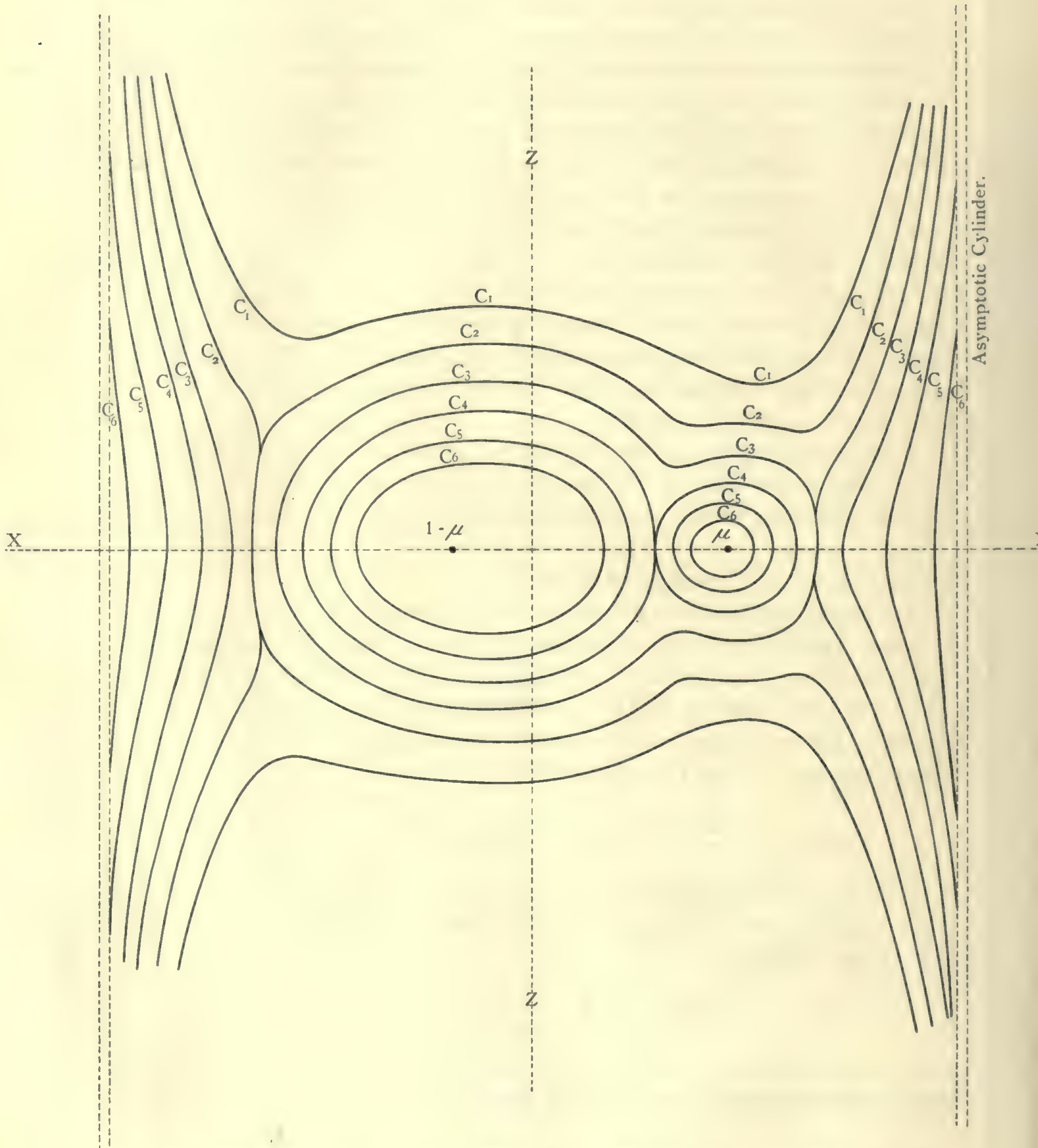


FIG. 19. THE SURFACES OF CONSTANT RELATIVE ENERGY AS TRACED ON THE *xz*-PLANE, THE RATIO OF THE MASSES BEING ABOUT AS 4 TO 1.

Consider next the effect of varying the constant C. It is evident from equation (222) that the larger C is the larger is the value of x corresponding to a given value of z, satisfying the equation; and consequently the smaller is ϵ. Therefore the larger values of C give lines nearer and nearer to the asymptotic cylinder. On the other hand small values of x and z satisfying (222) will make the first term relatively unimportant, and hence we may put the expression in the form

$$\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}=\frac{C}{2}-\epsilon\,;$$

which likewise represents a system of equipotential curves, with properties similar to those described above. The general character of these curves of constant relative energy is shown in the accompanying figure, but their forms are not calculated with great accuracy.

In like manner, when we put x equal to zero, we get the curves made by the intersection of the surfaces on the yz-plane:

$$y^2+\frac{2(1-\mu)}{\sqrt{x_1^2+y^2+z^2}}+\frac{2\mu}{\sqrt{x_2^2+y^2+z^2}}=C\,. \tag{223}$$

Large values of y and z satisfying this equation make the second and third terms unimportant, and therefore we may put

$$y^2=C-\epsilon\quad,\quad\text{or}\quad y=\sqrt{C-\epsilon}\,.$$

This is the equation of a pair of lines near the asymptotic cylinder, which is approached more and more as C increases. When one body, $1-\mu$, is much larger than the other, μ, the numerical value of x_2 is much greater than that of x_1; and therefore for small values of y and z satisfying the equation (223), the expression becomes

$$\frac{2(1-\mu)}{\rho_1}=C-\epsilon\,.$$

This is the equation of a circle, which enlarges as C decreases.

With these considerations before us, we see that the forms of the curves on the yz-plane are as shown in the accompanying figure.

From these observations it is easy to make out the general character of the surfaces of constant relative energy. For large values of C they consist of two distinct parts:

(1). A closed fold somewhat resembling a Jacobian Ellipsoid of three unequal axes around each body and pointed in each case end-on, with the extreme points tending to coalesce like the neck of an hour-glass.

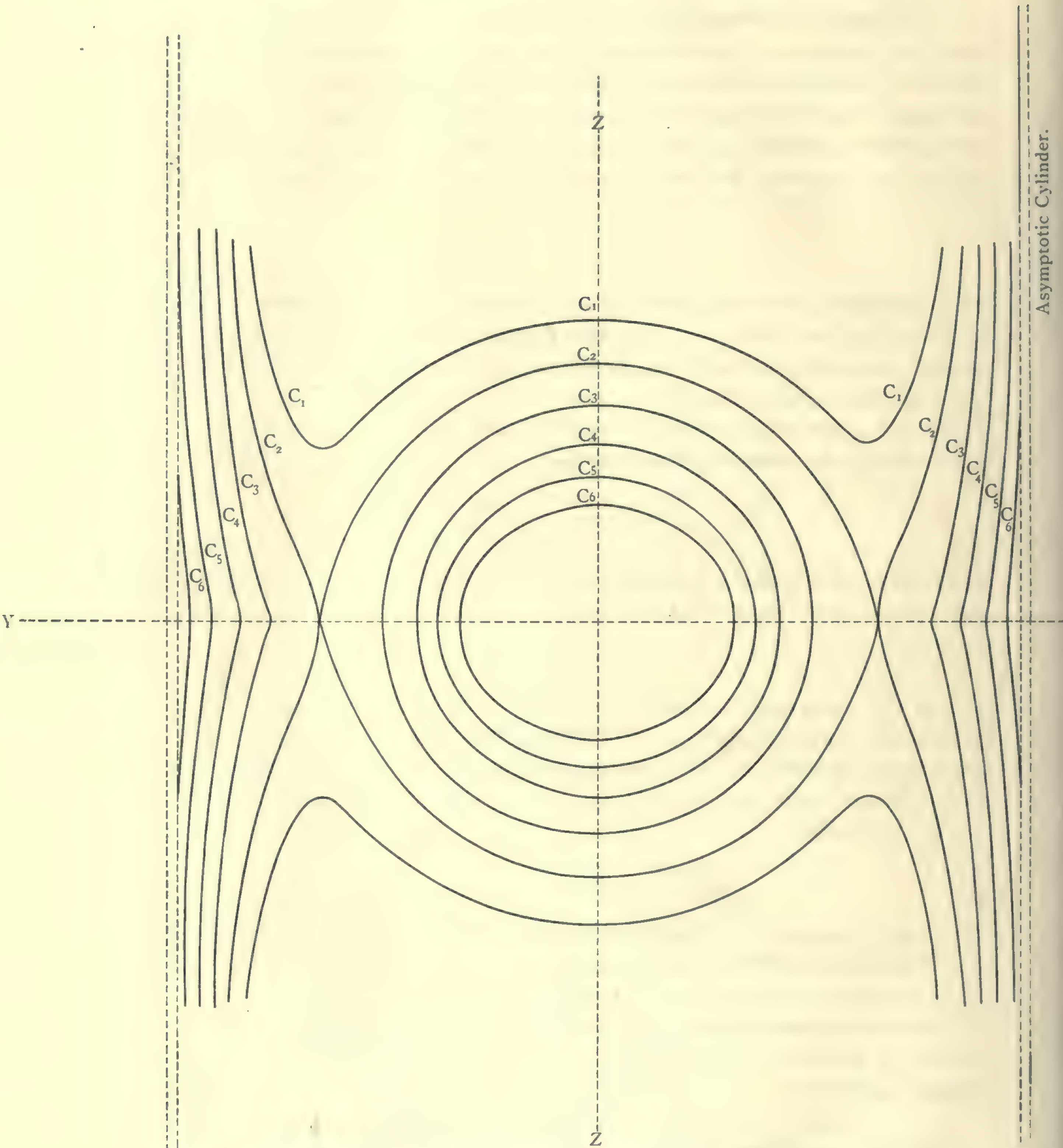


FIG. 20. THE SURFACES OF CONSTANT RELATIVE ENERGY AS TRACED ON THE yz-PLANE, THE RATIO OF THE MASSES BEING ABOUT AS 4 TO 1.

(2). A pair of curtains hanging down from the asymptotic cylinder and symmetrically arranged with the respect to the xy-plane. With smaller values of C these two types of surfaces approach each other, and finally coalesce in the xy-plane; the folds around the two bodies also unite into one surface enclosing both bodies. It would be very interesting to have the forms of these closed surfaces accurately calculated for various mass-ratios; for as we shall see hereafter true nebular fission depends on the forms of these surfaces and the initial distribution of the nebulosity.

§ 87. *On the Regions of Real and Imaginary Velocity and on the Velocity from Infinity.*

The expression for the square of the velocity is

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C. \tag{224}$$

We have seen that when C is large the energy surface is so constructed that the ovals about the bodies are distinct from the curtains which hang down from the asymptotic cylinder (cf. HILL's *Collected Works*, Vol. I, pp. 300–303). Now if the right member of this equation (216) is positive, the velocity will be real in the corresponding portion of relative space. Thus if it is positive at any point of a closed fold it will be positive for every other point within it, because the function changes sign only at a surface of zero relative velocity. In figure 18 representing the surfaces on the xy-plane, we see that on either side of SJ, the line joining Sun and Planet, there is a horse-shoe figure with lines around it. The space within is one of the regions of imaginary velocity, and the particle of course cannot cross it, for it is enclosed by surfaces of zero relative velocity.

A comet moving in a parabola corresponds to a body with velocity from infinity, and as this is sufficient to carry it back again, it will not become attached to the solar system unless considerably disturbed by the planets near perihelion passage. The surfaces around our actual planets are comparatively small, because their masses are very small compared to that of the sun.

Accordingly, we see that when C is so large that the folds around the bodies are closed, and the particle is within these closed regions at the origin of time, the system then moving for an instant as a single rigid body, it will always remain there. For it cannot escape without crossing the surface of zero relative velocity and that is impossible.* Taking the moon's mass to be insensible and the earth's orbit

* The action of a fourth body, however, might so disturb these surfaces as to enable the third body to escape; and conversely it might operate to effect the capture of the third body. We are here considering the case of three bodies only, with the motions of the two larger ones restricted to circles, and the third a particle of infinitesimal mass.

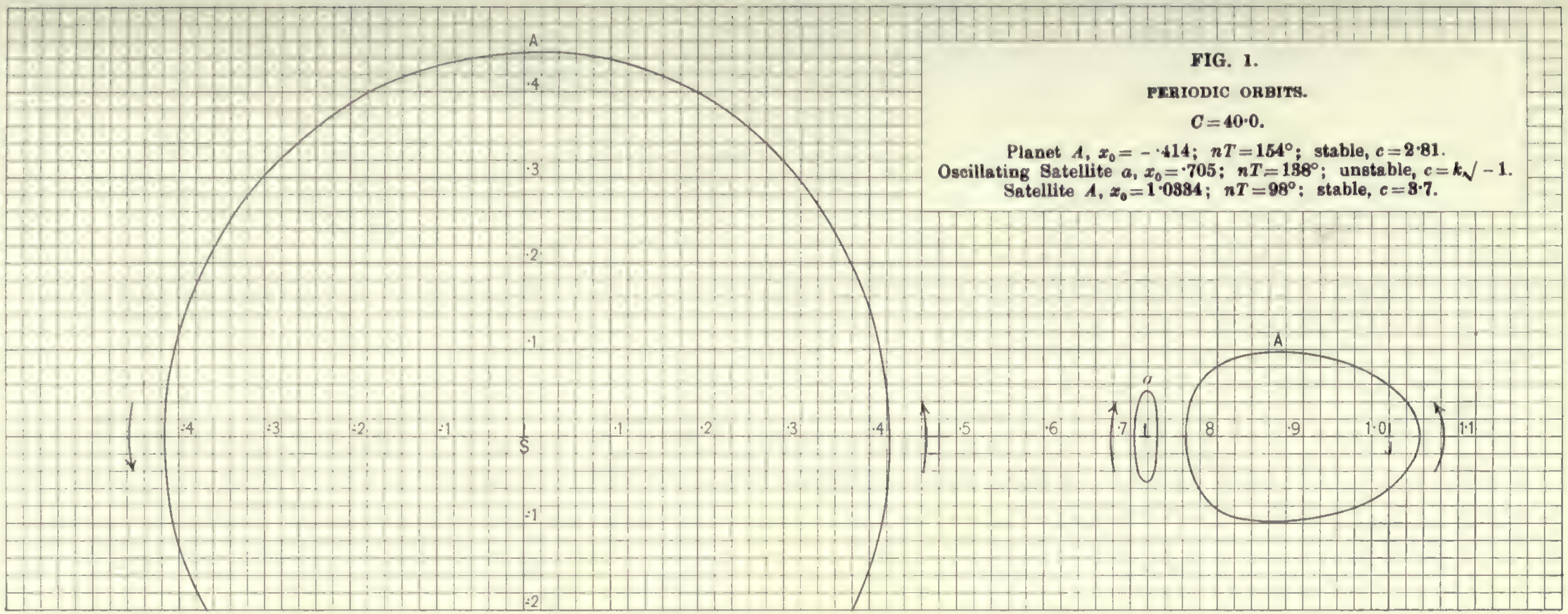
circular, Dr. Hill showed in his famous memoir on the Lunar Theory, in 1877, that the surface about the earth is closed, and consequently the moon cannot now escape from the earth, but has a superior limit to its maximum distance. This corresponds to a long lunation of 204.896 days calculated by Hill; but, as was first pointed out by Adams and afterwards by Poincaré, the maximum lunation is still longer. In his *Mécanique Céleste*, Tome, I, p. 109, Poincaré has shown that if the mean period of the moon relative to that of the sun exceeds 1:2.78, the curve found by Dr. Hill becomes looped, and such a moon would appear in quadrature six times during one revolution (cf. Hill's *Collected Mathematical Works*, Vol. I, pp. 326–334). A looped orbit similar to that found by Poincaré has been drawn also by Lord Kelvin, in the *Philosophical Magazine*, Vol. 34, p. 447. But Darwin has pointed out that both Kelvin and Poincaré have neglected the solar parallax in their calculation of these looped orbits; so that the results obtained do not correspond to the ideal solution of the problem under consideration.

In the same way, Moulton has calculated the value of the Jacobian Integral for the region about *Saturn*, and found that the ninth satellite, *Phœbe*, is safely within the enclosed folds about *Saturn*, and therefore revolves in stability. Likewise the closed folds about *Jupiter* have been found by Cowell and Crommelin to be large enough to include the orbit of the eighth satellite. Being once within this region, and having a rather small velocity to start with, it too, may always revolve in safety and cannot now escape from the planet's control. This subject is treated more at length in Chapter X of this work, and we therefore defer further discussion of it till other problems have been examined.

But although all these satellites are now under the control of their respective planets, and will remain so indefinitely hereafter, it does not follow that because they cannot now escape they have for that reason not originally come in from a great distance. On the contrary we shall see later that they have all been added to these systems from without, and that their orbits have since been reduced in size, and rounded up under the secular action of a resisting medium; so that although all the satellites once moved around the sun like the planets and comets now do, they suffered great transformation of their orbits when near the planets, by leaving the sun's control and revolving around them many times, and under the effects of resistance have become permanent satellites of the planets which finally captured them.

We give here for comparison with Hill's Periodic Orbit of the Moon a figure of the periodic orbit with loops found by Poincaré; also Plate V from Darwin's memoir which shows the forms of some other periodic orbits.

Of late years the subject of Periodic Orbits has occupied the attention of a



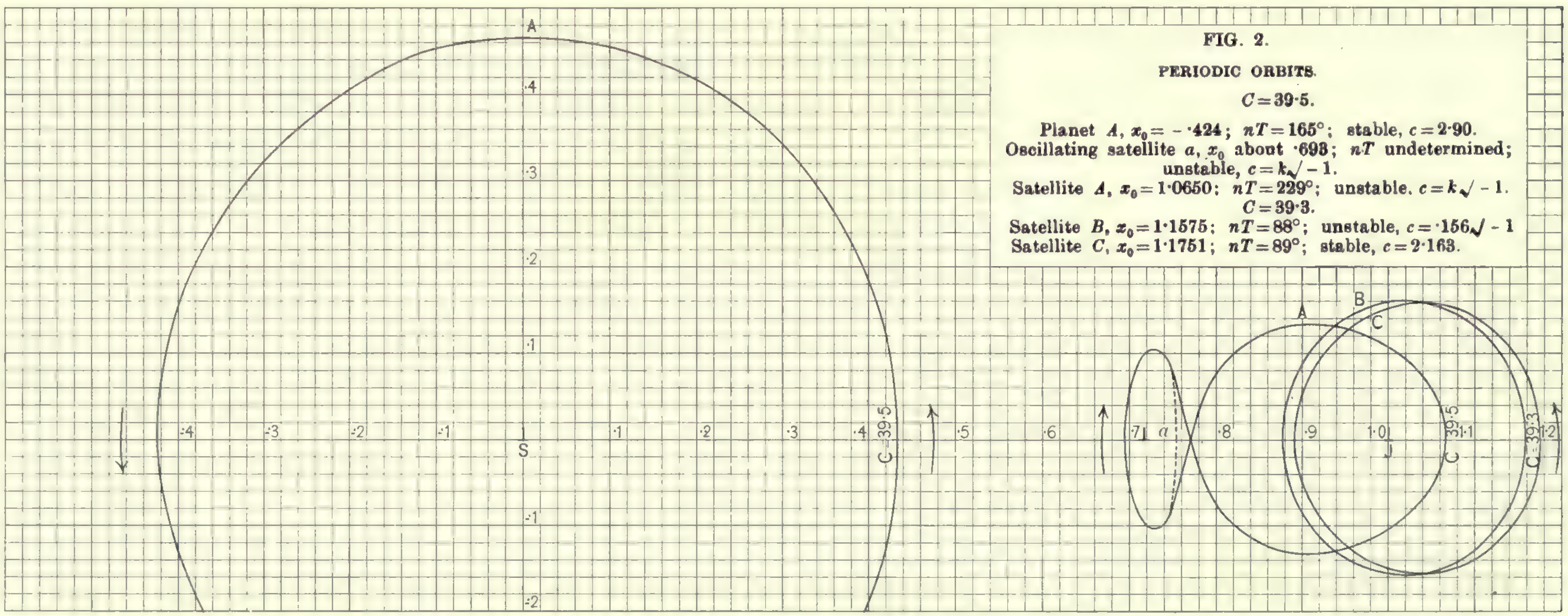


PLATE V. THE FIRST PLATE OF DARWIN'S MEMOIR, SHOWING SEVERAL TYPES OF PERIODIC ORBITS.

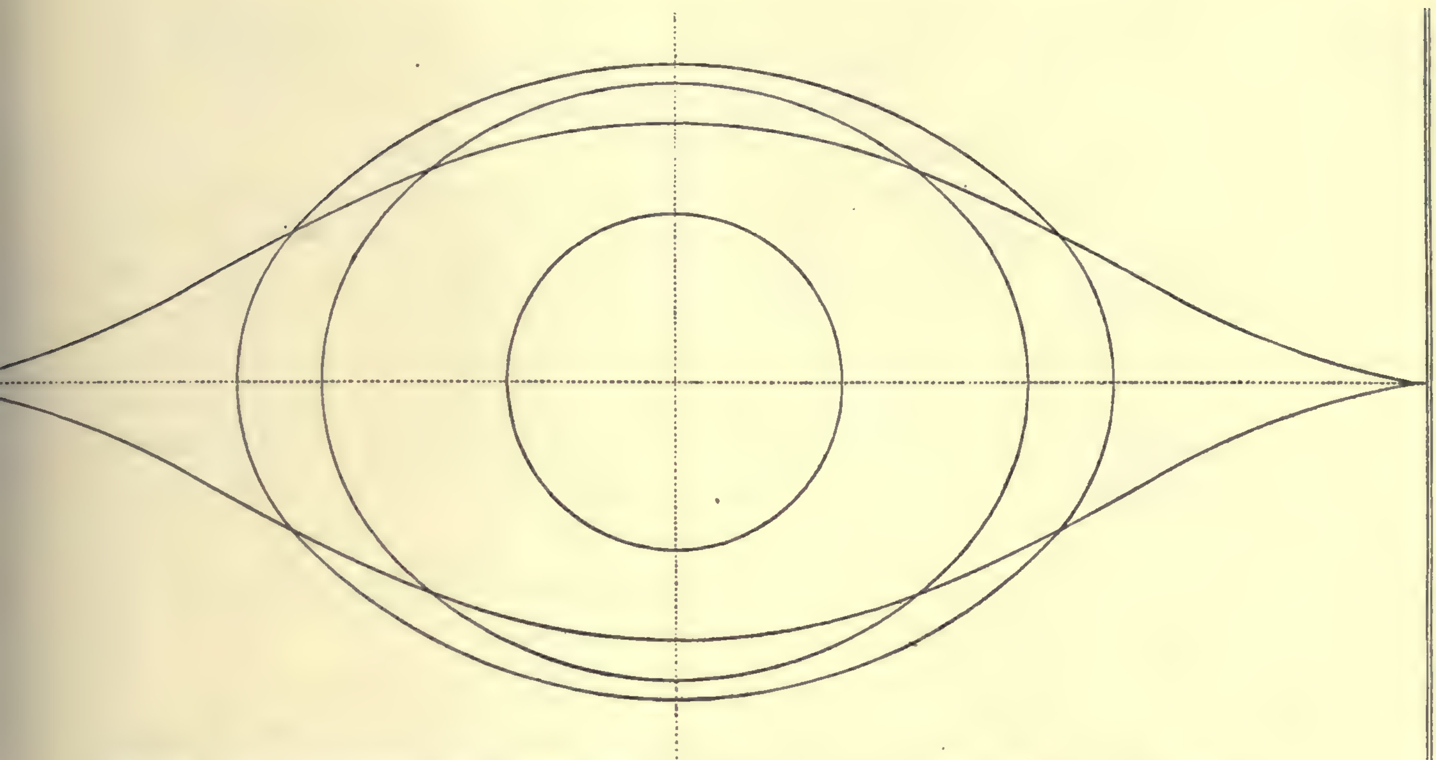


FIG. 21. HILL'S PERIODIC ORBIT OF MOON OF LONG LUNATION, TERMINATING WITH CUSPS, THE SUN BEING BELOW, AT RIGHT ANGLES TO THE LINE JOINING THE CUSPS.

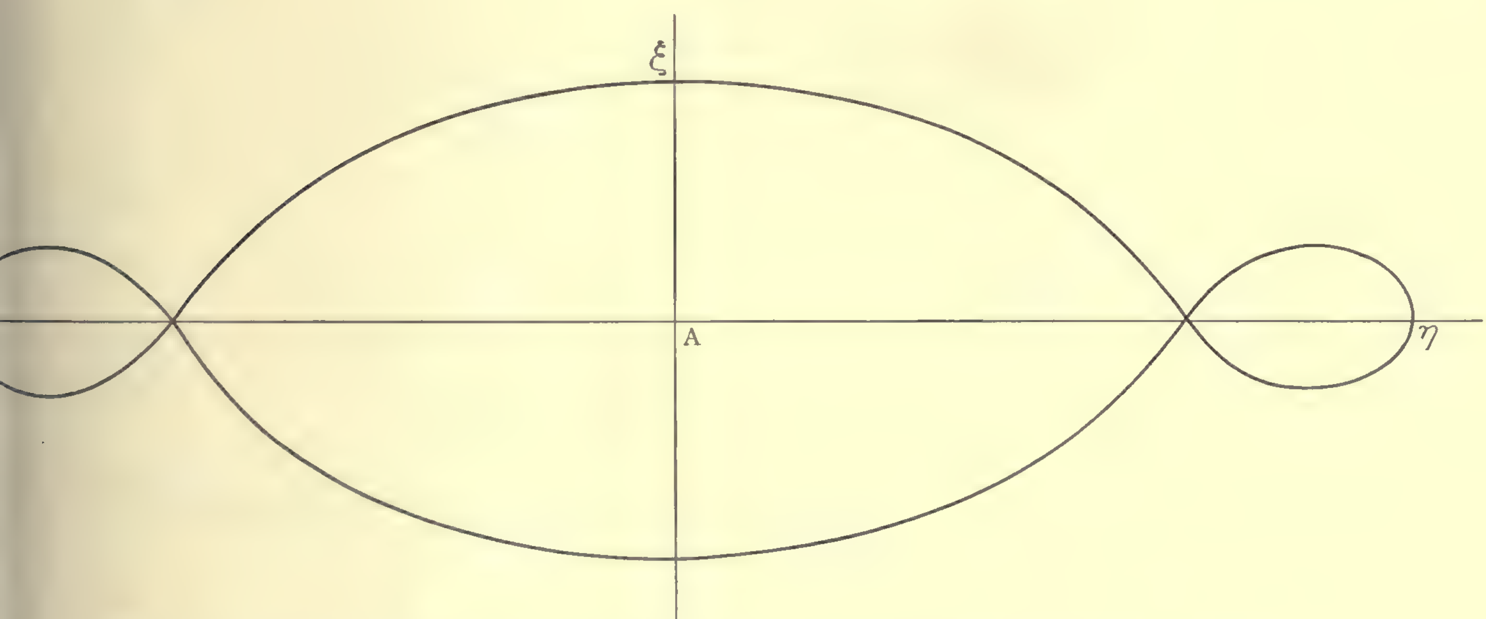


FIG. 22. POINCARÉ'S PERIODIC ORBIT OF MOON OF MAXIMUM LUNATION, WITH LOOPS, GIVING SIX QUADRATURES DURING ONE REVOLUTION.

number of eminent mathematicians, and the hope has been entertained that the solution of these problems would give us the Laws of Cosmical Evolution (cf. DARWIN's Presidential Address to the British Association at Cape Town, 1905). But it may well be doubted whether after all much new light is to be expected from this line of inquiry. If the views developed in Chapter X of this work be admissible, it will follow that *stable periodic orbits* seldom if ever develop in actual nature, because of the paramount part played by the resisting medium. All orbits are temporary, and subject to gradual transformation. And whilst the conditions that give rise to periodic orbits throw light on the stability of motion, and are therefore important, it will be found that the Laws of Cosmical Evolution cannot be thus deduced, but must be sought in the processes of nebular fission by the capture of satellites, as outlined in Chapter X.

§ 88. *The Principle of Least Action as Applied to the Dynamics of a Conservative System.*

MAUPERTUIS' celebrated principle of *Least Action* is stated by THOMSON and TAIT (*Natural Philosophy*, Part I, § 327) as follows: "Of all the different sets of paths along which a conservative system may be guided to move from one configuration to another, with the sum of its potential and kinetic energies equal to a given constant, that one for which the action is the least is such that the system will require only to be started with the proper velocities, to move along it unguided." If T denote the kinetic energy at any time between the epoch and t, m_i the mass of an element of the system, and v_i its velocity, then we shall have for the entire system

$$T = \tfrac{1}{2}\sum_{i=0}^{i=i} m_i v_i^2 . \tag{225}$$

And the action at the time t is

$$A = 2\int_0^t T d\tau = \int_0^t \sum_{i=0}^{i=i} m_i v_i^2 d\tau . \tag{226}$$

If ds denote the space described by a particle in time $d\tau$, so that $vd\tau = ds$, we shall have

$$A = \int_0^t \sum_{i=0}^{i=i} m_i v_i ds . \tag{227}$$

Now in Kinematics we have

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

and by differentiation we obtain the *acceleration in curvilinear motion*, namely:

$$\frac{d^2s}{dt^2} = \frac{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}. \tag{228}$$

Whatever be the direction of the motion or the curvature under any system of forces, this expresses the acceleration along the path, in tridimensional space, or the change of the velocity. Accordingly, if x_i, y_i, z_i be the coördinates of the mass m_i, we have

$$A = \int_0^t \sum_{i=0}^{i=i} m_i \left\{ \frac{dx}{dt} dx + \frac{dy}{dt} dy + \frac{dz}{dt} dz \right\}. \tag{229}$$

The action of the system is therefore equal to the *sum of the average momentums for the spaces described by the elements m_i, each multiplied by the length of its path.* In *Least Action* it is required to find by the method of variations

$$\delta A = \int_0^t \sum_{i=0}^{i=i} m_i \left\{ \frac{dx}{dt} d\delta x + \frac{dy}{dt} d\delta y + \frac{dz}{dt} d\delta z + \delta \frac{dx}{dt} dx + \delta \frac{dy}{dt} dy + \delta \frac{dz}{dt} dz \right\} = 0. \tag{230}$$

Now since

$$dx = \frac{dx}{dt} d\tau \quad , \quad dy = \frac{dy}{dt} d\tau \quad , \quad dz = \frac{dz}{dt} d\tau \, ,$$

and

$$\sum_{i=0}^{i=i} m_i \left\{ \frac{dx}{dt} \delta \frac{dx}{dt} + \frac{dy}{dt} \delta \frac{dy}{dt} + \frac{dz}{dt} \delta \frac{dz}{dt} \right\} = \delta T \, , \tag{231}$$

we have

$$\int_0^t \sum_{i=0}^{i=i} m_i \left\{ \delta \frac{dx}{dt} dx + \delta \frac{dy}{dt} dy + \delta \frac{dz}{dt} dz \right\} = \int_0^t \delta T d\tau \, . \tag{232}$$

If we integrate (230) by parts, we find

$$\left.\begin{aligned}\int_0^t \sum_{i=0}^{i=i} m_i \left\{ \frac{dx}{dt} d\delta x + \ldots \right\} = \left\{ \sum_{i=0}^{i=i} m_i \left(\frac{dx}{dt} dx + \ldots \right) \right\}_{t=t} \\ - \left\{ \sum_{i=0}^{i=i} m_i \left(\frac{dx}{dt} dx + \ldots \right) \right\}_{t=0} - \int_0^t \sum_{i=0}^{i=i} m_i \left(\frac{d^2x}{dt^2} \delta x + \ldots \right) d\tau \,.\end{aligned}\right\} \tag{233}$$

In this expression $d\frac{dx}{dt} = \frac{d^2x}{dt^2} d\tau$, etc.; and therefore the variation in the action from (230) is

$$\left.\begin{aligned}\delta A = \left\{ \sum_{i=0}^{i=i} \left(\frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right) \right\}_{t=t} - \left\{ \sum_{i=0}^{i=i} m_i \left(\frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right) \right\}_{t=0} \\ + \int_0^t \left\{ \delta T - \sum_{i=0}^{i=i} m_i \left(\frac{d^2x}{dt^2} \delta x + \frac{d^2y}{dt^2} \delta y + \frac{d^2z}{dt^2} \delta z \right) \right\} d\tau = 0.\end{aligned}\right\} \tag{234}$$

This expression, as Thomson and Tait remark, is entirely general and free from terminal or kinetic conditions. And since in the problem of *Least Action* the initial and final positions are invariable, the terminal variations δx etc., must vanish, and the two integrated expressions in the right member of (234) disappear. The equation of energy is

$$T + V = \tfrac{1}{2} \sum_{i=0}^{i=i} m_i \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} + V = E \,, \tag{235}$$

where T is the kinetic and V is the potential energy; and therefore in the present problem $\delta T = -\delta V$. Hence to make $\delta A = 0$, it is necessary and sufficient that the integral expression in (234) disappear; or putting for δT the equivalent—δV, the necessary and sufficient condition is

$$\sum_{i=0}^{i=i} m_i \left(\frac{d^2x}{dt^2} \delta x + \frac{d^2y}{dt^2} \delta y + \frac{d^2z}{dt^2} \delta z \right) + \delta V = 0 \,. \tag{236}$$

§ 89. Hamilton's *Principle, or the Stationary Condition, According to the Principles of the Calculus of Variations.*

In any unguided motion of a conservative system, the action from any one stated position to any other, though not necessarily a minimum, fulfills the *Stationary*

Condition, expressed by $\delta A = 0$, which secures either a minimum or maximum, or maximum-minimum, according to the principles of the Calculus of Variations. Put $L = T + U$, where T is the kinetic energy, and U is the *work function,* and and let V denote the potential energy; then we shall have $L = T - V$. It will be observed that U and V are functions of the coördinates, and not of the velocities. When the system passes from one configuration A, to another configuration B, by any varied paths whatever, the variations of the coördinates of these terminal positions being zero, we have clearly for the action in the interval t_1 to t,

$$\delta \int_{t_1}^{t} L dt = -C(\delta t - \delta t_1) , \tag{237}$$

$$\delta \int_{t_1}^{t} 2T dt = (t - t_1)\, \delta C . \tag{238}$$

If we suppose, that in pursuing their varied paths, the particles, without violating geometrical conditions, may be conducted with such velocities that the energy $C = T - U$ has a *given value,* then $\delta C = 0$, and the action by (238) is $\int_{t_1}^{t} 2T dt = 0$, which is a maximum-minimum, or is stationary in the actual path. This leads to the celebrated *principle of* HAMILTON,

$$\overset{CD}{\int_{t_1}^{t}} L dt - \overset{AB}{\int_{t_1}^{t}} L dt = 0 ; \tag{239}$$

which shows that the integral $\int_{t_1}^{t} L dt$ *has a stationary value for any part of an actual trajectory AB, as compared with neighboring paths CD, which have the same terminal points as the actual trajectory and for which the time has the same terminal values.* In the *Philosophical Transactions* for 1834–5, SIR W. R. HAMILTON has applied these methods to the motions of planets and comets, and worked out the results with considerable detail (cf. WHITTAKER'S *Analytical Dynamics,* p. 242).

§ 90. *Application of These Dynamical Principles to the Case of Nature Where the Systems are Non-Conservative.*

If now we introduce the action of a Resisting Medium, it is clear that the system will no longer be strictly conservative, but that the energy will gradually degrade. Since the free conservative system, when unguided, would pursue the path of Least Action, the resisted system evidently would pursue the path of *Least Resistance* to the otherwise free motion; and therefore the path of degradation would still be that of *Least Action.* This principle enables us to understand the nature of the transformations suffered by the orbits of planets, comets and satellites, during the evolution of the solar system. If we had complete knowledge of all the events in the past history of our system, and a perfect theory of the motions now going on, we might take the existing system and trace it back to the original state, by the principle of Least Action and HAMILTON'S principle. But as the resistances formerly encountered have now almost wholly disappeared, while the corresponding infinite series of perturbations by which the motions were modified could not be accurately restored, from the present state of the system, it is obvious that we could not trace the path of the system back through a very long period. In our attempts to understand the past history of the solar system we have therefore to fall back on the indications furnished by the integral equations for the surfaces of energy, from which the time is eliminated, save in the secular term depending on the action of the Resisting Medium. Since in non-conservative systems the energy must degrade, it is clear that the so-called constant of JACOBI'S integral must increase with the time. When the secular term αt is added to make the original integral correspond with the actual conditions of nature, the *complete integral* becomes

$$x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} = C + \alpha t\,. \tag{240}$$

As the secular term slowly increases with the time, there is a secular shrinkage of the energy surface corresponding to any particle; and sooner or later the particle passes within the closed folds about the sun or planet, and no longer circulates about both, but has its movement restricted to that of an inferior planet, or satellite. In this simple idea of the secular shrinkage of HILL'S surfaces of energy for any particle, is contained the whole theory of the origin of the satellites by capture; but we shall defer the full discussion of it to Chapter X, and meanwhile examine the theory of the capture of comets, which has long been familiar to astronomers.

CHAPTER IX.

Theory of the Capture of Comets and of the Transformation of Their Orbits by the Perturbations of the Planets.

§ 91. Tisserand's *Criterion for the Identity of Comets.*

In dealing with the motions of comets, we may always treat these small masses as mere particles, while the Sun and *Jupiter* may be taken for the two bodies referred to the rotating axes in the anlaysis of the preceding chapter. The neglect of the eccentricity of *Jupiter's* orbit is not important, but it has been taken account of by Callandreau in a mathematical paper entitled *Sur la Théorie dès Comètes Périodique* (*Annales de l'Observatoire de Paris,* Tome XX), and those who wish to consider the slight modification thus arising may consult this memoir.

Under the circumstances the Jacobian integral is valid for the motion of comets, and we have

$$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C\,. \tag{241}$$

If the constant C is found to be the same for two successive comets, they are, of course, identical. Tisserand has introduced this valuable criterion and shown how to find the value of C from the ordinary elements of a comet's orbit about the Sun. (*Bulletin Astronomique,* Tome VI, p. 289, and Traité *de Mécanique Céleste,* Tome IV, Chap. XII, §85). We shall now give some account of Tisserand's method of procedure. The co-ordinates of the comet referred to rotating axes, give the following relations, when referred to axes fixed in space:

$$\left.\begin{aligned} x &= \xi\cos t + \eta\sin t\,,\\ y &= -\xi\sin t + \eta\cos t\,,\\ z &= \zeta\,. \end{aligned}\right\} \tag{242}$$

$$\left.\begin{aligned} &\text{Therefore}\quad x^2 + y^2 = \xi^2 + \eta^2\,,\quad \text{and,}\\ &\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 + \xi^2 + \eta^2 - 2\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right). \end{aligned}\right\} \tag{243}$$

Omitting $x^2 + y^2 = \xi^2 + \eta^2$ from the two members of (241) we have

$$\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 - 2\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) = \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C\,. \tag{244}$$

But by the theory of elliptic motion the velocity and constant of areas are respectively

$$\left.\begin{aligned} V^2 = \frac{2}{\rho} - \frac{1}{a} &= \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2, \\ \xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} &= \sqrt{a(1-e^2)}\cos i\,. \end{aligned}\right\} \tag{245}$$

Therefore

$$\frac{2}{\rho} - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i = \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C\,. \tag{246}$$

And this equation must hold true of any two comets supposed to be identical. This expression is further simplified by the fact that *Jupiter's* mass is less than a thousandth part of the mass of the Sun, so that, in general, ρ is sensibly equal to ρ_1. It will be recalled that ρ is the radius vector of the comet's orbit about the Sun, when the action of the planet is neglected. And as the comet's orbit is found by observations taken as far as possible from both the Sun and *Jupiter*, we have sensibly $\frac{2}{\rho} = \frac{2}{\rho_1}$ and may neglect the sum of the small quantities $-\frac{2\mu}{\rho_1} + \frac{2\mu}{\rho_2}$. Thus from (246) we have simply

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = C\,. \tag{247}$$

Putting a' for the mean distance of *Jupiter*, we have for cometary orbits transformed by this planet

$$\frac{1}{a} + \frac{2\sqrt{a(1-e^2)}}{a'\sqrt{a'}}\cos i = \alpha\,, \tag{248}$$

which is a numerical expression always about 0.5. This is the form of the criterion developed by TISSERAND (*Mécanique Céleste,* Tome IV, p. 205), and extensively used by SCHULHOF and other recent investigators. Such a criterion is valuable in many researches on comets, because the orbits of these small bodies are greatly modified by the disturbing action of the planets. In all these transformations, however, the aphelion distance departs but little from that of the disturbing planet, because its sphere of influence is small compared to that of the Sun; and the motion is sensibly in an undisturbed orbit about the Sun, both before and after the encounter with the planet.

§ 92. *Analytical Method of Treating of the Capture of a Parabolic Comet.*

The results of LEUSCHNER'S recent researches indicate that comets with parabolic orbits are comparatively rare, and such motion can no longer be considered to represent the actual movements of the majority of comets; yet as this is the traditional theory and in many cases enables us to represent the motion fairly well, along the arc near the Sun, over which the observations extend, even when the orbit is an elongated ellipse, we shall briefly consider it, in order to afford the reader some conception of the mode by which a comet moving in a parabolic orbit may be captured by a planet such as *Jupiter*. The following discussion is essentially that given by TISSERAND, *Mécanique Céleste*, Tome IV, § 87, pp. 207–208.

The velocity in the case of elliptic motion about the Sun is

$$v^2 = k^2\left(\frac{2}{r} - \frac{1}{a}\right). \tag{249}$$

Let S and J be the positions of the Sun and *Jupiter*, respectively, and the distance of the comet μ_0 from the Sun $S\mu_0 = \rho_1$, from *Jupiter* $J\mu_0 = \rho_2$, and suppose that the angle $SJ\mu_0 = 90°$, and that the initial velocity v_0 of the comet is directed along the radius vector $\mu_0 J$, or makes with this line a very small angle. If therefore $a = \infty$, $\rho_1 = S\mu_0 = SJ = r$, nearly, because the comet is very near *Jupiter*; then it is clear that we shall have

$$v = k\sqrt{\frac{2}{r}}. \tag{250}$$

If we neglect the eccentricity of the orbit of *Jupiter*, so that $r = a$, the comet's velocity will be directed along the ray $\mu_0 J$, and have the value

$$v_0' = k\sqrt{\frac{2}{r} - \frac{1}{a}} = \frac{k}{\sqrt{r}}. \tag{251}$$

In the relative movement of the comet about *Jupiter*, we have therefore for the velocity

$$V_0 = v_0 - v_0' = k\frac{\sqrt{2} - 1}{\sqrt{r}}. \tag{252}$$

And the general formula

$$V^2 = k^2 m'\left(\frac{2}{R} - \frac{1}{A}\right), \tag{253}$$

when we put $R = \rho_2$ and $V = V_0$, gives

$$\frac{1}{m'}\left(\sqrt{2} - 1\right)^2 \frac{\rho_2}{r} = 2 - \frac{\rho_2}{A}. \tag{254}$$

Taking $\frac{1}{m'} = 1047$, and $\frac{\rho_2}{r} = 0.062$, since the sphere of *Jupiter's* activity is very small, we get from (254) very nearly

$$A = -\frac{\rho_2}{9}. \tag{255}$$

This expression of the semi-axis major of the relative orbit of the comet about *Jupiter* is negative, and the Jovicentric orbit is an hyperbola which departs but little from the ray $J\mu_0$. One branch of the hyperbola gives the first part of the motion, and after passing the planet also the second part of the motion, as the comet recedes from *Jupiter*. The comet enters the sphere of activity near the point μ_0, and since the distance R is still equal to ρ_2, the velocity V_1, calculated by formula (253), will be equal in absolute value to V_0, but of contrary sign. Thus we have

$$V_1 = k\frac{1 - \sqrt{2}}{\sqrt{r}}. \tag{256}$$

Combining this relative velocity V_1 with v_0' given by (251), we get the absolute velocity v_1 at the departure from the sphere of activity; thus we find

$$V_1 + v_0' = v_1 = k\frac{2 - \sqrt{2}}{\sqrt{r}}. \tag{257}$$

If now we substitute in (249) V_1 for v, and replace a by a_1, we get for the elliptic orbit of the comet after its passage near *Jupiter*,

$$k^2\frac{(2 - \sqrt{2})^2}{r} = k^2\left(\frac{2}{r} - \frac{1}{a_1}\right), \quad \text{or} \quad a_1 = r\frac{\sqrt{2} - 1}{4} = 5.20\,\frac{\sqrt{2} + 1}{4} = 3.14\,; \tag{258}$$

which corresponds to the orbits of comets like those of BRORSEN and WINNECKE.

After giving this analysis of the movement of a parabolic comet near *Jupiter*, TISSERAND remarks that the planet plays the role of capturer of comets; and that if the conditions are exactly reversed, the resulting elliptic orbit may evidently be restored to the parabolic form by the disturbing action of the planet. For, if the comet be started backward on the path it first traveled, with the planet in the same situation, it is evident that after passing *Jupiter* it will again return to infinity. And in the course of actual events, after a number of revolutions about the Sun, it may again approach *Jupiter* so as to enter his sphere of activity,

with a velocity smaller than that of *Jupiter*, since the semi-axis major of the comet $a_1 < r$. At the entrance into the sphere the absolute velocity is

$$k\sqrt{\frac{2}{r} - \frac{1}{a_1}} = k\frac{(2 - \sqrt{2})}{\sqrt{r}};$$

and the relative velocity at the entrance is $V_1 = k\dfrac{1 - \sqrt{2}}{\sqrt{r}}$, and at the exit the sign is simply changed, $k\dfrac{\sqrt{2} - 1}{\sqrt{r}}$. And combining these with the velocity of *Jupiter*, as in deriving (252), we get

$$v_0 - v_0' = k\frac{\sqrt{2} - 1}{\sqrt{r}}, \quad \text{or} \quad v_0 = k\frac{\sqrt{2} - 1}{\sqrt{r}} + \frac{k}{\sqrt{r}} = k\sqrt{\frac{2}{r}}; \tag{259}$$

which is the parabolic velocity of the comet after quitting *Jupiter's* sphere of activity.

Tisserand remarks that the conditions of the problem here treated are exceptional, but that there are many other positions of less exceptional character, in which similar changes would be produced. To show this requires an analysis similar to that given by the late Professor H. A. Newton in his valuable memoir "On the Capture of Comets by Planets, and Especially Their Capture by *Jupiter*," *Memoirs of the National Academy of Sciences*, Vol. VI. The subject has also been carefully treated by Callandreau in a paper entitled "*Sur la Théorie des Comètes Périodiques*" (*Annales de l'Observatoire de Paris*, Tome XX). Tisserand devotes considerable space to these theories, which are of high theoretical interest; but they are scarcely necessary to the continuity of the present work, and we content ourselves with the foregoing special cases.

The work of Newton, Callandreau and Tisserand shows not only the correctness of the theory that the orbits of comets are subject to transformation, but also the special circumstances under which it may arise. Newton concludes his memoir with the following analysis of probability: "If in a certain interval of time, a thousand million comets arrive, in parabolic orbits, passing nearer to the Sun than to *Jupiter*, 126 among them will be transformed into ellipses of which the time of revolution T will be less than $\frac{1}{2}T'$, the half period of the revolution of *Jupiter*; for 839 we shall have $T < T'$; for 1701 $T < \frac{3}{2}T'$, and finally for 2670 we shall have $T < 2T'$." Tisserand remarks that if the orbit of a comet is not transformed at a single approach to *Jupiter*, it may finally result from several such approaches. This conclusion is fully verified from another point of view by Darwin's researches on Periodic Orbits, *Acta Math.*, XXI, pp. 168–169.

§ 93. TISSERAND'S *Discussion of the Elements of the Comets of Jupiter's Group.*

TISSERAND gives the accompanying table of the elements of the comets of the *Jupiter* group (*Mécanique Céleste*, Tome IV, p. 205). As usual a denotes the semi-axis major; i the inclination; $\varpi - \Omega$ the angular distance from the node to the perihelion; l the longitude of the point of the orbit which is nearest to the orbit of *Jupiter*; $a\,(1 + e)$ and $a\,(1 - e)$ the aphelion and perihelion distances, respectively; α the numerical value of TISSERAND's criterion, always

Name of Comet	Apparition	a	i	$\varpi - \Omega$	l	$a(1+e)$	$a(1-e)$	α	$\varpi_1 - l$
			°	°	°				°
ENCKE	1795	2.21	14	182	335	4.09	0.33	0.580	+ 2
BLANPAIN*	1819	2.85	9	350	247	4.82	0.88	0.555	0
HELFENZRIEDER*	1766	2.93	8	177	80	5.45	0.41	0.487	− 9
TEMPEL	1873	3.00	13	185	125	4.65	1.35	0.571	+ 1
BARNARD*	1884	3.08	5	301	126	4.84	1.32	0.567	0
DE VICO	1844	3.10	3	279	162	5.02	1.18	0.556	+ 1
TEMPEL-SWIFT	1869	3.11	5	106	223	5.16	1.06	0.544	0
BRORSEN	1846	3.14	31	13	283	5.62	0.66	0.475	+13
WINNECKE	1858	3.14	11	162	113	5.50	0.79	0.512	−17
LEXELL*	1770	3.16	2	224	184	5.66	0.66	0.500	− 8
TEMPEL	1867	3.19	6	125	60	4.82	1.56	0.570	− 4
PIGGOTT*	1783	3.26	45	354	233	5.05	1.47	0.487	− 3
BARNARD*	1892	3.41	31	170	"	5.40	1.43	"	"
BROOKS*	1886	3.41	13	177	53	5.49	1.33	0.533	− 3
SPITALER*	1890	3.44	13	13	228	5.06	1.82	"	"
D'ARREST	1851	3.44	14	175	153	5.71	1.17	0.519	−10
TUTTLE*	1858	3.52	20	26	0	5.88	1.16	0.505	+21
FINLAY	1886	3.54	3	316	205	6.09	0.99	0.502	−17
WOLF	1884	3.58	25	173	210	5.58	1.58	0.518	−11
BIÉLA	1772	3.58	17	213	268	6.16	1.00	0.491	+22
HOLMES*	1892	3.62	21	12	"	5.11	2.14	"	"
BROOKS	1889	3.67	6	344	185	5.39	1.95	0.556	− 3
FAYE	1843	3.81	11	201	209	5.94	1.68	0.529	+21

* Denotes that only one apparition of the comet has been observed.

about 0.5; $\varpi_1 = \varpi + 180°$ is the longitude of the aphelion. It will be seen that the residuals in the column $\varpi_1 - l$ usually are quite small; which shows that the comets have been captured by *Jupiter*, and have since had their aphelia shifted but slightly. Finally, the first column gives the name of the comet, and the second the year of its appearance; while the star indicates that the comet has been seen at but one return. After giving this table TISSERAND draws from it the following conclusions:

"(1) All these comets are direct, and their orbits but little inclined to the ecliptic: the mean of the inclinations $= 14°$; the parabolic comets, on the contrary, are as often retrograde as direct.

"(2) The aphelion distances differ but little from the mean distance of *Jupiter* from the Sun.

"(3) Eighteen of the values of $\varpi - \Omega$ are equally close to 0° or 180°.

"The relation of the preceding comets to *Jupiter* appears to be beyond doubt."

He remarks that if one inquires why some have appeared but once, while others recently discovered have been observed at successive returns, an answer is furnished by what happened to the first comet of 1770, which was investigated by LEXELL. This early investigator remarked that in 1767 and in 1779 it passed near *Jupiter*, and that in consequence of the action of that planet its perihelion distance was decreased, so that it became visible and passed near the earth in 1770. When it next passed near *Jupiter* in 1779, the same attraction again increased the perihelion distance, and rendered the comet invisible, as it was prior to 1767. As we shall see more fully in § 95, this hypothesis of LEXELL, at the suggestion of LAPLACE, was carefully investigated by BURKHARDT. His calculations showed that in 1770 the comet was moving in an ellipse with mean distance of 5.06, and perihelion distance of 2.96; but in 1779 the powerful atraction of *Jupiter* again transformed the orbit till the semi-axis major was 6.37, and the perihelion distance 3.33. This shows why it had not appeared previous to 1770, and has not returned since 1779. BURKHARDT studied the motion of this comet as it moved in a hyperbola about *Jupiter*, and found that it passed through the system of satellites; without, however, producing the slightest disturbance in their motions, as was pointed out by LAPLACE in the *Mécanique Céleste*, Liv. IX, Chap. III, §§ 13–14. Yet LAPLACE found that the attraction of the earth had shortened the period of the comet by 2.046 days.

In his review of this question TISSERAND remarks (*Méc. Cél.*, Tome IV, p. 206) that the same thing happened to the Comet *Wolf* of 1884; at least LEHMANN-FILHÈS showed (*A.N.* 2953) that it passed very near *Jupiter* in 1875, and suffered such perturbations as to increase the perihelion distance to 2.55, which would render the comet invisible. D'ARREST likewise has shown (*A.N.* 1087) that when the comet of BRORSEN approached close to *Jupiter*, the former perihelion distance of 1.50 was reduced to less than half, so as to give the present value of 0.66. From these phenomena TISSERAND remarks that we are justified in concluding that the actual orbits of the periodic comets of this family may be traced back to the action of *Jupiter*, by which they have been captured.

§ 94. SIVASLIAN'S *Projection of the Orbits of the Family of Comets Belonging to the Jupiter Group.*

In *Popular Astronomy* for October, 1893, Professor W. W. PAYNE has given a projection of the orbits of the comets of the *Jupiter* group, made by A. G. SIVASLIAN at Goodsell Observatory of Carleton College, Northfield, Minnesota. The statement had been made by Professor C. A. YOUNG, in one of his works on Astronomy, that if the orbits of this family of comets were drawn they would be found so interlocked and confused as to be unintelligible. SIVASLIAN undertook the projection of the orbits at Professor PAYNE'S suggestion, and a few improvements in the diagram were afterwards made by DR. H. C. WILSON, the present Director of Goodsell Observatory. We give herewith the diagram of SIVASLIAN, as slightly corrected by DR. WILSON; and the Table of the Elements used, though not greatly different from those above quoted from TISSERAND, is also added for the sake of completeness.

This diagram presents to the eye an impressive picture of what *Jupiter* has done for the comets of the solar system. It will be seen that these bodies are generally thrown within *Jupiter's* orbit, leaving their aphelia near the disturbing planet by which their paths were transformed. *This must necessarily happen, since the sphere of the planet's influence is small;* and when a comet leaves this sphere, it revolves in an ellipse about the Sun and thus returns to the place of disturbance.

But this lucid diagram gives us something more than a grand illustration of what *Jupiter* has done for our periodic comets; it also furnishes us a magnificent model of what goes on in every other planetary system among the fixed stars where the disturbing body has a similar orbit, distance, and mass-ratio to that of *Jupiter* to the Sun.

And it is easy to see from the discussion of the restricted problem of three bodies, given in Chapters VIII and X, that if the system be made up of two equal or comparable stars, the comets will wander back and forth from the sphere of activity of one sun to that of the other, much more than is the case in our solar system, with one body relatively so small as *Jupiter* is found to be. This comparative smallness of *Jupiter's* mass enables the sphere of the Sun's control to extend very close up to the planet; and the whole region within the orbit is dominated by the Sun's attraction. *Jupiter* disturbs and transforms the orbits of comets from time to time, when they cross his orbit, but if they are once well within they may afterwards revolve in comparative stability. Thus the region occupied by the body of the asteroids is one of essential stability, while those

ELEMENTS OF THE JUPITER FAMILY OF PERIODIC COMETS

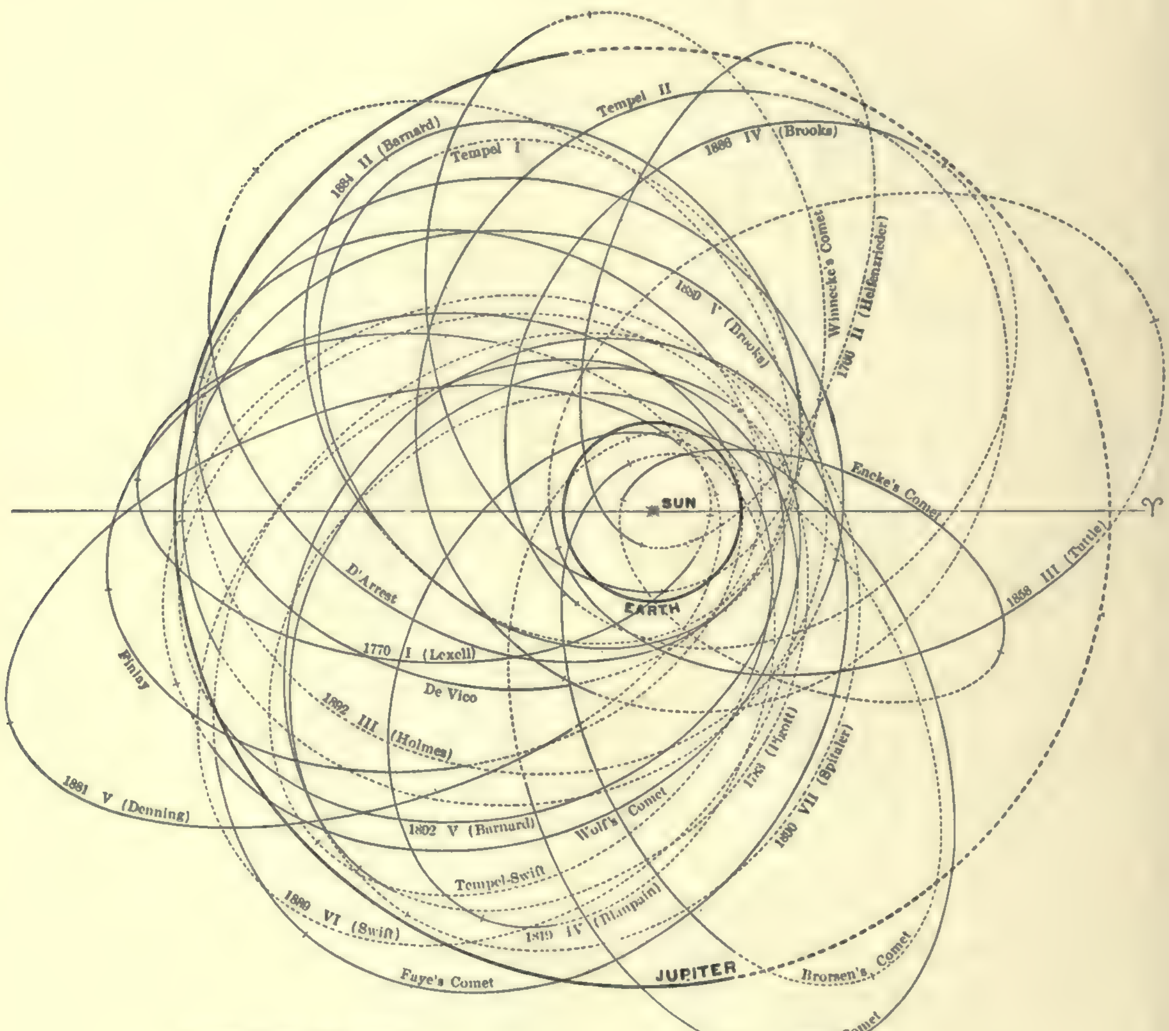
COMETS OF WHICH MORE THAN ONE APPARITION HAS BEEN OBSERVED.

Name	Period in yrs.	Time of Perihelion Passage.	Perihel'n Distance	Aphelion Distance	e	π (° ′ ″)	☊ (° ′ ″)	i (° ′ ″)	Mean Eqninox	Calculator and Reference.
ENCKE	3.303	1891 Oct. 17.986	0.34047	4.09489	0.84647	158 38 46	334 41 27	12 54 58	1891.0	BACKLUND, *Astr. Gesel. J.* 27-1
TEMPEL	5.211	1889 Feb. 2.101	1.38660	4.66545	0.55210	306 8 3	121 9 17	12 45 5	1890.0	SCHULHOF
BRORSEN	5.456	1890 Feb. 4.104	0.58776	5.61038	0.81034	116 23 10	101 27 34	29 23 48	1890.0	E. LAMP, *A.N.* 2933
TEMPEL-SWIFT	5.534	1891 Nov. 14.083	1.08660	5.17088	0.65270	43 14 16	296 31 15	5 23 14	1891.0	BOSSERT, *Astr. Gesel. J.* 27-1.
WINNECKE	5.818	1892 June 30.893	0.88642	5.58313	0.72297	276 11 9	104 4 59	14 31 31	1890.0	HAERTDL, *A.N.* 3062 *Astr. Gesel. J.* 28, p. 142
TEMPEL	6.507	1885 Sept. 25.734	2.07332	4.89733	0.40513	241 21 50	72 24 9	10 50 27	1880.0	GAUTIER, *A.N.* 2656.
BIELA Nucleus 1	6.587	1852 Sept. 23.718	0.86016	6.16732	0.75520	109 5 20	245 49 34	12 33 28	1852.0	D'ARREST, *A.N.* 933
BIELA Nucleus 2	6.629	1852 Sept. 22.952	0.86059	6.19687	0.75512	108 58 17	245 58 29	12 33 50		
FINLAY	6.627	1893 July 12.176	0.98912	6.06371	0.71951	7 41 34	52 27 43	3 2 2	1893.0	SCHULHOF, *A.N.* 3171
D'ARREST	6.691	1890 Sept. 17.493	1.32404	5.77776	0.62713	319 14 34	146 16 32	15 42 41	1890.0	LEVEAU, *Astr. Gesel. J.* 26-1
WOLF	6.821	1891 Sept. 3.473	1.59285	5.60058	0.55714	19 11 38	206 22 29	25 14 33	1891.0	Thraen, *A.N.* 3050, Berberich, *A.J.* 253, *Astr. Gesel. J.* 27-1
FAYE	7.566	1881 Jan. 22.671	1.73814	5.97009	0.54902	50 48 47	209 35 25	11 19 40	1880.0	MÖLLER, *Berl. Jahrb.* 1882

COMETS OF WHICH ONLY ONE APPARITION HAS BEEN OBSERVED

Name	Period in yrs.	Time of Perihelion Passage.	Perihel'n Distance	Aphelion Distance	e	π (° ′ ″)	☊ (° ′ ″)	i (° ′ ″)	Mean Eqninox	Calculator and Reference.
1819 IV, (BLANPAIN)	4.810	1819 Nov. 20.252	0.89256	4.806	0.68675	67 18 48	77 13 57	9 1 16	1819	ENCKE
1766 II, (HELFENZRIEDER)	5.025	1766 Apr. 26.995	0.39898	5.468	0.86400	251 13 0	74 11 0	8 1 45	1766	BURCKHARDT
1884 II, (BARNARD)	5.398	1884 Aug. 16.483	1.27969	4.872	0.58395	306 11 20	5 8 38	5 27 33	1884.0	EGBERT, *A.N.* 2657
1844 I, (DE VICO)	5.459	1844 Sept. 2.484	1.18632	5.015	0.61737	342 30 48	63 49 38	2 54 46	1844	BRUNNOW
1886 IV, (BROOKS)	5.595	1886 June 6.691	1.32772	4.976	0.57874	230 16 51	53 28 57	12 43 26	1886.0	S. OPPENHEIM
1770 I, (LEXELL)	5.626	1770 Aug. 13.547	0.67431	5.652	0.78684	356 16 27	131 59 34	1 34 31	1770	LE VERRIER
1783, (PIGOTT)	5.888	1783 Nov. 19.937	1.45929	5.062	0.55246	50 17 25	55 40 30	45 6 54	1783	C.H.F. PETERS
1892 V, (BARNARD)	6.309	1892 Dec. 11.05	1.42911	5.396	0.58123	16 52 36	206 38 45	31 12 28	1892.0	KRUEGER, *Astr. Gesel. J.* 28 p. 145
1890 VII, (SPITALER)	6.378	1890 Oct. 26.601	1.81791	5.061	0.47144	58 25 58	45 5 52	12 50 44	1890.0	SPITALER, *A.N.* 3011
1858 III, (TUTTLE)	6.609	1858 May 2.974	1.14922	5.894	0.67368	200 46 27	175 4 9	19 30 2	1858.0	SCHULHOF
1892 III, (HOLMES)	6.909	1892 June 13.238	2.13940	5.116	0.41024	345 53 12	331 42 12	20 47 23	1892.0	SCHULHOF, *A.N.* 3140
1889 V, (BROOKS)	7.073	1889 Sept. 30 012	1.95023	5.419	0.47070	1 26 17	17 58 45	6 4 10	1890.0	Chandler, *A.J.* 205, *A.N.* 2952, *Astr. Gesel. J.* 28-1
1881 V, (DENNING)	8.343	1881 Sept. 12.834	0.72384	7.503	0.82403	18 10 5	66 9 2	6 53 26	1881.0	CHANDLER, *A.N.* 2406
1889 VI, (SWIFT)	8.534	1889 Nov. 29.572	1.35367	6.998	0.67585	40 15 2	330 36 2	10 14 54	1890.0	HIND, *Astr. Gesel. J.* 27-1

POPULAR ASTRONOMY.



Jupiter's family of Comets.

From a drawing by A. G. Sivaslian,
Northfield, Minn.

Goodsell Observatory,
Carleton College.

FIG. 23. SIVASLIAN'S PROJECTION OF THE ORBITS OF JUPITER'S FAMILY OF COMETS.

which still travel across *Jupiter's* orbit will suffer further transformation in the same way that the comets do.

And just as it is possible for *Jupiter* under unusually favorable conditions to send a comet off on a parabolic path never to return to our system, so also this same thing might happen to an asteroid, if the same unusual conditions should arise, of which, however, the probability is very slight. As we shall see elsewhere in this volume, the asteroids have been gathered into the zone which they now occupy mainly by the action of *Jupiter* and of the resisting medium formerly pervading the solar system. Similar zones of asteroids and families of comets may be supposed to exist in other planetary systems having large planets such as *Jupiter*.

If, on the other hand, the mass distribution be double, as is often the case with double stars, planets can exist chiefly near each star, and the zone of stability for these captured bodies, such as comets and asteroids, will be still nearer to the two suns composing the system. So far as we can see, all the orbits which wind about from one star to the other will be temporary and non-periodic; so that a rapid series of changes will occur, till the small bodies are brought permanently under the control of one star or the other. When stable motion thus becomes possible, a system of planets may develop, just as systems of satellites have developed about our planets in the solar system. If, however, the binary orbit is very eccentric, the stability of such planetary systems would be greatly endangered, and probably orbits of a permanent and orderly character could be developed only very near each mass.

The elements used by SIVASLIAN in working out the accompanying diagram, and given in the table, were believed to be complete in 1893, but some additional members of the *Jupiter* group have been discovered since. PROFESSOR R. T. CRAWFORD, of the University of California, has kindly sent the author the following list of additional members of the *Jupiter* group:

	Period.
1896 *d* (GIACOBINI)	9.00 years.
1896 *g* (PERRINE)	6.67 "
1904 *e* (BORELLY)	7.30 "
1906 *e* (KOPFF)	6.67 "
1906 *h* (METCALF)	6.89 "

It will be noticed that the number of members of *Jupiter's* family thus increases quite rapidly; which is due, principally, to the extension of astronomical research by an increased number of observers using the superior appliances and instruments of our time. The most improved methods of calculation are now

promptly applied to the observations by LEUSCHNER, CRAWFORD and other investigators, and the theory of comets is thus greatly extended. But even at the present time it cannot be assumed that we know more than a part of the comets which have been captured by *Jupiter*. This known group, however, is sufficiently large to show the part played by this great planet in the transformation of the orbits of comets. In most cases the new members of *Jupiter's* group are not recent additions to the solar system, but had previously escaped detection, owing to their faintness and unfavorable situation with respect to the Earth. Yet there are cases of genuine *new comets*, from remote regions of space, as well as of old ones whose orbits are so transformed as to render them visible to observers.

§ 95. *Researches of* BURKHARDT, LAPLACE *and* LEVERRIER *on* LEXELL'S *Comet of* 1770.

The permanent disappearance of Comet I 1770, discovered on the night June 14–15, by MESSIER, at Paris, and at first taken to be a part of the nebula in the constellation *Sagittarius*, but subsequently found to be a comet, and carefully investigated by LEXELL, the friend and associate of EULER, in St. Petersburg, and found to revolve in an elliptic orbit with a period of 5.585 years, justly excited the wonder of astronomers, and has therefore been the subject of several profound researches, of which we shall give a brief account. When nearest the Earth, July 1, 1770, the nucleus had an angular diameter of 80″, and the tail extended 2° 23′, or nearly five times the Moon's apparent diameter; and the distance was then only 2,315,200 kilometers (1,324,500 miles), about five and a half times the distance of the Moon. The comet was lost in the Sun's rays July 4th, but again found on August 4th, and observed till October 3d, when the distance became so great that it was rendered invisible in the small telescopes then in use. Its orbit about the Sun was sensibly disturbed by the attraction of the Earth, and LAPLACE has calculated (*Mécanique Céleste*, Liv. IX, Chap. II, § 13) that the period of the comet's revolution was thus diminished by 2.046 days; and, as the Earth's motion was not disturbed by so much as 2″.8, he found that the mass of the comet was less than $\frac{1}{5000}$ of that of the Earth.

When the comet failed to reappear, in accordance with the orbit computed by LEXELL, which was carefully confirmed by other calculators, great surprise was expressed by astronomers; but LEXELL remarked that this body had passed very near *Jupiter* in 1767, and again in 1779, and that the path had probably been greatly transformed by the disturbing action of that great planet, which could thus make the comet visible in 1770, and again render it invisible after 1779.

Such a daring hypothesis as this could not be accepted without a critical inquiry based on exact calculation; and the perturbations of LEXELL's Comet in its encounters with *Jupiter* was, therefore, proposed as a prize subject by the Paris Academy of Sciences.

At the suggestion of LAPLACE, who had developed the theory of such an encounter, from a method first outlined by D'ALEMBERT (*Opuscules*, Tome VI, p. 304), BURKHARDT undertook the investigation, won the prize, and at the same time confirmed the sagacious suggestion of LEXELL (cf. Memoirs of the Paris Academy, 1806, pp. 20 *et seq.*). BURKHARDT found that the comet had entered the sphere of *Jupiter's* attraction January 18.358, 1767, and quitted it at noon on May 9, 1767, thus remaining under the domination of the planet for a period of 110.642 days, and describing about it an hyperbola with semi-axis major of — 0.0220462, and eccentricity of 1.86220. The period of revolution, after the first encounter with *Jupiter*, was found to be 2050.095 days; and when entering *Jupiter's* control the second time, in 1779, 2042.682 days. The hyperbola described about *Jupiter* in 1779 had a semi-axis major of — 0.0205086, and an eccentricity of 1.26586.

In the second encounter, the comet had come under *Jupiter's* control June 20, at midday, and quitted his sphere of activity on October 3.9320, after an interval of 105.932 days. BURKHARDT found the perihelion distance after 1779 to be 3.3346, which has been substantially confirmed by later investigators; and this accounts for the comet's failure to reappear.

This celebrated comet was afterwards very carefully investigated by LEVERRIER (*Comptes Rendus*, Tome XIX, p. 559; Tome XXV, pp. 561, 917, and Tome XXVI; also in *Annales de l'Observatoire de Paris*, Tome III). CLAUSEN, BRÜNNOW, and others, have also studied its motion and they agree with LEVERRIER that the comet is not identical with any other known body, and is now lost. The perihelion distance prior to 1767 BURKHARDT had found to be 5, with a semi-axis major of 13; but LEVERRIER showed the latter to have been less than 5. When D'ARREST examined BURKHARDT's manuscript he found an accidental error which brought his work into substantial agreement with that of LEVERRIER (cf. *A.N.* 1087).

The accompanying diagram represents the general aspects of the orbits of LEXELL's Comet before and after its encounters with *Jupiter*. At the points of nearest approach to *Jupiter* the relative orbit is always concave to the planet; but as the distance increases it finally becomes only slightly so, since the velocity of approach under the Sun's attraction was sufficient to carry it through the region near the planet in an hyperbola, about *Jupiter* as the center of attraction.

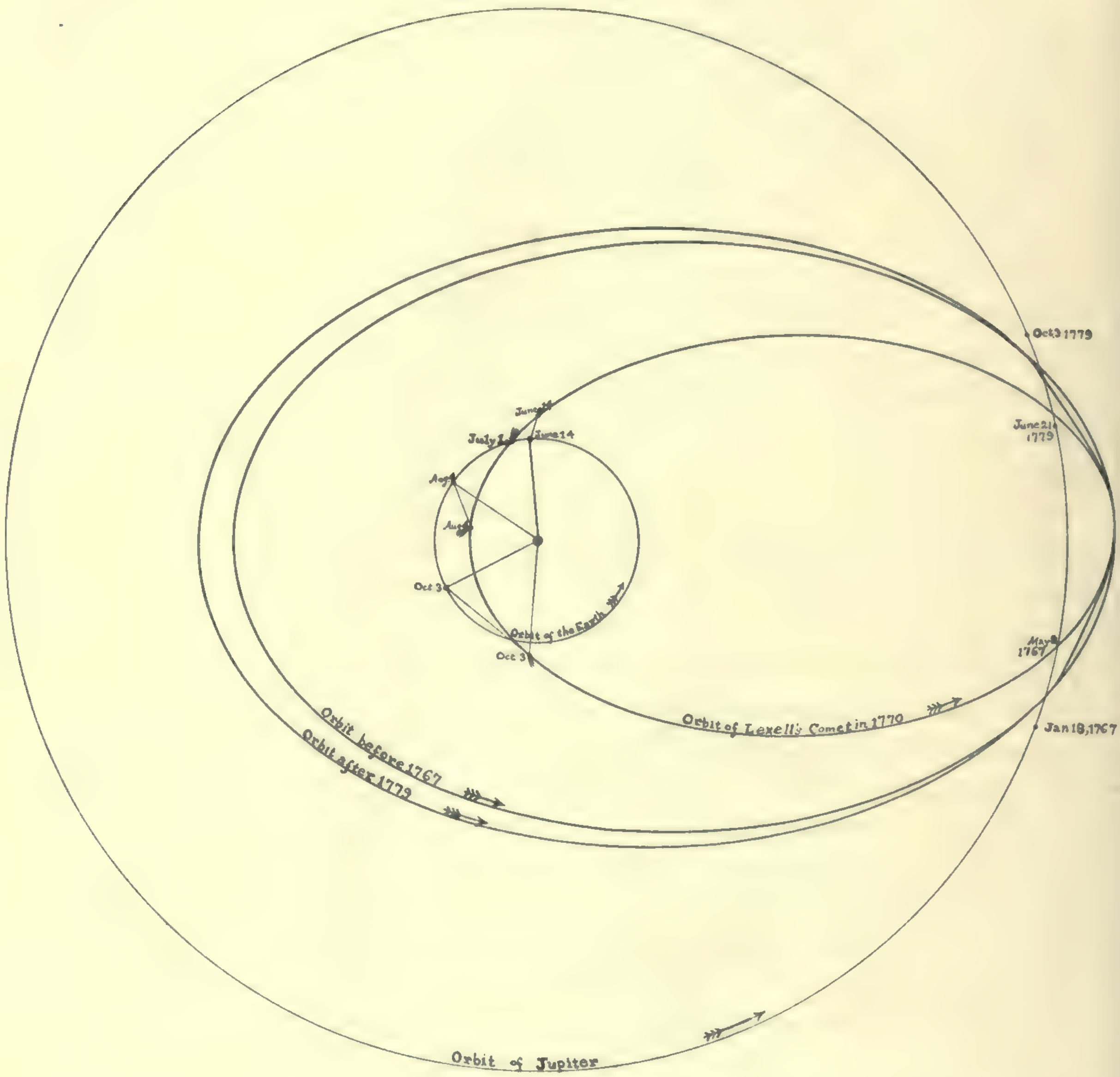


FIG. 24. DIAGRAM SHOWING THE TRANSFORMATIONS OF THE ORBIT OF LEXELL'S COMET, 1767–1779.

For further details concerning this comet the reader may consult also Professor Poor's valuable memoir: "Researches as to the Identity of the Periodic Comet of 1889–1896–1903 (Brooks) with the Periodic Comet of 1770 (Lexell)" (*Contributions from the Observatory of Columbia University*, No. 22, 1904).

In concluding his account of Lexell's Comet (*Mécanique Céleste*, Liv. IX, Chap. III, § 14), Laplace justly remarks: "It follows, from the calculations of the preceding chapter, that this comet passed directly through the space where *Jupiter* and his satellites were then situated: and yet it does not appear that the comet produced the slightest alteration in the motion of these bodies."

"*It not only happens that the comets do not trouble the motions of the planets and satellites, by their attractions; but if, in the immensity of past ages, some of the comets have encountered them, which is very probable, it does not seem that the shock can have had much influence on the motions of the planets and satellites.* It is difficult not to admit that the orbits of the planets and satellites were nearly circular at their origin, and that the smallness of their ellipticity, as well as their common direction from west to east, depend upon the primitive state of the planetary system. The action of the comets, and their impact upon those bodies, have not varied these phenomena; yet if one of them, with a mass equal to that of the Moon, should encounter the Moon, or a satellite of *Jupiter*, there is not the least doubt that it would render the orbit of the satellite very eccentric. Astronomy also presents to us two other phenomena, which seem to date their origin from that of the planetary system, and which would have been altered by a very small shock. We here allude to the equality in the rotatory motions of the Moon, and the librations of the three inner satellites of *Jupiter*. It is evident, from the formulas explained in the fifth book and in the preceding book that the shock of a comet, whose mass was only $\frac{1}{1000}$ part of that of the Moon, would be sufficient to give a very sensible value to the actual libration of the Moon, and to that of the satellites. We may, therefore, rest assured relative to the influence of the comets, and astronomers have no reason to fear that their action can impair the accuracy of astronomical tables."

§ 96. *Poor's Researches on Comet* 1889 *V* (Brooks).

In a series of important researches, beginning in 1889 and extending well into the nineties, Professor Charles Lane Poor, then of Johns Hopkins University, now of Columbia University, New York, devoted a number of years to the critical investigation of Comet 1889 V, which had been discovered by Brooks, at Geneva, New York, July 6, 1889. It had been found to have a period of about

seven years, and CHANDLER had shown that in 1886 it passed so near to *Jupiter* that the orbit was entirely changed by the encounter with that giant planet; the previous orbit having been quite large, with a period, he believed, of about twenty-seven years. It was at first inferred that the transformation of the orbit was similar to that which happened to LEXELL's Comet in 1767; in fact, that this was a return of the long-lost comet of LEXELL, and it was often referred to as the LEXELL-BROOKS Comet.

But DR. POOR's profound investigation — one of the most remarkable in the annals of Astronomical Science — finally led him to the conclusion that Comet 1889 V (BROOKS) was not identical with the lost comet of LEXELL, though the original orbit prior to the encounter with *Jupiter* in 1886 was somewhat similar. POOR's calculations were verified by the reappearance of the comet in 1896, near its predicted place, and by the second reappearance in 1903, so that the supposed identity with LEXELL's Comet was abandoned. The accompanying illustrations of the way in which Comet 1889 V was transformed by *Jupiter* have been calculated and drawn by DR. POOR, and are sufficiently remarkable to interest every one who follows the results obtained in the sublime Science of Celestial Mechanics. Few results are more astonishing than the transformations here brought out, and they will always be reckoned among the most beautiful achievements of modern Astronomy.

When nearest *Jupiter*, POOR showed that the path was an hyperbola passing between the planet and the first satellite, and, indeed, somewhat within the orbit of BARNARD's Satellite V. Yet no derangement of the motions of any of these satellites occurred, unless it was that of the Fifth, which had not yet been discovered. This absence of disturbance of the Galilean satellites showed that the comet's mass is insensible. As the comet described an arc of 95° about the planet, at about the same distance as the Fifth Satellite, without any known derangement of its motion, the mass probably is so rare that it would interpose but little resistance to so small a body as the Fifth Satellite.

The accompanying diagram illustrates DR. POOR's calculation of the comet's path through the satellite system. *Jupiter* has an oblateness of $\frac{1}{15.53}$ (cf. *A.N.* 3670, p. 405), and this ellipticity of the planet's figure exercises a considerable influence, and must be carefully taken account of in tracing the comet's path back through the appulse with *Jupiter* in 1886. The slight uncertainty in the exact path when very near *Jupiter* and troubled by the effect of the oblateness is the principal source of doubt as to the identity with LEXELL's Comet of 1770.

In *A.N.* 4321 DR. GUSTAV DEUTSCHLAND has reinvestigated this problem of evaluating the effect of the planet's oblateness, by improved mathematical

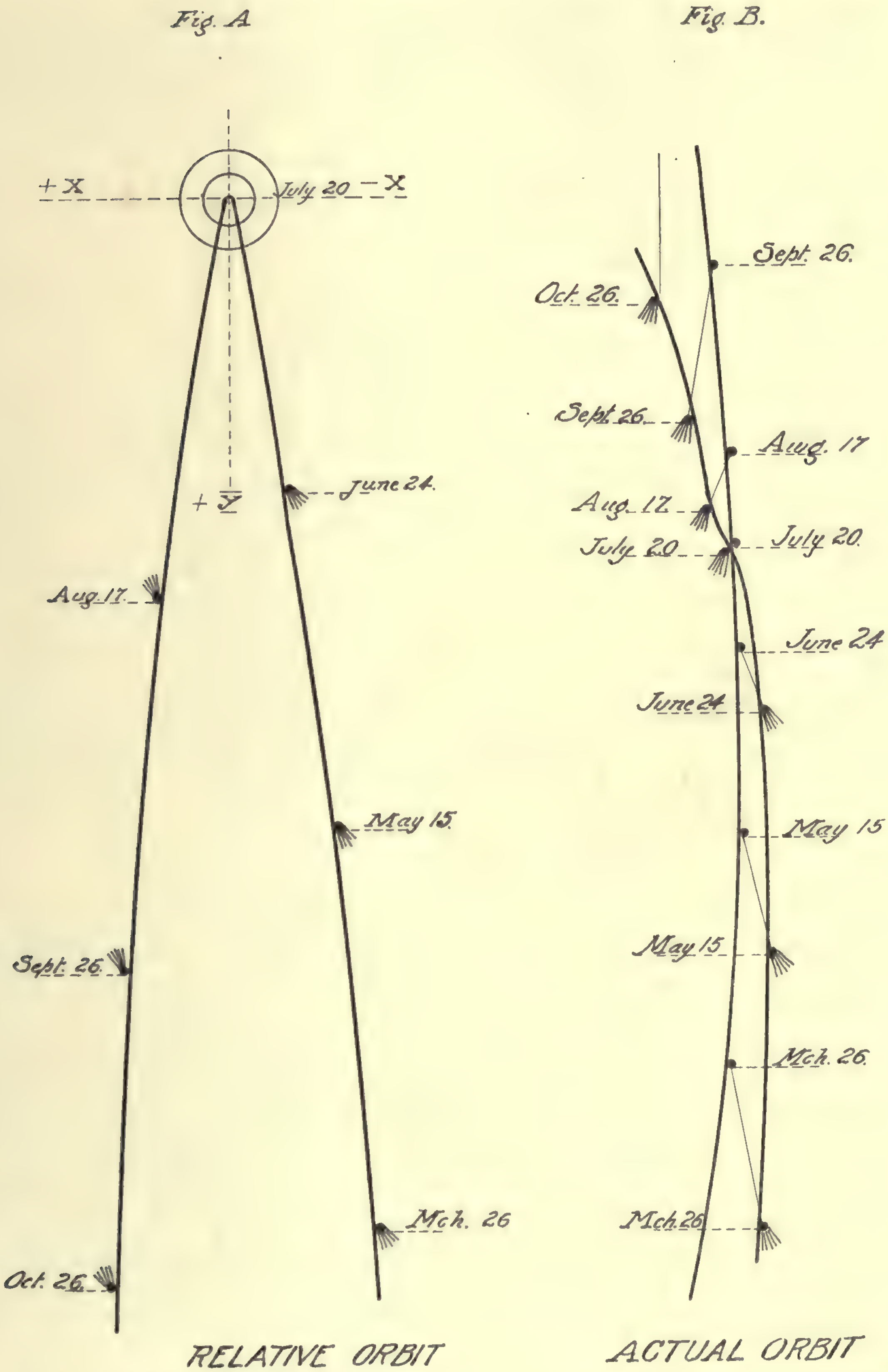


FIG. 25. POOR'S RELATIVE AND ACTUAL ORBIT OF COMET 1889 V (BROOKS), ABOUT JUPITER, 1886.

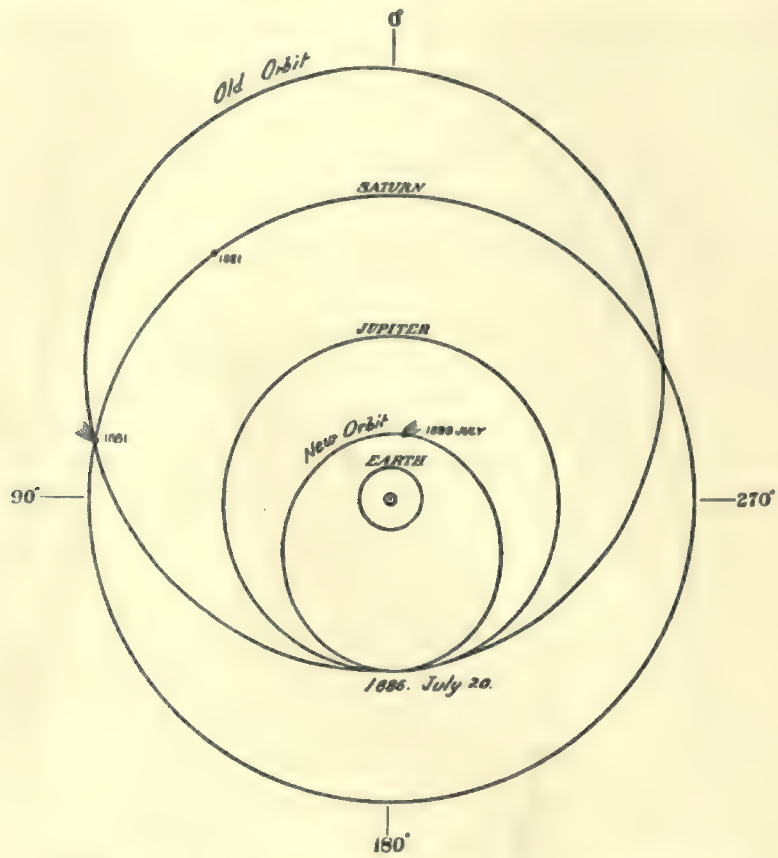


FIG. 26. TRANSFORMATION OF THE ORBIT OF COMET 1889 V (BROOKS) BY THE ENCOUNTER WITH JUPITER, JULY 20, 1886.

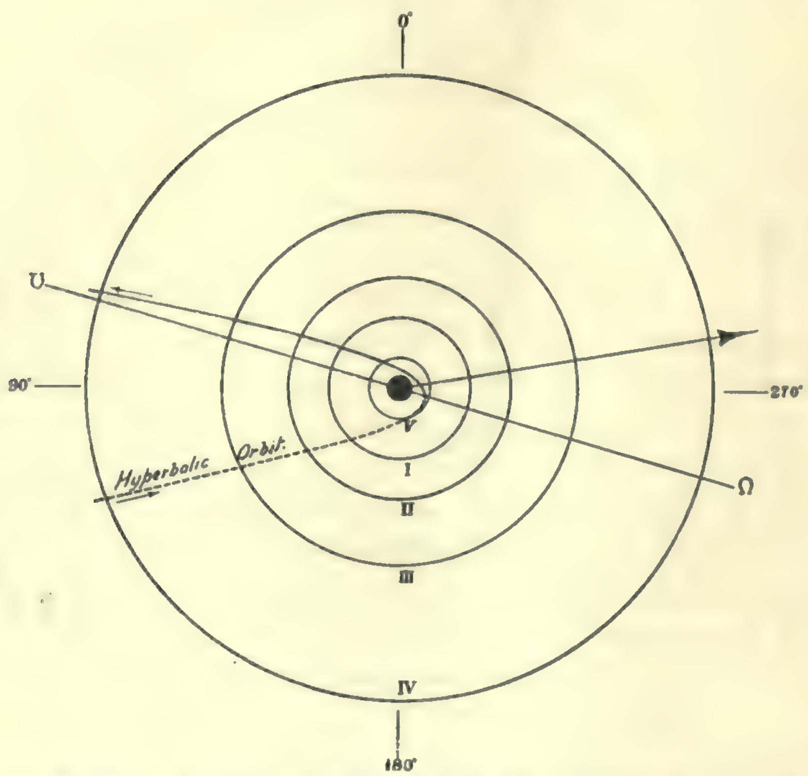


FIG. 27. POOR'S HYPERBOLIC ORBIT OF COMET 1889 V (BROOKS) THROUGH THE SYSTEM OF JUPITER'S SATELLITES.

methods, and is led to the conclusion that after all the Comet 1889 V (BROOKS) may well be identical with that of LEXELL. There is an appreciable difference between the hyperbolic elements found by POOR and those found by DEUTSCHLAND, as will be seen in the following table (cf. *A. N.* 4321, p. 7) :

	POOR.	DEUTSCHLAND
Epoch	1886, July $18^{d}\ 12^{h}\ 0^{m}$	1886 July $18^{d}\ 12^{h}\ 0^{m}$
$\pi =$	$282^\circ\ 37'\ 56''.1$	$282^\circ\ 33'\ 55''.7$
$\Omega =$	254 32 10.8	254 27 14.9
$i =$	61 28 21.2	61 29 31.6
$e =$	1.0112997	1.0123784
$\nu =$	0.0082420	0.0095846
$N =$	−0.0149655	−0.0173187

DEUTSCHLAND finds that the perijovium of the orbit of the comet had a declination of only 26° 28′, and thus the path was approximately symmetrical with respect to the spheroid of *Jupiter*. The distance of the comet on July $20^{d}\ 7^{h}\ 22^{m}.2$ he finds to have been only 1.14 equatorial radii of the planet, or 30,000 kilometers within the orbit of Satellite V. During a two-hour interval, in this region of close proximity to *Jupiter*, the comet swept over an arc of 95°.

PROFESSOR POOR first pointed out the necessity of the determination of the effects of the oblateness of figure on the comet's path in the immediate vicinity of the planet; and the work of DEUTSCHLAND, carried out in accordance with that suggestion, has now given the theory of the motion of this comet a greater degree of accuracy and interest than that of any other comet yet investigated.

§ 97. *Spheres of Activity of the Planets.*

It was first pointed out by D'ALEMBERT in his *Opuscules* (Tome VI, p. 304) that a planet may be supposed to have a sphere of activity within which the relative motion of a passing comet may be regarded as affected only by the planet's attraction, while the Sun acts merely as a disturbing body. The radius of this sphere of activity depends on the mass of the planet, and its distance from the Sun; and the limit fixed by LAPLACE has been generally adopted by astronomers. Let r and r' be the radii vectores of the planet and comet referred to the center of the Sun; then the Sun's action on the planet will be proportional to $\frac{1}{r^2}$ and that of the planet on the comet proportional to $\frac{m'}{(r'-r)^2}$. When the comet is without the sphere of activity of the planet (cf. LAPLACE, *Mécanique Céleste* Liv. IX, Chapter II, § 10), the quantity $\frac{1}{r^2}$ must greatly exceed $\frac{m'}{(r'-r)^2}$. But

when the comet is within the planet's sphere of activity the disturbing action of the Sun upon the planet and comet is proportional to

$$\frac{1}{r^2} - \frac{1}{r'^2} = \frac{(r' - r)}{r^2 r'} \cdot \frac{(r' + r)}{r'}.$$

And when r' is nearly equal to r, or $\frac{r' + r}{r}$ is nearly equal to 2, this disturbing action becomes $\frac{2(r' - r)}{r^2 r'} = \frac{2(r' - r)}{r^3}$, very nearly. This supposes that the expression $\frac{2(r' - r)}{r^3}$ is very small compared to $\frac{m'}{(r' - r)^2}$, or

$$\frac{1}{r^2} : \frac{m'}{(r' - r)^2} = \frac{m'}{(r' - r)^2} : \frac{2(r' - r)}{r^3} \tag{260}$$

And this proportion gives

$$r' - r = \rho = r \sqrt[5]{\frac{1}{2} m'^2}. \tag{261}$$

This is the radius of the planet's sphere of activity, according to LAPLACE's method, by which $\frac{m'}{(r' - r)^2}$ is taken to be a mean proportional between $\frac{1}{r^2}$ and $\frac{2(r' - r)}{r^3}$.

LEVERRIER has developed somewhat more fully LAPLACE's original conception of a sphere of activity for each planet. TISSERAND treats the theory and gives the following numerical values (*Mécanique Céleste*, Tome IV, p. 201) for the several members of the solar system, the semi-axis major of the Earth's orbit being unity:

	Radius of Sphere of Activity. ρ		Radius of Sphere of Activity. ρ
Mercury	0.001	*Jupiter*	0.322
Venus	0.004	*Saturn*	0.363
The Earth	0.006	*Uranus*	0.339
Mars	0.004	*Neptune*	0.576

The relatively larger spheres of activity of the outer planets is to be explained by the feebleness of the Sun's attraction at that great distance. It thus appears that *Neptune* has the largest of all the spheres of activity; yet as *Jupiter* is so much nearer the Sun it exerts the leading influence in transforming the orbits of bodies such as comets and asteroids, because a planet such as *Neptune* is so far away that few bodies come near enough to enter his sphere of activity, while in the case of *Jupiter* the chance that a body will enter the sphere of his influence in revolving around the Sun is always considerable.

§ 98. *On the Division and Disintegration of Comets.*

Closely connected with the sphere of activity of the planets is another related question of great importance, namely, the disintegration of comets and the production of meteoric swarms. This enables their nebulosity to be scattered over the solar system, so as to offer resistance to the motions of the planets and satellites, and thus act as a resisting medium. Now disintegration depends primarily on tidal action in the cometary mass; and is thus directly as the square of the radius of the comet, and mass of the Sun or planet, as the case may be, and inversely as the cube of its distance. For the most important term in the tide-generating potential is

$$V = \frac{3mr^2}{2\rho^3}\left(\cos^2 z - \frac{1}{3}\right), \tag{262}$$

where ρ is the distance of the disturbing body, m its mass, and r the radius of the comet, in the body of which the tide is raised, and z the angle between r and ρ.

This tendency to the disruption of comets will become quite considerable when they pass close to large bodies like the Sun and *Jupiter*, since the comet has so little mass that it is practically powerless, by virtue of the feeble attraction on its own particles, to resist disruption. Accordingly it is in this way that we are to explain the observed disruption of comets after they have passed near the Sun or *Jupiter*. As the comets revolve in eccentric orbits and thus come near the sun, there is a tendency, in the course of ages, for all comets to become gradually disrupted, and their matter more or less diffused over the solar system, where it acts as a slight resisting medium to retard the motions of revolving bodies. This subject has been especially investigated by KIRKWOOD and SCHIAPARELLI, and more recently by CALLANDREAU and TISSERAND.

CALLANDREAU suggests that the "comet groups" have arisen in this manner; also that *Jupiter's* family of comets has been greatly multiplied by this process of breaking up. Perturbations would in time separate the elements of the fragments still more widely, and give the appearance of separate bodies moving in neighboring orbits. In several cases, the comets have been observed to be divided, with nuclei traveling in parallel orbits, but so far separated that they could not again become united under the feeble power of their mutual gravitation. Undoubtedly the number of comets in a family has thus been greatly increased. The connection of this process of disintegration with the production of meteoric swarms is generally recognized. It was, in fact, pointed out as long ago as December, 1861, by KIRKWOOD, who showed in the *Danville Quarterly Review*, that the di-

vision of BIÉLA's Comet observed in 1845–6 (cf. CLERKE's *History of Astronomy During the Nineteenth Century*, pp. 96–97) was the logical outcome of a general process in nature. With prophetic foresight he asked: "May not our periodic meteors be the debris of ancient but now disintegrated comets, whose matter has become distributed round their orbits?" (cf. *Nature*, Vol. VI, p. 148).

It was not felt to be surprising that the matter in the tails of comets should be permanently lost to the nucleus by the repulsive forces emanating from the Sun; but, rather, that the heads also should have suffered disintegration and diffusion along the orbit, yet this was fully explained by tidal action in masses with such feeble power of resistance. Thus in time a swarm becomes diffused over the whole path, and constitutes a ring or girdle, such as that producing the August meteors. On this account LEVERRIER recognized, in 1867, that the *Perseids* constitute an older swarm than the *Leonids*. The late MISS CLERKE has discussed these problems in an interesting and lucid manner in her well-known *History of Astronomy During the Nineteenth Century*, pp. 323–340. The following is her suggestive account of the observed disruption of BIÉLA's Comet (loc. cit. p. 96): "The return of the same body in 1845–6 was marked by an extraordinary circumstance. When first seen, November 28, it wore its usual aspect of a faint round patch of cosmical fog; but on December 19, MR. HIND noticed that it had become distorted somewhat into the form of a pear; and ten days later, it had divided into two separate objects. This singular duplication was first perceived at New Haven, in America, December 29, by MESSRS. HERRICK and BRADLEY, and by LIEUTENANT MAURY, at Washington, January 13, 1846. The earliest British observer of the phenomenon (noticed by WICHMANN the same evening at Königsberg) was PROFESSOR CHALLIS. 'I see *two* comets!' he exclaimed, putting his eye to the great equatorial of the Cambridge Observatory on the night of January 15; then, distrustful of what his senses had told him, he called in his judgment to correct their improbable report by resolving one of the dubious objects into a hazy star. On the 23d, however, both were again seen by him in unmistakable cometary shape, and until far on in March (OTTO STRUVE caught a final glimpse of the pair on the 16th of April), continued to be watched with equal curiosity and amazement by astronomers in every part of the northern hemisphere. What SENECA reproved EPHORUS for supposing to have taken place in 373 B.C. — what PINGRÉ blamed KEPLER for conjecturing in 1618 — had then actually occurred under the attentive eyes of science in the middle of the nineteenth century!"

§ 99. *Researches of* Newton, Adams *and* Schiaparelli *on the Origin of the November Meteors.*

Shortly before the great star shower of November, 1866, the problem of the probable periodicity of these displays was subjected to a most searching examination by several eminent astronomers, but especially by Professor H. A. Newton, of Yale University, Professor Schiaparelli, of Milan, and Professor J. C. Adams, of Cambridge, England. These showers of stars had been recorded by miscellaneous chroniclers since the classic period, and during the nineteenth century a systematic search of old records was made by Edward Biot, Quetelet, Humboldt, Newton, and others, with the result that a regular periodicity was established. On November 12, 1799, a remarkably brilliant shower was witnessed by Humboldt and Bonpland, in South America. The account given in Humboldt's *Travels*, Vol. I, pp. 351–353, runs thus: "The night of the 11th of November was cool and extremely fine. From half after two in the morning, the most extraordinary luminous meteors were seen in the direction of the east. M. Bonpland, who had risen to enjoy the freshness of the air, perceived them first. Thousands of bolides and falling stars succeeded each other during the space of four hours. Their direction was very regular from north to south. They filled a space in the sky extending from due east 30° to north and south. In an amplitude of 60° the meteors were seen to rise above the horizon at E.N.E. and at E., to describe arcs more or less extended, and to fall towards the south, after having followed the direction of the meridian. Some of them attained a height of 40°, and all exceeded 25° or 30°. There was very little wind in the low regions of the atmosphere, and that little blew from the east. No trace of clouds was to be seen. M. Bonpland states that, from the first appearance of the phenomenon, there was not in the firmanent a space equal in extent to three diameters of the moon, which was not filled with bolides and falling stars every instant. The first were fewer in number, but as they were of different sizes, it was impossible to fix the limit between these two classes of phenomena. All these meteors left luminous traces from five to ten degrees in length, as often happens in the equinoctial regions. The phosphorescence of these traces, or luminous bands, lasted seven or eight seconds. Many of the falling stars had a very distinct nucleus, as large as the disc of *Jupiter*, from which darted sparks of vivid light. The bolides seem to burst as by explosion; but the largest, those from 1° to 1° 15′ in diameter, disappeared without scintillation, leaving behind them phosphorescent bands (trabes) exceeding in breadth fifteen or twenty minutes. The light of these meteors was white and not reddish, which

must doubtless be attributed to the absence of vapour and the extreme transparency of the air. For the same reason, within the tropics, the stars of the first magnitude have, at their rising, a light decidedly whiter than in Europe."

"Almost all the inhabitants of Cumana witnessed this phenomenon, because they had left their houses before four o'clock, to attend the early morning mass. They did not behold these bolides with indifference; the oldest among them remembered that the great earthquakes of 1766 were preceded by similar phenomena. The Guaiqueries in the Indian suburb alleged 'that the bolides began to appear at one o'clock; and that as they returned from fishing in the gulf, they had perceived very small falling stars towards the east.' They assured us that igneous meteors were extremely rare on those coasts after two o'clock in the morning.

"The phenomenon ceased by degrees after four o'clock, and the bolides and falling stars became less frequent; but we still distinguished some to northeast by their whitish light, and the rapidity of their movement, a quarter of an hour after sunrise." (HUMBOLDT'S Personal Narrative of Travels to the Equinoctial Regions of America, translated by ROSS, BOHN'S Library, London, 1852).

These lucid observations of HUMBOLDT, made at Cumana, Venezuela, are almost identical with the accounts I have often heard my Father, the Honorable NOAH SEE (1815–1890), give of the meteoric shower of November 12, 1833, as it was observed by him in the mountains of Virginia. He was at the time on a hunting expedition, camping under the open sky, and about two o'clock in the morning of the 12th the shower was noticed to be in full progress, and so brilliant that the meteors illuminated the sky and seemed to fill it as completely as a storm of snowflakes frequently do the air. Although my Father, Grandfather, and their party witnessed the display with entire equanimity, recognizing from the first that it was of cosmical origin, it frightened the negroes so badly that they believed the end of the world had come.

The researches of PROFESSOR H. A. NEWTON, in 1864, verified the prediction of OLBERS that a shower would occur in 1866 or 1867, and the conclusions of the astronomers were in due time verified by the phenomena. The shower was witnessed in Europe on November 13, 1866, but was not conspicuous in America; while on the same date the following year, the display was brilliant enough to excite general public interest in America, but was scarcely noticeable in Europe. When the period for the next expected display came around in 1899 and 1900, interest was alive to the expected event, but it almost failed to appear. No conspicuous display occurred on either continent. Whether the falling off in the brilliancy of these showers in 1866 and 1899, compared to those of 1799 and

1833, was due to a derangement of the swarm below its usual path, by some two million miles, as DOWNING and STONEY concluded from a study of the perturbations, or to a gradual wasteage of the meteors, cannot yet be decided; but the chances are that future displays will be less brilliant than those of the past.

PROFESSOR SCHIAPARELLI investigated the meteroic orbit from the radiant of the shower of November 13, 1866, and found that there was a relation between the orbit of the shower and that of a comet discovered by TEMPEL at Marseilles, in December, 1865, and afterwards independently discovered by TUTTLE at the Naval Observatory in Washington. The meteoric swarm with a period of 33¼ years was moving in an elongated elliptic orbit extending beyond *Uranus*, and crossing the orbit of the Earth at the point which our planet occupies at the epoch of the shower.

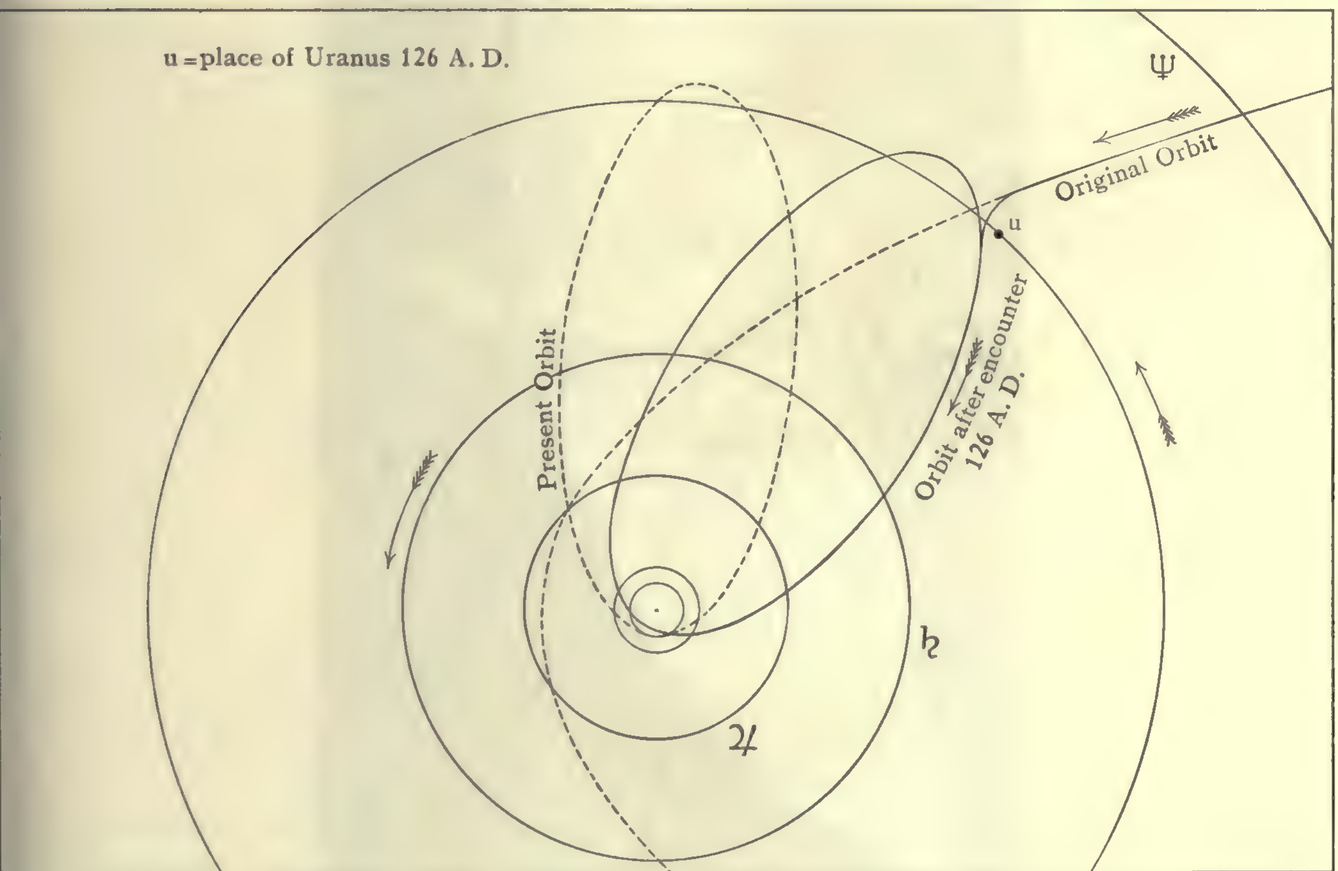


FIG. 28. LEVERRIER'S THEORY OF THE CAPTURE OF TEMPEL'S COMET BY URANUS, 126 A.D.

§ 100. *Connection Between Comets and Meteoric Swarms Established by the Researches of* LEVERRIER, OPPOLZER, SCHIAPARELLI *and* KLINKERFUES.

As soon as it became known that a comet was moving in the same path as the meteors, great interest was awakened among astronomers. LEVERRIER presented a very complete investigation of the orbit of the meteoric shower to the Paris Academy of Sciences, January 21, 1867. OPPOLZER, of Vienna, had been occu-

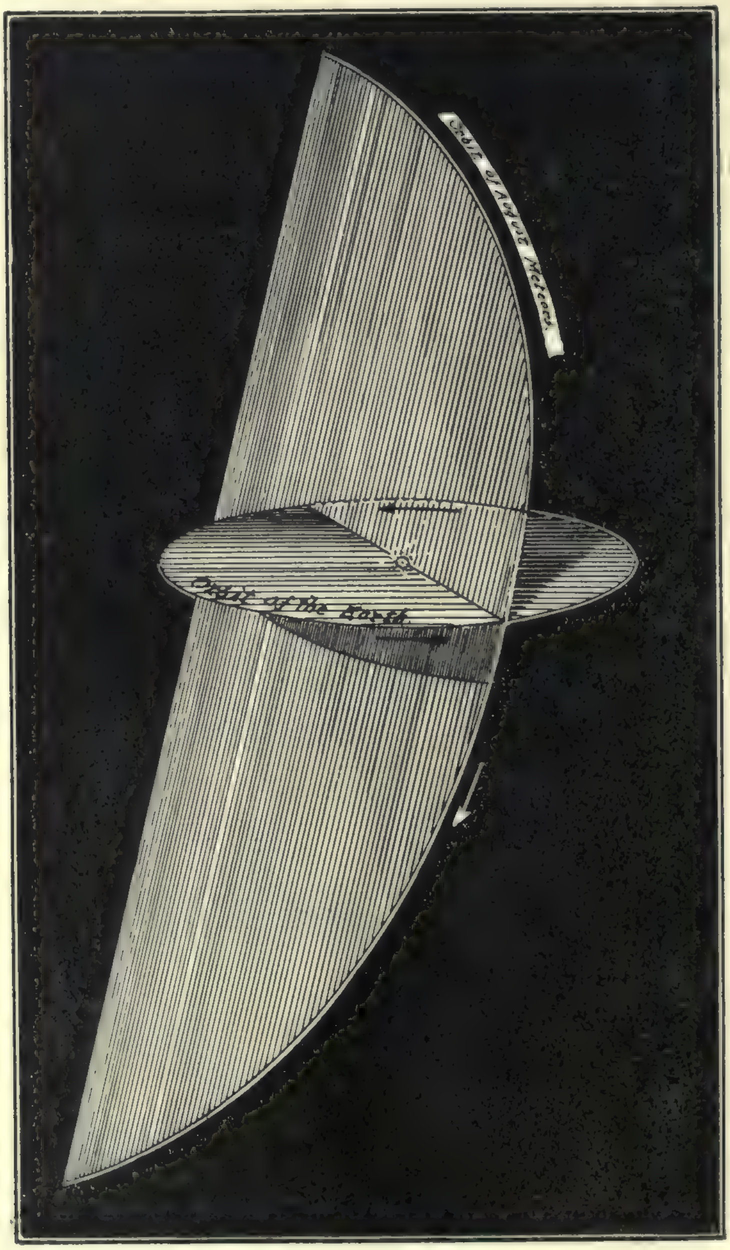


FIG. 29. ORBIT OF THE AUGUST METEORS.

pied with a critical investigation of the orbit of TEMPEL'S Comet, and on January 28, 1867, published in the *Astronomische Nachrichten* the elements at which he had independently arrived. The following table shows the elements of the comet found by OPPOLZER and of the meteoric swarm calculated by LEVERRIER. The agreement is sufficiently impressive to tell its own story:

	OPPOLZER.	LEVERRIER.
Period of Revolution	33.18 years	33.25 years
Eccentricity	0.9054 "	0.9044 "
Perihelion Distance	0.9765 "	0.9890 "
Inclination of Orbit	162° 42′	165° 19′
Longitude of Node	51° 26′	51° 18′
Longitude of Perihelion	42° 24′	Near Node

The only elements which differ materially are the inclinations, and this arose from a defect in LEVERRIER'S assumed position of the radiant. ADAMS, by an independent calculation, made the inclination of the orbit of the meteoric swarm 163° 14′, which agreed sufficiently well with OPPOLZER'S inclination of TEMPEL'S Comet to leave no doubt that the two bodies were following the same orbit in space. *Hence it was confidently concluded that the November meteoric shower arose from the earth encountering a swarm of particles following* TEMPEL'S *Comet in its orbit.* The theory in Fig. 28 is from YOUNG'S *General Astronomy.*

SCHIAPARELLI had been occupied with the August meteors which have their radiant in the constellation *Perseus*, and are often very brilliant. He now found the August meteors to be moving in the same orbit as the Comet II, of 1862, and that the meteors are scattered over a great part of the elliptic path, which extends some twenty astronomical units beyond *Neptune*, the period, according to OPPOLZER, being 124 years.

COMPARISON OF ELEMENTS OF THE COMET AND METEOR SWARM.

	OPPOLZER Comet II, 1862	SCHIAPARELLI August Meteors
Perihelion Distance	0.9626	0.9643
Inclination of Orbit	113° 35′	115° 57′
Longitude of Node	137° 27′	138° 16′
Longitude of Perihelion	344° 41′	343° 28′

PROFESSOR KLINKERFUES, of Göttingen, observed a striking display of meteors, on November 27, 1872, when BIÉLA'S Comet was known to be near the Earth, and on November 30, he telegraphed the English astronomer POGSON, at Madras: "*Biéla* touched Earth November 27; search near *Theta Centauri.*" KLINKERFUES hoped thus to detect the "anti-radiant" as the comet receded from the Earth, and in this brilliant idea he was successful. Bad weather hindered POGSON for a day and a half, but on December 2d, sure enough, he saw BIÉLA'S Comet in the predicted position.

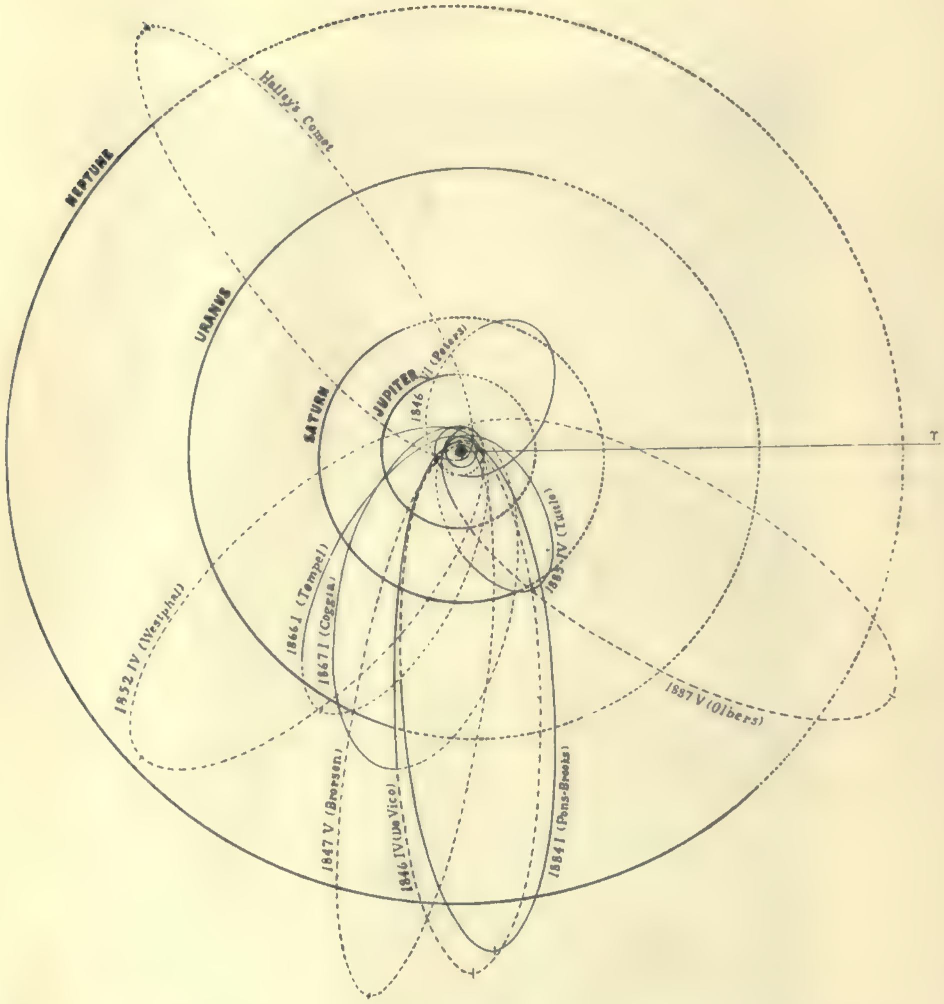


Fig. 30. Diagram of the Comet Families of Saturn, Uranus and Neptune. The Inclinations Are Here Neglected (*Popular Astronomy*, December, 1909).

PLATE VI.

DONATI'S COMET, 1858, OCT. 4 (BOND). GREAT COMET OF 1882 (SEE). PATH OF HALLEY'S COMET NEAR THE EARTH, 1910.

CHAPTER X.

Dynamical Theory of the Capture of Satellites and of the Division of Nebulae under the Secular Action of a Resisting Medium.*

§ 101. *Introductory Considerations.*

Highly important dynamical considerations, based on the mechanical principle of the conservation of areas, in the form of a criterion proposed by Babinet, in 1861, have been adduced in the *Astronomische Nachrichten*, No. 4308, and in the *Publications of the Astronomical Society of the Pacific*, No. 125, April 1, 1909, to show that the planets and satellites of the solar system have never been detached from the central bodies which now govern their motions, by acceleration of rotation, as was supposed by Laplace, and for a long time very generally believed by astronomers; but, on the contrary, that all of these bodies have been captured, or added from without, and have since had their orbits reduced in size and rounded up under the secular action of a resisting medium. The argument there set forth and reproduced with some additions in Chapter XV of this volume is very brief, but sufficient to be conclusive on the main points; for the dynamical rigor and universal validity of Babinet's criterion admits of no dispute, and the data given in the Table of Calculated Rotation Periods, when combined with the Observed Periodic Times, show such enormous disparity that it is clear that centrifugal force could not have been effective in detaching these masses.

For the centrifugal force varies as the square of the velocity divided by the radius; and if we form a table of the squared velocities from the periods as given in *A.N.* 4308, pp. 187–190, the radius of the rotating central mass being by hypothesis the same as that of the radius vector of the body revolving about it, we shall find the differences $R_c^2 - P_o^2$ so very large that any supposition that centrifugal force, due to axial rotation, has exercised a sensible influence will

* The substance of this chapter was communicated to the *Astronomische Nachrichten* for publication, May 6, 1909, and the rigorous demonstration of the capture of the satellites announced by a cablegram in *A.N.*, 4323. An additional communication was made to the Astronomical Society of the Pacific, June 25, 1909 (c.f. *A.N.* 4341–2).

have to be abandoned. The efficiency of the resulting centrifugal force, compared to that which would detach the body, would be measured by the fraction γ in the expression

$$1 - \frac{P_0^2}{R_e^2} = 1 - \gamma ;$$

which in all cases proves to be quite insignificant. Thus, in the case of the inner ring of *Saturn*, where this fraction is the largest, on account of the rapid rotation of that planet, the value of γ never exceeds about a seventh part; in other words, the centrifugal force, due to axial rotation, would have to be about seven times larger than it is, in order to detach a satellite from the equator of *Saturn*, if the planet rotated with its present moment of momentum, but the globe were expanded to fill the sphere enclosed by the inner ring.

One might, indeed, imagine that the central bodies have gradually increased in mass with the lapse of ages, so that at one time the central attraction was much less than at present, and the periodic time of a satellite, therefore, much longer, corresponding to a smaller orbital centrifugal force and possibly a nearer approach to equality with the lessened centripetal force of the planet; yet by no change of this kind will it be found possible to establish an approximation to equality between the centrifugal and centripetal forces, and we shall be compelled to admit that LAPLACE'S hypothesis is wholly inadmissible, even for the most favorable case in the solar system.

As the planets and satellites have therefore all been captured and have since had their orbits reduced in size and rendered rounder and rounder by the secular action of the nebular resisting medium formerly pervading our system, it becomes desirable to indicate somewhat more fully how this great transformation has come about. In particular, it becomes advisable to set forth with more detail how the satellites were captured, and how nebulae inevitably divide under the natural operation of their own gravitation; and that, too, without the intervention of the conditions of fluid equilibrium under hydrostatic pressure, as heretofore very generally assumed by mathematicians.

§ 102. JACOBI'S *Integral and the Equation of Relative Energy.*

In order to bring out the effects of a resisting medium upon the capture of satellites and upon the division of nebulosity between rival centers of attraction, it is necessary to recall very briefly some of the results of the researches of mathematicians upon the Problem of Three Bodies, in the restricted case where one

body is a particle and the other two revolve in circles about their common center of gravity.

For the sake of continuity in the development of the argument, we shall first recall very briefly the required equations established in Chapter VIII. In any system, whatever be the relative magnitude of the two principal bodies, we may always adopt a special system of units, and take the sum of the masses to be unity or put

$$M + m = 1 = (1 - \mu) + \mu . \tag{263}$$

And we may take the unit of distance such that the space between the centers of gravity of M and m shall be unity, or

$$a = \sqrt{\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2} = 1 , \tag{264}$$

where ξ_1, η_1, ξ_2, η_2 are the co-ordinates of the centers of gravity of the bodies here assumed to move in the $\xi\eta$-plane. And we may put the constant of attraction

$$k^2 = \frac{4\pi^2}{\tau^2} \cdot \frac{a^3}{M + m} = n^2 a^3 = 1, \tag{265}$$

where τ is the periodic time, also taken to be unity.

Now, if the smaller mass be placed on the x-axis at the initial epoch, and new axes (xyz), with the same origin, be imagined to rotate at the same rate that the two bodies $1 - \mu$ and μ revolve in circles under their mutual attraction about the common center of gravity, we shall have the well-known differential equations of motion for the particle

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \frac{\partial\Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \frac{\partial\Omega}{\partial y}, \\ \frac{d^2z}{dt^2} \qquad\quad &= \frac{\partial\Omega}{\partial z}; \end{aligned}\right\} \tag{266}$$

where

$$2\Omega = x^2 + y^2 + \frac{2(1 - \mu)}{\rho_1} + \frac{2\mu}{\rho_2}, \tag{267}$$

$$\left.\begin{aligned} \rho_1 &= \sqrt{(x - x_1)^2 + y^2 + z^2} , \\ \rho_2 &= \sqrt{(x - x_2)^2 + y^2 + z^2} . \end{aligned}\right\} \tag{268}$$

Here ρ_1 and ρ_2 are the radii vectores of the particle referred to the two revolving

centers of attraction, and the fixed co-ordinates are connected with those referred to the rotating axes by the relations

$$\left.\begin{aligned} \xi &= x\cos t - y\sin t\,, \\ \eta &= x\sin t + y\cos t\,, \\ \zeta &= z\,. \end{aligned}\right\} \tag{269}$$

As Ω is a function of x, y, z only and of constant quantities, the equations (266) admit of an integral, when they are successively multiplied by $\frac{2dx}{dt}$, $\frac{2dy}{dt}$, $\frac{2dz}{dt}$ respectively, and the products added. For this gives

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\int\left\{\frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} + \frac{\partial\Omega}{\partial z}\frac{dz}{dt}\right\}dt = 2\Omega - C = V^2$$

$$= x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - C\,. \tag{270}$$

This is the integral obtained by JACOBI (*Comptes Rendus de l'Académie des Sciences*, Tome III, p. 61), and, therefore, called by DR. G. W. HILL, the JACOBIAN Integral (cf. *Collected Mathematical Works of* G. W. HILL, Vol. I, p. 244). This integral $V^2 = 2\Omega - C$ represents the velocity of the particle under the attraction of the two bodies, and therefore has been called by LORD KELVIN (*Philosophical Magazine*, Vol. 34, Nov., 1892, p. 447) and afterwards by SIR GEORGE DARWIN, the *equation of relative energy*.

§ 103. *Surfaces of Zero Relative Velocity.*

The JACOBIAN Integral $V^2 = 2\Omega - C$, when put equal to zero, gives the surfaces corresponding to zero relative velocity of the particle:

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2 + z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2 + z^2}} - C = 0\,. \tag{271}$$

These surfaces were first discussed by DR. G. W. HILL in his celebrated "*Researches in the Lunar Theory*" (cf. HILL's *Collected Mathematical Works*, Vol. I, pp. 294–304), but they have since been treated by POINCARÉ, DARWIN, and other mathematicians.

The forms of the curves on the co-ordinate planes are easily found by the usual process in the theory of surfaces. Thus, for the xy-plane, we put $z = 0$, and have

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C\,. \tag{272}$$

which has been treated quite fully by PROFESSOR SIR G. H. DARWIN in his celebrated memoir on "Periodic Orbits" in the *Acta Mathematica*, Vol. XXI, 1897. If we put $y = 0$, we get the curves on the xz-plane:

$$x^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2+z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2+z^2}} = C. \tag{273}$$

As the object of this brief discussion is to point out the practical use of these results in Astronomy, rather than to examine the details from a mathematical standpoint, we pass at once to the curves in the xy-plane as drawn by DARWIN. These are given in Fig. 18, p. 170. The two bodies conveniently designated as the *Sun* and *Jove*, respectively, have in this case masses in the ratio of ten to one, $1 - \mu = \frac{10}{11}$, $\mu = \frac{1}{11}$, or the sun has ten times the mass of the planet. In describing the figure DARWIN says (loc. cit., pp. 112–113): "An important classification of orbits may be derived from this figure. When C is greater than 40.1821 the third body must be either a superior planet moving outside of the large oval, or an inferior planet moving inside of the larger internal oval, or a satellite moving inside of the smaller internal oval; and it can never exchange one of these parts for either of the other two. The limiting case $C = 40.1821$ gives superior limits to the radii vectores of inferior planets and of satellites, which cannot sever their connections with their primaries.

"When C is less than 40.1821 but greater than 38.8760, the third body may be a superior planet, or an inferior planet or satellite, or a body which moves in an orbit which partakes of the two latter characteristics; but it can never pass from the first condition to any of the latter ones.

"When C is less than 38.8760 and greater than 34.9054, the body may move anywhere save inside of a region shaped like a horse-shoe. The distinction between the two sorts of planetary motion and the motion as a satellite ceases to exist, and if the body is started in any one of these three ways it is possible for it to exchange the characteristics of its motion for either of the two other modes.

"When C is less than 34.9054 and greater than 33, the forbidden region consists of two strangely shaped portions of space on each side of SJ.

"Lastly, when C is equal to 33, than which it cannot be less, the forbidden regions have shrunk to a pair of infinitely small closed curves enclosing the third angles of a pair of equilateral triangles erected on SJ as a base."

If we consider equation (271), we see that when $z = 0$, and x and y are

small, the first and second terms become relatively insignificant, and there only remains

$$\frac{1-\mu}{\sqrt{(x-x_1)^2+y^2}}+\frac{\mu}{\sqrt{(x-x_2)^2+y^2}}=\frac{C}{2}-\epsilon=\frac{1-\mu}{\rho_1'}+\frac{\mu}{\rho_2'}; \tag{274}$$

which shows that the surfaces approximate equipotential surfaces about the two masses $1-\mu$ and μ. In like manner, if $y=0$, and x and z are small, equation (273) gives

$$\frac{1-\mu}{\sqrt{(x-x_1)^2+z^2}}+\frac{\mu}{\sqrt{(x-x_2)^2+z^2}}=\frac{C}{2}-\epsilon=\frac{1-\mu}{\rho_1''}+\frac{\mu}{\rho_2''}. \tag{275}$$

Thus we see that the curves on the xz-plane are similar to those on the xy-plane, and both resemble equipotential surfaces. For small values of the co-ordinates the surfaces of relative energy are nearly symmetrical about the axis of x, and may be produced approximately by revolving the curves in the xy-plane about the axis SJ.

In tri-dimensional space these surfaces of zero velocity are suspended like narrow curtains from an asymptotic cylinder with radius equal to $\sqrt{C}$; for as z^2 increases indefinitely equation (271) approaches the limit $x^2+y^2=C$, and the radius of the asymptotic cylinder therefore is $\sqrt{C}$, as was long ago pointed out by Dr. Hill.

We need not here dwell on the outer parts of the above figure, which have been fully discussed by Hill, Darwin, Poincaré, and others; but shall invite attention to the interior region, where the surfaces are closed about each body, or run together in the form of an hour-glass or pear-shaped figure with equal or unequal bulbs, according as the masses $1-\mu$ and μ are equal or unequal, respectively. In the general case they are unequal; but as numerous systems of double stars have a comparatively equable distribution of mass, this special case is deserving of some attention.

Now, as to the form of the closed and connected surfaces of the hour-glass or pear-shaped figure in tri-dimensional space, it is sufficient, as already remarked, to imagine the inner parts of Darwin's Fig. 1,* up to $C=38.88$, revolved about the x-axis coinciding with SJ; and then imagine the surfaces of revolution very slightly flattened in the direction parallel to the z-axis, to take into account the small distortion due to centrifugal force incident to revolution of the system in the xy-plane. For the surface $2\Omega-C=0$ involves the rotation potential, and is not quite the same in all directions, but a little larger in the plane of motion.

Thus we see that the two bodies have separate closed folds around each center of attraction, and besides a series of pear-shaped or hour-glass surfaces about

* This is our Fig. 18, page 170, to which the reader is referred.

both bodies; and, moreover, these surfaces are symmetrical about the x-axis, coinciding with SJ, except for the effects of rotation, which makes these surfaces a little flattened in the direction parallel to the z-axis. The effect of distortion on these otherwise symmetrical surfaces is quite analogous to the flattening of figures of equilibrium of rotating masses of fluid. The figures of equilibrium assumed by a mass of incompressible fluid when subjected to the mutual gravitation of its particles and endowed with a rotatory motion, have been studied by mathematicians for more than two centuries, and of late years have been especially investigated by POINCARÉ and DARWIN, by the most powerful methods of modern mathematical analysis. These eminent mathematicians and others have established the existence of a series of pear-shaped figures of equilibrium for a rotating mass of fluid, with striking resemblance to the surfaces here discussed. The calculation of the figures of equilibrium, however, is more difficult than that of the energy surfaces, because in a fluid mass under its own gravitational attraction, the forces of each element contribute to the shaping of the figure of the mass, and the figure in turn determines the intensity of the forces tending to modify the fluid surface; whereas in the case of the energy surfaces the attractive forces may be regarded practically as centered in two points, since the non-sphericity of the figures of the masses may generally be neglected.

§ 104. *How a Particle May Move About S and J Separately and Collectively.*

The nature of the surfaces closed about each body separately and about the two collectively is now clear, and as we have shown that these closed surfaces are nearly symmetrical about the line SJ in all directions in space, it is evident that, for particles confined within these folds, the motion in tri-dimensional space will be essentially similar to that of a particle moving in the xy-plane. Thus DARWIN's classification of orbits, just given in § 103, becomes applicable to all motion in which C is greater than 38.88, and practically for all motion in which C is less than 34.91, when we disregard the outer parts of the figure, as not here under consideration.

Accordingly, it is clear that a particle, with C less than 38.88, may move about the two bodies separately, pass freely from one space to the other, and around both together; or again it may move about either of the two bodies separately and go between them in such a way that, as it quits the control of S, it may pursue a retrograde path about J.

DARWIN has discussed this motion in a characteristically lucid manner as follows: "Being ignorant of the nature of the orbits of which I was in search, I

determined to begin by a thorough examination of one case. It seemed likely that the most instructive results would be obtained from cases in which it should be possible for an inferior planet and satellite to interchange their parts. Now, when C is greater than 38.8760 but less than 40.1821, the two inferior ovals of the curve of zero velocity coalesce into the shape of an hour-glass, and thus interchange of parts is possible. I therefore began by the consideration of the case where C is 39, and traced a large number of orbits which start at right angles to SJ, and in some cases I also traced the orbit with reference to axes fixed in space.

"The two curves which represent the orbit in space and with reference to the moving plane, contain a complete solution of the problem.

"For if the curve on the moving plane be drawn as a transparency, and if the *Sun* in the two figures be made to coincide, and if the transparent figure be made to revolve uniformly about the *Sun*, the intersection of the two curves will give the position of the body both in time and place.

"In order to exhibit this I show in Fig. 2 a certain orbit with reference to axes fixed in space and also the same orbit referred to rotating axes. In the former figure the simultaneous positions of the planet and of *Jove* are joined by dotted lines. It is interesting to observe how the body hangs in the balance between the two centers, before the elliptic form of the orbit asserts itself, as the body approaches the *Sun*.

"This figure, and others of the same sort, are instructive as illustrating the usual sequence of events in orbits of this class.

"If a planet be started to move about the *Sun* in an orbit of a certain degree of eccentricity, it will at first move with more or less exactness in an ellipse with advancing perihelion. But as the aphelion approaches conjunction with *Jove* the perturbations will augment at each passage of the aphelion. At length the perturbation becomes so extreme that the elliptic form of the orbit is entirely lost for a time, and the body will either revert to the *Sun*, or it will be drawn off and begin a circuit round *Jove*. In either case after the approximate concurrence of aphelion with conjunction, the orbit will have lost all resemblance to its previous form.

"The Fig. 2 exhibits the special case in which the body only makes a single circuit round *Jove*, and where the heliocentric elliptic orbit before and after the crisis has the same form; the perihelion has, however, advanced through twice the angle marked ω on the figure. In general the body would, after parting from the *Sun*, move several times round *Jove* until a concurrence of apojove with conjunction produced a severance of the connection, but in the figure this concurrence happens after the first circuit. If the neck of the hour-glass defining the

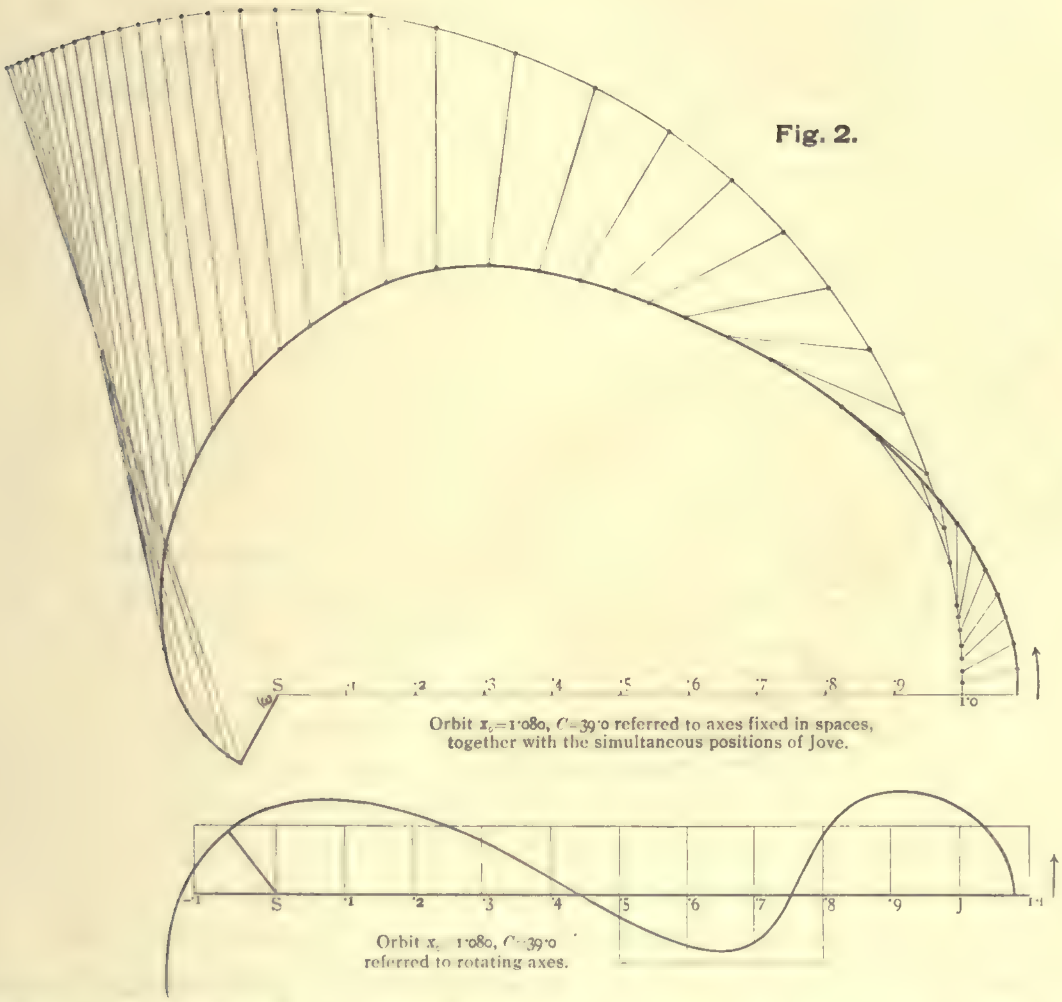


Fig. 31. Darwin's Fig. 2, Showing Orbit of Particle Passing from Jove to the Sun (*Acta Mathematica*, Vol. XXI, p. 169).

curve of zero velocity be narrow, the body may move hundreds of times round one of the centers before its removal to the other." (cf. "*Periodic Orbits*," pp. 168–170).

§ 105. *Non-Periodic Orbits Passing from Jove to the Sun.*

Without going into the details of the nature of this movement, or into the laborious numerical processes by which the paths have been calculated by DARWIN, we may reproduce, in Fig. 32, one of his most interesting figures showing some of the non-periodic paths extending from *Jove* to the *Sun*.

These cases are, of course, ideal and somewhat arbitrary, with the masses always in the ratio of 10 to 1, and, therefore, with the planet relatively about 100 times larger than any found in the solar system; but they throw a flood of light upon the processes at work in actual nature.

In the passage partially quoted above ("*Periodic Orbits*" pp. 170–171), DARWIN concludes his discussion as follows:

"It seems likely that a body of this kind would in course of time find itself in every part of the space within which its motion is confined. Sooner or later it must pass indefinitely near either to the *Sun* or to *Jove*, and as in an actual planetary system those bodies must have finite dimensions, the wanderer would at least collide with one of them and be absorbed. We thus gain some idea of the process by which stray bodies are gradually swept up by the *Sun* and planets.

"It might be supposed that all possible orbits for any value of C will pass through a similar series of changes and that the bodies which move in them will be thus finally absorbed. LORD KELVIN is of opinion that this must be the case, and that all orbits are essentially unstable. This may be so when sufficient time is allowed to elapse, but we shall see later that, even when the hour-glass has an open neck, there are still stable orbits, as far as our approximation goes. The only approximation permitted in this investigation is the neglect of the perturbation of *Jove* by the planet. For a very small planet the instability must accordingly be a very slow process, and I cannot but believe that the whole history of a planetary system may be comprised in the interval required for the instability to render itself manifest. Henceforward then I shall speak as though the stability of stable orbits were absolute, instead of being, as it probably is, only approximate."

This work of DARWIN supplements and generalizes in a most impressive manner the work of astronomers on such bodies as LEXELL's Comet of 1770, and

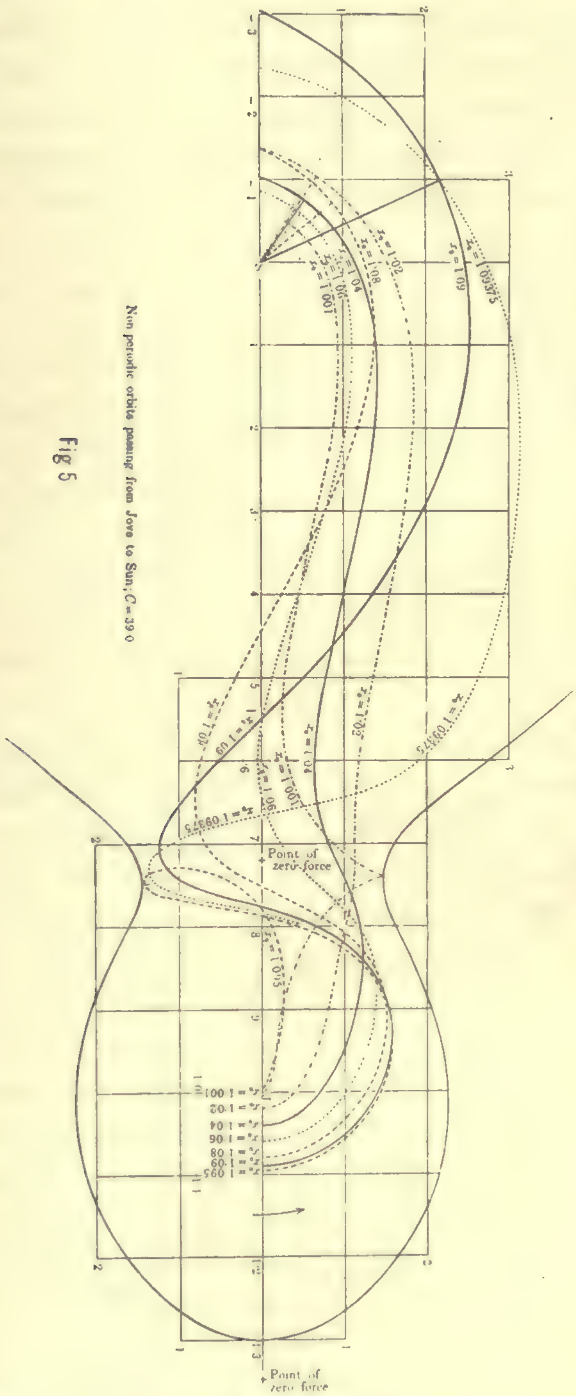


FIG. 32. DARWIN'S FIG. 5, SHOWING VARIOUS NON-PERIODIC ORBITS OF PARTICLES PASSING FROM JOVE TO THE SUN (*Acta Mathematica*, VOL. XXI, P. 177).

the whole series of periodic comets, when they pass near a planet such as *Jupiter*.* It is scarcely necessary to recall the early work of BURKHARDT and LAPLACE, or the more modern researches of LEVERRIER, ADAMS, NEWTON, SCHIAPARELLI, CALLANDREAU and TISSERAND, as this is well known to astronomers. But it may be pointed out that DARWIN's results are beautifully confirmed by the movement of BROOKS' Comet, 1889 V, which was laboriously investigated by DR. CHARLES LANE POOR, and observed in the places predicted by him in 1896 and 1903.

§ 106. *Part Played by the Resisting Medium in the Capture of Satellites.*

In the foregoing discussion of the surfaces of Zero Relative Velocity and Constant Relative Energy, it is assumed that all the motion takes place in empty space, and is, therefore, wholly free from conditions, or from any kind of obstruction. *The differential equations and the resulting* JACOBIAN *Integral rest on this hypothesis.* This same assumption pervades the whole Science of Modern Dynamics, and lies at the basis of the work of all the great mathematicians who have dealt with the Problem of Three Bodies and Periodic Orbits. The premise thus tacitly permeating so many branches of Mathematical Science is approximately correct for the present state of the solar system and for short intervals of time; but is false and incorrect when extended to the past history of our system, and to long intervals of time, because of the general prevalence of a resisting medium in space, which adds a small additional term to the differential equations and renders the accepted form of JACOBI's Integral incomplete. Thus the complete †Jacobian Integral might be taken to be of the form

$$\dot{x}^2 + \dot{y}^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2+y^2+z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2+y^2+z^2}} = C + at; \tag{276}$$

the additional term of secular character, which we call at, representing the average rate of increase of the constant of relative energy, as the particle revolves against resistance and steadily drops nearer and nearer to the centers of attraction. The resisting medium with the lapse of ages has exercised the greatest influence in modifying the orbits of the heavenly bodies; and even now it is only in exceptional cases that this cause has wholly disappeared from our solar system or from the other similar systems existing in the sidereal universe.

* There is, however, one considerable difference between the motion of a comet such as LEXELL's, and that of the infinitesimal satellites considered by DARWIN; namely, the satellites move in a rotating space, with periods which make it possible for them to pass easily from Sun to planet and *vice versa*, while the comet's period usually is considerably shorter than that of the planet, and the orbit does not rotate, except slowly under the perturbative action of the planet.

† Neglecting the effect of the resisting medium upon the planet's mean distance, which is diminished but little compared to the much greater decrease in the mean distance of the particle.

Let us now ascertain what modification in the ideal results of pure Dynamics is required to take account of this neglected physical cause, and see if the inference just drawn is justifiable.

(1). In the first place, we notice the familiar result, that when a body revolves about the Sun or about a planet, wholly within the closed folds of the surfaces of zero velocity, the resisting medium decreases the instantaneous velocity at every point of the orbit, and, as the central attraction remains unchanged, the path therefore curves more rapidly than it otherwise would have done, and the result is that the body falls nearer the centers of attraction. This effect was known in the time of NEWTON, for the case of a single central body, and applied by him to the great comet of 1680, which passed so near the surface of the Sun, and was supposed to have suffered some resistance from the Sun's atmosphere. The decrease in the mean distance of all the planets due to resistance was distinctly recognized by EULER, in 1749 (cf. *Phil. Trans.*, 1749, pp. 141–142, and a paper by the author on EULER's Remarks in *A.N.* 4334). But the secular decrease in the eccentricity seems to have been first established by LAPLACE, in 1805, in *Lib. X*, Chap. VII, § 18, of the *Mécanique Céleste.* In *A.N.*, 4308, the writer has shown that this is the true origin of the remarkable circularity of the orbits of the planets and satellites of the solar system.

(2). If such results follow for the resisted particle when it moves within the closed surfaces near either body, it is obvious that a similar result will take place when the particle circulates within the hour-glass or pear-shaped space enclosing both bodies. For when the particle has its instantaneous velocity diminished by resistance, it will drop nearer the attracting center or centers. This, of course, really increases the velocity, because the constant of relative energy is greatest near the bodies and decreases as we go outward, as shown in DARWIN's Fig. 1. Therefore we may say generally that resistance causes the particle to drop nearer and nearer one or both masses, and thus it may finally pass within the closed folds about one or the other of the two bodies, and acquire in time a constant of relative energy such that it can never escape from these regions, in which the control is vested in a single mass.

(3). Therefore, it follows that when our Moon is once safely under the control of the Earth (cf. HILL's *Collected Mathematical Works*, Vol. I, pp. 297–301), and the other satellites of the several planets are under their respective controls, they must forever remain within these folds or closed surfaces, and their radii vectores will have superior limits. As far back as 1877 HILL remarked that this condition is fulfilled by all existing satellites of the solar system, and was necessary for their stability (cf. "Researches in the Lunar Theory," HILL's *Collected Works*, Vol. I, p. 297).

(4). But it does not follow, as PROFESSOR F. R. MOULTON and others have erroneously claimed (cf. *Astrophysical Journal*, Vol. XXII, No. 3, October, 1905, pp. 177–178), in the case of *Phœbe*, that because these satellites are now safely under control of their several planets, and cannot escape, so, also, conversely, they can never have come to the planets from a remote distance. In such reasoning the effects of the resisting medium is wholly neglected, and this completely invalidates all the argument based on the false premise that space is empty and the motion of the satellite unconditioned, according to the ideal conceptions of pure Dynamics. This example affords us a good illustration of the improvement which may result in the equations of Dynamics from their application to the motions of the heavenly bodies, under the actual conditions of Nature.

In order to leave no possible doubt of the entire rigor of the theory here developed, it seems advisable to introduce some additional considerations. For it might be held by those who look at the problem from a purely mathematical point of view that as JACOBI'S Integral applies only to the case of free motion under purely gravitational attraction in empty space, the resisting medium violates the conditions on which the integral is based; or that the differential equations would thus become quite different, and the integral given by JACOBI have no application. The following simple considerations will show that this criticism, should it be seriously entertained by anyone, is quite devoid of foundation.

Suppose the whole time from t_0 to t_i to be divided into an infinite series of infinitesimal intervals $\tau_1 = t_1 - t_0$, $\tau_2 = t_2 - t_1$, $\tau_3 = t_3 - t_2$, $\tau_4 = t_4 - t_3$, ... $\tau_i = t_i - t_{i-1}$ which may be made less than any assignable quantity however small. Then, during the intervals with odd subscripts, $\tau_1 , \tau_3 , \tau_5 , \ldots \tau_\nu$ we may suppose the space where the bodies are moving to be entirely empty, and no resistance of the motion will occur. The above differential equations and the resulting Jacobian Integral will, therefore, hold rigorously true during these intervals. And for any interval τ_ν there will be the appropriate surfaces of relative energy and of zero relative velocity. During the other intervals, with even subscripts, $\tau_2 , \tau_4 , \tau_6 , \ldots \tau_{\nu+1}$, we may suppose all the resistance to occur, and in these intervals we may imagine the HILL surfaces to disappear; just as if the view of them was illuminated by a rapidly succeeding series of flash lights, the circuit being broken and the lights put out every time a collision with a particle began, but closed and the lights restored the instant the collision ceased. As the nebulosity in space is supposed to be made up of cosmical dust, conceived as a discontinuous granular medium, with absolutely free space between the particles, this arrangement corresponds precisely with the actual conditions in Nature.

Accordingly in looking at the diagram of such a system, we should not see

a continuous series of HILL surfaces, but a rapidly succeeding series of flashes of them, so close together in time that they would make an impression upon the retina of the eye which would be almost if not quite absolutely continuous. When the illumination is restored as rapidly as the successive particles are encountered in space, the image of the HILL surfaces would be not only continuous to the eye, but also very nearly so to the mind. If we watched these surfaces, as they are flashed before us for a long time, placing beneath our projected image of them a fixed *reseau* representing these surfaces as they were at the initial epoch, t_0, we should at length perceive that they were undergoing a slight secular shrinkage. The resistance could, of course, be supposed to be so small, or so distributed in the system that it exerted little or no influence on the motion of a heavy body like the planet *Jove*.

Thus the theory of the secular shrinkage of the surfaces of relative energy and of zero relative velocity, under the action of *a discontinuous resisting medium*, is established with all possible mathematical accuracy. And in adding the secular term to JACOBI'S original integral, we have made no approximation whatsoever, but represented the actual phenomena of Nature, with all the rigor of the Infinitesimal Calculus. For, without doubt, the heavenly bodies generally are resisted in their orbital revolutions by the widespread diffusion of nebulosity, in the form of discontinuous cosmical dust, throughout the celestial spaces, and the secular effect* of the continued action of this medium is essentially that set forth above.

§ 107. *How the Motions of the Satellites Become Retrograde.*

From the results indicated in *A.N.* 4308, and the foregoing considerations, we see how all the planets and satellites have been captured — the former now revolving under the control of the Sun as superior or inferior planets, the latter drawn down very close to the several planets, where alone stable motion is possible. The great preponderance of the Sun's control is due to its larger mass; but small, closed surfaces exist about each planet, and in the case of *Jupiter*, *Saturn*, *Uranus* and *Neptune*, these surfaces are of considerable size.

If our system was once pervaded by a resisting medium, with an infinite number of particles and a lesser number of small bodies circulating everywhere in the spaces where the planets now move, just as the comets still do; then it is obvious that some of them would pass continually from the control of the Sun to that of these several planets; and under the influence of resistance they would, sooner or later, be captured.

* Further considerations confirming this conclusion are set forth at the close of § 112.

PROFESSOR SIR G. H. DARWIN remarks in the passage above quoted: "If the neck of the hour-glass defining the curve of zero velocity be narrow, the body may move hundreds of times round one of the centers before its removal to the other." *And if the dynamical condition imposed by the resisting medium is introduced, he might well have added, a permanent capture will result. Thus Jove would acquire a real satellite; and when the major axis and eccentricity of the orbit had been reduced by the secular action of the resisting medium about the planet, the orbits of such bodies would be similar to those of the satellites now observed.*

It is especially important to point out that in passing from the Sun to *Jove* the orbits may be either retrograde or direct; for the body may enter the neck of the hour-glass in such a way as to cross over the line SJ before coming completely under *Jove's* control. In general, such a crossing satellite would probably be comparatively remote from the planet. But if it should come nearer to the planet, where most of the satellites have direct motion, in the denser revolving vortex of nebulosity about the central nucleus, the chances are that it would not survive, but in time be precipitated upon the planet, or disintegrated into dust when it came within ROCHE'S limit.

Therefore it is not remarkable that only two known retrograde satellites have survived, while those with direct motion are more than ten times more numerous. Nor is it surprising that these retrograde satellites are on the outskirts of the systems of *Jupiter* and *Saturn*, for at this distance from these centers the resisting medium must have been of extreme tenuity. The fact that considerable eccentricities survive, 0.44 in the case of *Jupiter* VIII, and 0.22 in the case of *Phœbe*, both confirms this mode of capture, and indicates that the density of the medium at that great distance must have been quite small.

On the other hand, the greater roundness of the orbits of the other satellites nearer the planets which revolved in a denser medium, naturally follows, as well as their direct revolution. It should not, however, be concluded that retrograde motion near these planets is wholly excluded, but merely that the chances of its surviving are very slight.

§ 108. *Theoretical and Observed Distances of Satellites in the Solar System.*

In the theory of the closed surfaces about the planets it is shown that the superior limit of distance for a satellite which can just be retained by a planet is found by the condition that the neck of the hour-glass figure shall contract till the two separate surfaces meet in a point between the Sun and planet, on the line SJ. This leads to a quintic equation, which, for small bodies, such as those

in our solar system, is shown to be capable of reduction to the simple form (cf. DARWIN'S "*Periodic Orbits,*" pp. 108–109):

$$\rho_2 = \frac{1}{\left(3\nu + 1\right)^{\frac{1}{3}} + \frac{1}{3}}, \tag{277}$$

where ν is the mass of Sun in terms of the planet's mass as unity, or the reciprocal of the planet's mass as ordinarily expressed.

The following table gives the principal data for the planets and satellites of the solar system, and shows the comparative magnitude of the closed surfaces about the several bodies, and what parts of these spaces are known to be occupied. These closed spaces are not quite spherical, as one may notice from DARWIN'S Fig. 1, reproduced on p. 170, but this is a detail which we need not dwell on here. The value of ρ_2 given in the table is the maximum value.

TABLE OF SATELLITE DISTANCES IN THE SOLAR SYSTEM.

Planet	Reciprocal of Adopted Mass (cf. *A.N.* 3923)	Satellite	Observed Distance in Kilometers	Theoretical Limit in Kilometers	Theoretical Limit in Astr. Units	Constant of Jacobian Integral
Mercury	14868548.	....	....	163086.	0.001094	3.000072
Venus	408134.	....	....	1008152.	0.006747	3.000776
The Earth	328715.	*The Moon*	384400.	1497577.	0.010013	3.000898
Mars	3089967.	*Phobos*	9377.	1083118.	0.0072419	3.000202
		Deimos	23475.			
Jupiter	1047.35	V	180936.	51940750.	0.347283	3.038735
		I	421632.			
		II	670859.			
		III	1070067.			
		IV	1882150.			
		VI	11456800.			
		VII	11891000.			
		VIII	27475000.			
Saturn	3500.00	*Mimas*	185465.	69210900.	0.4627540	3.017937
		Enceladus	237942.			
		Tethys	294555.			
		Dione	377258.			
		Rhea	526847.			
		Titan	1221340.			
		Hyperion	1479622.			
		Iapetus	3559253.			
		Phœbe	12886600.			
Uranus	22780.	*Ariel*	191312.	69637300.	0.465605	3.005238
		Umbriel	266526.			
		Titania	437174.			
		Oberon	584626.			
Neptune	19313.	Satellite	355518.	115234000.	0.770473	3.005940

The value of the constant in the Integral of JACOBI could be determined by several methods, but we shall consider only the simplest of the processes, depending on the use of bi-polar co-ordinates. The centers of the bodies $1-\mu$ and μ will be taken as the poles, and the radii vectores ρ_1 and ρ_2 the distances from these poles, the origin, as before, being at the center of gravity of the system. Then, since the distance of the bodies from the center of gravity is inversely as their masses, it is evident that we shall have $\overline{O\mu} = 1-\mu$, $\overline{O(1-\mu)} = -\mu$; and the values of y referred to the two poles are:

$$\left.\begin{aligned} y^2 &= \rho_1^2 - (x+\mu)^2 = \rho_1^2 - x^2 - 2\mu x - \mu^2, \\ y^2 &= \rho_2^2 - \{x-(1-\mu)\}^2 = \rho_2^2 - x^2 + 2(1-\mu)x - (1-\mu)^2. \end{aligned}\right\} \tag{278}$$

Equating the right members of these expressions, we find $2x = \rho_1^2 - \rho_2^2 + 1 - 2\mu$. The original equations (278) then give

$$x^2 + y^2 = (1-\mu)\rho_1^2 + \mu\rho_2^2 - \mu(1-\mu); \tag{279}$$

and the integral of JACOBI giving the curves on the xy-plane becomes

$$(1-\mu)\left(\rho_1^2 + \frac{2}{\rho_1}\right) + \mu\left(\rho_2^2 + \frac{2}{\rho_2}\right) = C + \mu(1-\mu) = C'. \tag{280}$$

Now, since $\mu \leqq \frac{1}{2}$, and the first term, therefore, is always positive, this expression shows that C' is always greater than $\mu\left(\rho_2^2 + \frac{2}{\rho_2}\right)$, for all real and positive values of the radii vectores ρ_1 and ρ_2. If fixed numerical values of C' be adopted, and the arbitrary values of ρ_2 assigned, equation (280) enables us to find the corresponding values of ρ_1. This is the same as drawing arbitrary circles about the planet and calculating the corresponding radii vectores of the circles about the center of the Sun, and from their intersections finding points on the curves traced by the energy surfaces in the xy-plane. Equation (280) may be written in the form

$$\left.\begin{aligned} &\rho_1^3 + a\rho_1 + b = 0, \\ &a = -\frac{C}{1-\mu} + \frac{\mu}{1-\mu}\left(\rho_2^2 + \frac{2}{\rho_2}\right), \\ &b = 2. \end{aligned}\right\} \tag{281}$$

As $\mu \leqq \frac{1}{2}$, and C' is greater than $\mu\left(\rho_2 + \frac{2}{\rho_2}\right)$, a is always negative; and the cubic in ρ_1, is shown in the Theory of Equations to come under CARDAN'S *irreducible case,* all the roots being real and positive, with the following trigonometric solution (cf. CHAUVENET'S *Trigonometry,* p. 100):

$$\left.\begin{aligned}
\sin\phi &= \frac{b}{2}\sqrt{\frac{27}{-a^3}}\ , \quad \phi \leqq \frac{\pi}{2}\, ,\\
\rho_{1,1} &= 2\sqrt{\frac{-a}{3}}\sin\frac{\phi}{3}\, ,\\
\rho_{1,2} &= 2\sqrt{\frac{-a}{3}}\sin\left(60^\circ - \frac{\phi}{3}\right),\\
\rho_{1,3} &= -2\sqrt{\frac{-a}{3}}\sin\left(60^\circ + \frac{\phi}{3}\right).
\end{aligned}\right\}\qquad (282)$$

where $\rho_{1,1}$, $\rho_{1,2}$, $\rho_{1,3}$ are the three roots of the cubic, corresponding to the different values of the angle ϕ which have the same sine. The negative root $\rho_{1,3}$ may, of course, be neglected. In this case

$$-4a^3 \geqq 27b^2 \ ; \text{ or since } b = 2\ ,\ -a^3 \geqq 27\ , \text{ or } a + 3 \leqq 0\ . \qquad (283)$$

The limit of this inequality is $a' = -3$, and if we use this in the second equation of (281), we have the extreme values of ρ_2 for which the roots are real:

$$\left.\begin{aligned}
&\rho_2^3 + a'\rho_2 + b' = 0\ ,\\
&a' = -\frac{C'}{\mu} + \frac{3(1-\mu)}{\mu}\, ,\\
&b' = 2\ .
\end{aligned}\right\}\qquad (284)$$

If $a' \leqq -3$, the roots are real, and

$$-\frac{C'}{\mu} + \frac{3(1-\mu)}{\mu} = -3\ , \quad \text{or} \quad C' = 3\ .$$

This is the minimum value of C' that will permit the curves to have real points in the xy-plane; and for these values, $\rho_1 = 1$, $\rho_2 = 1$, satisfy the equation (280), and the surfaces just touch the xy-plane at the points which form equilateral triangles upon SJ as a base. It is easily shown that when the mass μ is very small, as is the case in the solar system, we may expand the expression for ρ_2 in a power series in $\mu^{1/3}$, namely,

$$\rho_2 = a_1\mu^{1/3} + a_2\mu^{2/3} + a_3\mu^{3/3}\ \ldots\ .$$

For points on the x-axis between x_2 and x_1, the coefficients are found to be

$$\left.\begin{aligned}
&a_1 = \frac{3^{2/3}}{3}\ ;\quad a_2 = \frac{-3^{1/3}}{9}\ ;\quad a_3 = -\frac{1}{27}\, ,\ \text{etc.,}\\
\text{and}\quad &\rho_2 = \mu^{1/3}\left\{\frac{3^{2/3}}{3} - \frac{(3\mu)^{1/3}}{9} - \frac{\mu^{2/3}}{27}\ \ldots\right\},\\
&\rho_1 = 1 - \rho_2\ .
\end{aligned}\right\}\qquad (285)$$

Having found ρ_2 and ρ_1 by these equations, the corresponding value of C' is found by (280), which also gives the constant of the Jacobian integral $C = C' - \mu(1 - \mu)$. The formulæ for investigating the details of the curves of zero velocity throughout their whole course need not be given here, as the subject is an extensive one; they will be found in works on periodic orbits and kindred subjects.

§ 109. *On the Future Search for New Satellites About the Planets.*

If we study the data in the foregoing table, we shall perceive that the closed surfaces about some of the planets are almost entirely vacant, or so little traversed by the known satellites, that it is quite probable that other satellites may yet be discovered in the vacant spaces. In the case of *Mercury* the problem of searching for satellites is, no doubt, hopeless, because of the smallness of the closed space, and the difficulty of recognizing faint objects about a planet so near the Sun.

Venus holds out better prospects of possible discoveries; for the closed surface about this planet has more than twice the diameter of the orbit of our Moon, and is, therefore, ample for holding one or more satellites. As *Venus* admits of prolonged photographic search when at elongation, this method would be worthy of trial.

The possible satellites moving about the Earth, other than the Moon, hardly require discussion, but as the closed space beyond the Moon is very ample, it is by no means improbable that a small body may yet be found there.

We next turn to *Mars*, and there we find a large closed space, apparently unoccupied, except by the small satellites very near the planet, which were discovered by the late Professor Asaph Hall, in 1877. If photography were applied to the outer region of this space, when *Mars* is very near the Earth, as it has been this year, it is quite possible, and even probable, that another faint body or two would be found to attend this interesting planet.

In the case of *Jupiter*, the search might be extended considerably further than it has yet been; for the outer half of the closed space is still unoccupied by any known satellite, and large vacant regions exist also among the known satellites, especially between the Fourth and Eighth. Of course, it does not follow that all the vacant spaces are really occupied; neither is the existence of pairs of satellites close together wholly excluded, as we see by the distances of the Sixth and Seventh satellites recently discovered by Perrine at the Lick Observatory.

The sphere of *Saturn's* possible domain for satellites is even larger than that

of *Jupiter*, and is still relatively less occupied. New satellites are, therefore, most likely to be found in this grand system.

In the case of *Uranus* there is a large domain apparently vacant on the outside, all the known satellites being concentrated quite near the center. Is it not likely that other bodies will yet be found at greater distance?

Similar remarks apply to *Neptune*, which, owing to its great distance from the Sun, has the largest closed space of any of the planets. As enough nebulosity was gathered into these remote planets to give them considerable masses, and the nebular resistance was great enough to round up their orbits, it seems almost certain that they must have several satellites still undiscovered. Photography can be applied here with full effect, and persistent searches with the most sensitive plates, the longest exposures and most powerful photographic telescopes is to be recommended.

§ 110. *Dynamical Theory of the Division of Nebulae.*

That there is in Nature a general dynamical process by which condensing nebulae divide into fairly equal parts and form double stars has long been held by the present writer (cf. *Inaugural Dissertation*, Berlin, 1892). This conclusion was first reached from the study of the brightness and probable masses of binary stars about May, 1886; and in the latter part of 1889 the double nebulae depicted by SIR JOHN HERSCHEL, in the *Philosophical Transactions of the Royal Society*, for 1833, were connected with the figures of equilibrium or rotating masses of fluid investigated by POINCARÉ (*Acta Mathematica*, Vol. VII), and DARWIN (*Phil. Trans., Roy. Soc.*, 1887). The belief that such a division of the nebulae takes place was necessary to associate the double stars with the nebulae from which it was held they had arisen by gradual condensation. This relationship between the double stars and nebulae in process of division had been impressed upon me from the forms of the equipotential surfaces which might be constructed about equal and unequal masses.

At that time the nature of the nebulae was not very well understood, and I could do no better than fall back on the figures of equilibrium of rotating masses of fluid, calculated by mathematicians, as the nearest known approach to the natural process of nebular division. Yet as the nebulae evidently were not homogeneous, the analogy was never felt to be entirely satisfactory, and was used only as a rough approximation to the true process of Nature. I have long looked forward to a further development of the theory of nebular fission. It was worked out last year along with other results on the origin of the solar system (cf. *A.N.*,

4308), but, owing to severe illness in the early part of this year (1909), an earlier opportunity has not occurred of presenting it to the public.

Let us return to the figure of the hour-glass or pear-shaped space about the Sun and *Jove*, remembering that it is nearly a figure of revolution about the x-axis SJ. Imagine this whole space filled with the nebulosity such as we see in comets, or in the nebulae of space. This is conceived to be an excessively tenuous medium essentially devoid of hydrostatic pressure, with the individual particles pursuing their own orbits and seldom coming into collision with others.

Then, as the medium will at length become densest about the central masses S and J, it is evident that all particles moving in orbits about these bodies will be confined to the hour-glass space already explained. They may traverse it in various ways, but usually will move in planes not departing greatly from that of the circular orbit of S and J. Now, as all move against resistance, they will steadily drop down nearer and nearer the two centers of attraction. In a moderately short time many of the particles will enter *Jove's* sphere of influence to depart no more, while still more will pass wholly under the influence of the Sun. The rest will circulate about the two masses or pass between them, and pursue retrograde orbits about one center or the other. The condensation at these centers will increase steadily owing to the effects of resistance, and we shall have two equal or unequal masses, according to the original supply of nebulosity as it flows into these centers.

Usually there will be a large central condensation or Sun and attendant planets; but, in some cases, there may be a nearly equal division of the nebulosity, as in the double stars.

The distribution of mass in a system depends on initial conditions. If, as a nebula coils up and condenses under gravity, there is a considerable companion nucleus already begun, the supply of nebulosity may be such as to give a pair of nearly equal masses, with smaller satellites near each body. In some cases there will be a single remote body, and another closer pair of stars, as in triple-star systems; while in yet others the division will give quadruple and multiple stars and clusters of higher order. Such equable division takes place where the original nebula is so widespread as to permit the development of multiple centers of attraction, all of large size and comparable in mass.

§ 111. *The Formation of Double and Multiple Stars and of Planetary Systems.*

It is evident from the above considerations that the resulting mass-ratio in a system depends on the supply of nebulosity and the original nuclei already

begun and slowly developing in the nebula when it was still of vast extent and great tenuity. The resisting medium operates to build up the nuclei already started, and, as the sphere of influence of each nucleus is thus extended, the power of capturing additional nebulosity steadily increases. In a system of two principal bodies the planet may, in some cases, thus rival the Sun; but, in general, the Sun's influence will predominate, while the remaining nebulosity will be divided among a number of planets all comparatively small.

Thus we are justified in believing that since the stars have, in general, resulted from the condensation of nebular vortices, or whirlpool nebulae, nearly all the stars have systems of planets circulating about them; but it is obvious that the double stars and spectroscopic binaries are the only attendant bodies which are sufficiently luminous or sufficiently massive to be detected, with our existing instruments, at the great distance of the fixed stars.

According to this view there will exist in the heavens all sizes of attendant bodies from infinitesimal planets, such as those observed in the solar system, to double stars with equal or comparable companions. In systems with double or multiple distribution of mass, the planets which are developed, if they are to continue to revolve in approximately stable orbits, will have to be near the large masses, so as to keep within the closed surfaces about these centers of attraction, or at great distances from both of them. Probably both of these classes of planets may be inferred to exist in the immensity of space. But it is worth while to notice that in *double-star systems with very eccentric orbits* the regions of stability about each mass is considerably narrowed, because what corresponds to the closed Hill surface is of extremely variable radius, being very large when the stars are in apastron and very small when at periastron. The constants of energy, if we may still use that expression, for particles revolving in such eccentric systems, would thus be very fluctuating, and the destructive tendency much greater than in systems with approximately circular orbits.

§ 112. *Analogy Between the Dynamical Division of a Nebula Under the Secular Action of a Resisting Medium and the Rupture of the Figure of Equilibrium of a Rotating Mass of Fluid Calculated by Mathematicians.*

The form of the energy surfaces about a revolving Sun and planet has been shown to be closely similar to that of a rotating and condensing mass of fluid kept in equilibrium under the pressure and attraction of its parts. The mathematical difficulties encountered in the investigation of the figures of equilibrium have proved to be nearly insuperable, and the results necessarily have been

restricted to the case of fluid which is both homogeneous and incompressible, which does not accord with the conditions in actual Nature. Under the circumstances it has been very difficult for the mathematician to attack the more general problem of the division of heterogeneous compressible masses, such as the nebulae have long been supposed to be.

If the above line of treatment be admissible, we see that elaborate mathematical treatment of this problem is now rendered unnecessary by the retarding and degrading influence of the resisting medium. To see what will happen in any case, all that is required is to calculate the form of the energy surfaces and draw limiting surfaces of energy with the pear-shaped figure connecting the Sun and planet. The division of the nebulosity is then necessarily effected automatically by the resisting medium, and is in accordance with the total supply of material and its distribution in the system at some initial epoch. Of course the exact form of the pear-shaped surface depends on the mass of the planet and its distance when the system is started, and it changes somewhat with the development of the two bodies. This is a comparatively simple conception and it probably will be capable of much more lucid treatment than that depending on figures of equilibrium of rotating masses of fluid. Nor will the variations of density in the nebular medium in general exert any unfavorable influence on the final result. It is, moreover, in accord with actual conditions in Nature, so far as these may be inferred from the study of the nebulae, and from the theory of gases. It is not necessary to take up here the problem of the density of the nebulosity in the pear-shaped or hour-glass space; but we may remark that the problem has been treated from different standpoints by the following well-known authors:

(1). J. Homer Lane (*American Journal of Science*, July, 1870), who considers the gaseous theory of the Sun's constitution, and develops the theory of a gaseous mass in convective equilibrium.

(2). Lord Kelvin (*Popular Lectures and Addresses*, pp. 376–429), who treats the gravitational theory of the Sun's heat, and (in *Proc. Roy. Soc.* of Edinburgh, Vol. XXVIII, Part IV, March 9, 1908) solves the problem of a spherical gaseous nebula, under several hypotheses.

(3). A. Ritter, who treats various problems in Wiedemann's *Annalen*, 1878–1882.

(4). G. W. Hill, in *Annals of Mathematics* (Vol. 4, No. 1, February, 1888).

(5). G. H. Darwin, *Phil. Trans. Roy. Soc.*, Nov. 15, 1888. He treats the mechanical condition of a swarm of meteorites from the gaseous standpoint.

(6). T. J. J. See, who treats very fully the monatomic theory in *A.N.*, 4053, November, 1905.

Among these several investigators DARWIN is the only one who considers the mass to be a swarm of meteorites; but even he adopts the theory of gases, which, however, is only partially valid for nebular conditions. Probably the law of density found by the present writer in *A.N.*, 4053, which makes the density increase quite slowly and becomes exactly six times the mean density at the center, is that which will, on the whole, accord best with the conditions of the nebulae. In the present discussion it is sufficient to remark that the density of the nebulosity or cosmical dust certainly increases toward each center, but the rate of increase is likely to be less rapid than in the case of ordinary gases, which makes the central density about twenty-three times the mean density.

Leaving the settlement of these details to the future, it is evident that the process of automatic division, by degradation of energy, under the action of a resisting medium, always going on in a nebula, is comparatively simple, and easily understood in connection with the closed surfaces which operate to capture the particles of nebulosity. This slow dynamical process by which particles are gathered in one by one takes, indeed, a very long time, because it depends upon the degradation of the energy under resistance, but its mode of operation is sure, and the final outcome beyond doubt.

Though this new line of thought deprives us of the principles of hydrostatic pressure, heretofore largely invoked in these researches on cosmical evolution, and we have to give up the historical point of view as largely inapplicable to the nebulae; yet as it can no longer be held that the attendant masses are detached by rotation proceeding from the center of the system, as formerly believed, perhaps we shall have less need for the theory of hydrostatics. The figures of the connecting energy surfaces, as defining the boundaries of capture, may well take the place of the figures of equilibrium of rotating masses of fluid. The method of attack here adopted is, therefore, much simpler, and also possibly less exact than those followed by POINCARÉ and DARWIN; and, although the conclusions drawn from the two lines of investigation are similar, one cannot help thinking that this latter process conforms much more closely to the law of Nature than that based on figures of equilibrium. This process of automatic division we may call *Nebular Fission*, in contrast to the process of *Fluid Fission*, found from the researches of mathematicians on the figures of equilibrium of rotating masses of fluid.

It is easily shown that if the resistance is proportional to the surface or cross section, the effect on two unequal spheres of homogeneous density is inversely as their radii, and thus the large body experiences a very small, and the small body a very large, change. Both bodies approach the sun, but the smaller one so much

more rapidly than the larger one that it is generally permissible to neglect the change in the mean distance of the planet. There are some additional considerations which render this approximation still more exact than it seems: (1) The orbit of the satellite is generally eccentric, and in crossing the orbit of the planet the tendency of the perturbations is to throw its orbit entirely within that of the planet; (2) Just as the action of the planet throws the satellite within, so also the reaction of the satellite throws the planet slightly but correspondingly out. Accordingly this reaction of the small bodies thrown within counteracts in some degree the effects of resistance in decreasing the mean distance of the planet. And the addition of a secular term to JACOBI's integral is more accurate than might be inferred from a superficial examination of the subject.

In conlcusion, it only remains to add that the writer's indebtedness to the researches of HILL, POINCARÉ and DARWIN, for valuable suggestions in connection with the problems of Cosmical Evolution, has been sufficiently pointed out in the papers published during the past eighteen years; but he may here again emphasize the profound significance of the famous "Researches in the Lunar Theory," of the classic "*Méthodes Nouvelles de la Mécanique Céleste*," and of the celebrated Memoir on "Periodic Orbits," without which, it is to be feared, the problems here treated would have remained insoluble. Imperfect as this feeble effort may be, he entertains the hope that it has considerably cleared up the problem of the capture, and transformation of the orbits of satellites; and of the Fission of Nebulae under the Secular Action of a Resisting Medium, on which Cosmical Evolution so largely depends.

CHAPTER XI.

Origin of the Lunar-Terrestrial System by Capture, with Further Considerations on the Theory of Satellites and on the Physical Cause which has Determined the Directions of the Rotations of the Planets About Their Axes.*

§ 113. *Comparison of the Moon with Other Satellites of the Solar System.*

In *A.N.*, 4308, the writer has adduced a general argument tending to show that the planets and satellites of the solar system have in no case been detached from the central masses which now govern their motions, but have all been captured, or added from without, and have since had their orbits reduced in size and rounded up under the secular action of the nebular resisting medium formerly pervading our system. And in *A.N.*, 4341–2, and in the foregoing Chapter, an outline of the dynamical basis of this new theory of the origin of our satellite systems has been developed in sufficient detail to render it intelligible. The methods there given appear to be entirely rigorous, and sufficiently general to be convincing without the examination of particular phenomena, except in the case of the Earth and Moon, which is the only planetary sub-system about which any doubt could arise.

The principal circumstance which might make our Moon seem different from the other satellites is its relatively large mass, which amounts to 81.45 of the mass of the Earth. (cf. *A.N.*, 3992, p. 117). This long ago led Professor Sir G. H. Darwin, and others, to the belief that its mode of origin probably was quite different from that of the other satellites of the solar system. But the considerations adduced by former writers rest on the hypothesis that our Moon, and the other satellites, have all been detached from the central masses which now govern their motions; whereas, in *A.N.*, 4308, and in the preceding Chapter, this hypothesis has been shown to be no longer admissible. If our

*Given here substantially as communicated to the *Astronomische Nachrichten*, May 22, 1909. The capture of the Moon by the Earth was first announced May 24, by a cablegram printed in *A.N.* 4325. A further communication on the subject was made to the Astronomical Society of the Pacific, June 25, 1909 (cf. *A.N.* 4343).

reasoning, that the satellites have been captured, is valid, it becomes advisable to examine the special case of the Moon with some care, and to inquire whether the Moon is, after all, relatively so large, or the Earth merely comparatively small. In the following table will be found what I believe to be the best available diameters of the satellites of the solar system.

TABLE OF SATELLITE DIAMETERS.

Planet.	Satellite.	Diameters in Kilometers	Mass in Terms of the Earth's Mass as Unity.	Density.
The Earth	*The Moon*	3480.5	1:81.45	3.31
Mars	*Phobos*	58		
	Deimos	16		
Jupiter	V	50		
	I	*3145	1:111.2	3.29
	II	2817	1:135.5	3.76
	III	4770	1:38.75	2.70
	IV	4408	1:146.5	1.90
	VI	160		
	VII	50		
	VIII	50		
Saturn	*Mimas*	35.1	1:143200	1.8
	Enceladus	528	1:42100	1.8
	Tethys	866	1:9450	1.8
	Dione	1032	1:5642	1.8
	Rhea	1331	1:2632	1.8
	Titan	*5049	1:49.4	1.79
	Hyperion	315	1:197600	1.8
	X	300	1:200000	1.8
	Iapetus	1314	1:1053	4.77
	Phœbe	320	1:200000	1.8
Uranus	*Ariel*	1030	1:5700	1.83
	Umbriel	835	1:10670	1.83
	Titania	1350	1:2522	1.83
	Oberon	1295	1:2856	1.83
Neptune	Satellite	2962	1:238.7	1.83

**A.N.* 3764

§ 114. *Further Considerations on the Capture of the Satellites.*

In the foregoing Chapter on the "Dynamical Theory of the Capture of Satellites" (cf. also *A.N.*, 4341–2), it has been shown that all the satellites of the solar system are well within DR. G. W. HILL's closed surfaces about the several planets; and it is made quite clear how these bodies have been brought within these folds

by the secular action of the nebular resisting medium formerly pervading our planetary system. As is there pointed out, this disturbing cause has the effect of adding a secular term to the Jacobian Integral, which thus becomes of the form:

$$x_i^2 + y_i^2 + \frac{2(1-\mu)}{\sqrt{(x_i - x_1)^2 + y_i^2 + z_i^2}} + \frac{2\mu}{\sqrt{(x_i - x\ )^2 + y_i^2 + z_i^2}} = C_i + a_i t_i \left| \begin{array}{l} i = \infty \\ i = 0 \\ t = \text{time} \end{array} \right| . \tag{286}$$

In accordance with the usual notation of Dynamics, the subscript i may be used in this equation; for it will hold for an infinite number of particles of nebulosity in the system, and each particle will have its own surfaces of zero relative velocity. The infinite family of energy surfaces, for all the particles of nebulosity in the system, is defined by the expression

$$\sum_{i=1}^{i=\infty} \left\{ x_i^2 + y_i^2 + \frac{2(1-\mu)}{\sqrt{(x_i - x_1)^2 + y_i^2 + z_i^2}} + \frac{2\mu}{\sqrt{(x_i - x_2)^2 + y_i^2 + z_i^2}} \right\} = \sum_{i=1}^{i=\infty} \left\{ C_i + a_i t_i \right\} \tag{287}$$

The secular co-efficient is different for different particles, even when the co-ordinates are the same; because it depends on the velocity and direction of motion at the initial epoch. It will be determined by the resistance encountered along the actual path, and as infinite variation in the trajectory is possible, the value of coefficient a_i cannot be exactly specified for any given case. It is easy to see, however, that it will always be a finite one-valued function. In the long run it will be positive, though, through the accidental collisions of the particle with others having different velocities and directions, it may temporarily become negative. If $a_1, a_2, a_3 \ldots . a_i$ be the values which this coefficient acquires at the epochs $t_1, t_2, t_3 \ldots . t_i$, owing to accidental collisions of the particle, some being positive and others negative, it is clear that for a long interval of time we may take

$$a_i = \frac{1}{i} \sum_{i=0}^{i=\infty} a_i . \tag{288}$$

For any given path, starting at an initial epoch, t_0, this function will always be definite and comparatively small; but as the collisions are countless, and the values of the terms in the series $a_1, a_2, a_3 \ldots . a_i$ will vary from one particle to another, according to the path, no two of the coefficients a_i can be expected to be the same. We may form some idea of the numerical values of these coefficients by taking $a_1 = 0.000\,000\,01$, and $t_1 = 10{,}000{,}000$ years. Then, for a particle with such a path, the second member of equation (278) will, after the lapse of ten million years, have increased by 0.1. This will bring the Hill surface of the particle considerably nearer the central masses than it was

at the outset; so that in time it will become closed for that particle about one of the bodies, and the particle will, therefore, become a permanent satellite of the Sun or of the planet.

Moreover, as the numerical value of the coefficient α_i fluctuates somewhat with the time, owing to collisions, it is clear that the HILL surface is not strictly of constant dimensions, but varies slightly, according to the nature of the collisions which the particle suffers in its path about S and J.

§ 115. HILL'S *Closed Surface About the Earth.*

We shall now consider somewhat more fully the problem of the origin of the terrestrial Moon. From the data given by the table in the preceding Chapter, on the "Dynamical Theory of the Capture of Satellites," we see that in this case the closed suface extends to about 1497577 kilometers from the center of the Earth, or about four times the present distance of the Moon. This is the maximum value corresponding to the part of the surface nearest the Sun, and, of course, other parts of the surface are considerably nearer the Earth; which agrees very well with DR. HILL'S estimate of the extent of this surface in his "Researches in the Lunar Theory," pp. 300–301–334, where he finds the Moon of maximum lunation to be 204.896 days.

It is true that in his *Mécanique Céleste*, Tome I, p. 109, POINCARÉ has traced a looped orbit of even wider extent and longer period, and LORD KELVIN has drawn an orbit of similar type in the *Philosophical Magazine* for November, 1892, p. 447; but PROFESSOR SIR G. H. DARWIN justly points out (cf. *Periodic Orbits*, p. 192), that both of these eminent mathematicians have neglected the solar parallax, so that the solutions given do not quite correspond with the ideal conditions of the problem. We are, of course, concerned here only with the space within the cusps as given by DR. HILL, and not at all with the loops found by POINCARÉ and KELVIN.

If our Moon has therefore been captured by the Earth, it has at length come well within HILL'S closed surface. In fact, the Moon revolves at a distance corresponding to the inner fourth of the possible radius. The same thing is true of the other satellites of our solar system, and they, too, are near the central portions of their several closed surfaces.

DR. HILL remarks that "If the body whose motion is considered, is found at any time within the first fold (the closed space about the Earth), it must forever remain within it, and its radius vector will have a superior limit." Neglecting the secular effects of the resisting medium upon JACOBI'S Integral, which

has not been considered by previous writers, MOULTON, and others, have drawn the unwarranted conclusion that because a satellite cannot now escape from a planet, so, also, conversely, such a satellite cannot have come to its planet from a great distance (cf. *Astrophysical Journal*, Vol. XXII, No. 3, October, 1905, pp. 177–178). But in the preceding Chapter on the "Dynamical Theory of the Capture of Satellites," we have established the erroneous character of this reasoning. Probably a considerable number of astronomers and mathematicians have been misled by this deceptive argument, which has the appearance of sound mathematics, but is easily shown to lead to false conclusions.

In no other way can we account for the failure of previous writers to recognize a truth which is of the first order of importance in our theories of the heavenly motions, and which alone gives us a clear insight into the nature of cosmical evolution. This process by which satellites are captured and reduced to order and stability by revolving against resistance, is undoubtedly one of Nature's greatest laws, and it operates uniformly throughout the physical universe.

§ 116. *Physical Grounds for Classifying the Moon with the Other Satellites, All of Which Have Been Captured.*

It will be seen from the foregoing table that two of *Jupiter's* satellites, III and IV, are considerably larger than our Moon; while *Saturn's* satellite *Titan* is much larger. *Jupiter's* satellites I and II have diameters nearly as large as that of the Moon, and the same is true of the satellite of *Neptune*, to which, however, considerable uncertainty attaches, owing to the great distance of that planet. In all cases where the satellites present no telescopic discs the diameters are calculated from the brightness, the albedo being taken to be the same as that of the planets about which they revolve, and the density one-third that of the Earth.

If, therefore, two satellites larger than the Moon, and two almost as large, exist in the system of *Jupiter;* and if *Titan* in the system of *Saturn* is much larger, while the satellite of *Neptune* is almost as large, and the two larger satellites of *Uranus* probably have diameters about half as large; it cannot really be said that, when judged by the size of the satellites observed in other parts of the solar system, our Moon is abnormally large. *The real fact is that the Earth is comparatively small.* And this makes the moon seem relatively large, and gives rise to a mass ratio of 1:81.45, which is much the largest in the solar system, *Jupiter* being 1:1047.35 of the Sun's mass, and *Titan* only 1:4700 of the mass of *Saturn.* So far as one may judge from these considerations, therefore, there is nothing improbable in the view that the Moon, too, was captured by the Earth.

If we recall that our planet is considerably the most massive body within the orbit of *Jupiter*, and that the Sun's enormous mass has been built up by the gathering in of small bodies, many of them certainly as large as the satellites, and perhaps even as large as the terrestrial planets, it will be seen that the capture of the Moon by the Earth presents no inherent improbability. The throwing of hundreds of small planets within the orbit of *Jupiter* (cf. *A.N.*, 4308), and the capture of dozens of periodic comets in the same way, affords us a good idea of the state of the solar system in the remote past. As the illustrious EULER remarked before the cosmogonic theories of KANT and LAPLACE were proposed, the Earth itself, at one time, moved as far out as where the Asteroids now circulate; and, we may add, in an orbit of considerable eccentricity. That such a planet as the Earth should capture a companion planet (for the Moon is nothing but one of the neighboring planets which were once so numerous in our system), is perfectly natural, and now demonstrated to be entirely within the range of possibility.

§ 117. *The Chief Objection to the Theory that the Moon was Captured Based on* DARWIN'S *Researches on Tidal Friction and Cosmogony.*

The chief objection to the theory that the Moon was captured is based on DARWIN'S celebrated researches on "Tidal Friction and Cosmogony" (*Proc. and Phil. Trans. Roy. Soc.*, 1878–1882).

The present writer has studied this work closely during the past twenty years and considers that the conclusions drawn by DARWIN are quite justifiable in the premises. On the traditional view that the satellites were detached from the planets which now govern their motions, as taught by LAPLACE, and his successors, for more than a century, no other outcome than that traced by the masterly hand of SIR GEORGE DARWIN was possible. But if our point of view is now changed, and we see clearly that all the other satellites were captured, the question naturally arises whether any good grounds can be adduced to show that the Moon should be considered to be an exception in the cosmogony of the solar system. After very careful consideration of all the relations involved, it seems to me that we shall have to give up this idea, and regard the moon as in the same class with the other satellites.

It is true that DARWIN'S work appears to be put together very powerfully by the relations he has brought out between such elements as the Earth's time of axial rotation, the obliquity of the ecliptic, the eccentricity of the lunar orbit, etc., and the secular changes of these elements during past ages. With admirable

philosophic frankness SIR GEORGE asks whether all these apparent confirmations of his theory can be accidental. If we still believed the satellites were formed by any kind of separation or process of detachment, as was taught by LAPLACE, we should unhesitatingly answer by saying that the relationships which DARWIN has so skillfully traced could not well be the result of chance. But with the whole point of view now changed, and the capture of the satellites shown to be possible, in the way above described — by the extension of the methods of HILL, POINCARÉ, and DARWIN, the latter's work being especially useful and suggestive, all of which have come into use since the work on "Tidal Friction and Cosmogony" was published thirty years ago — it is difficult to escape the impression that the relationship there brought out will, after all, prove to be largely, or wholly, accidental.

It might be best to leave the settlement of this question to the future, and avoid drawing hasty conclusions on so weighty a matter. For the probabilities in the case will appear different to different minds. Some will, no doubt, prefer the traditional view, and believe that the Moon has been detached from the Earth, while others will think it more probable that, like the other satellites, it came to us from the planetary spaces, and has since neared the terrestrial globe about which it revolves. In any case, tidal friction has exercised some influence on the past history of the lunar terrestrial system; but here, as elsewhere in Nature, the influence of the resisting medium has largely counteracted the secular effects of tidal friction. If the Moon came from the heavenly spaces, the eccentricity of the lunar orbit is more likely to be the survival of an original eccentricity than a development due to tidal friction, because in this event the latter cause will have been much less powerful than has been heretofore supposed.

If the Moon was captured, and not detached from the Earth, as DARWIN supposed, there would be no necessary relationship, and but little exchange need have taken place, between the moment of momentum of the Earth's axial rotation (0.7044) and the moment of momentum of the Moon's orbital motion (3.384). And the great moment of momentum of the whole Lunar-Terrestrial system might be the more easily explained. The Moon's great distance and relatively large mass is favorable to a large orbital momentum, and thus it might well be 4.8 times that of the Earth's axial rotation (cf. Appendix to THOMSON & TAIT's *Nat. Philos.*, Vol. I, Part II, p. 508), even if the latter had not been decreased and the former increased by tidal friction. In fact, this very large moment of momentum of the Moon's orbital motion is a very suspicious circumstance, and is not easily explained except on the supposition that it points directly to the capture of our satellite. If so, we shall have to give up the accepted view

that the Earth formerly rotated so rapidly that it was highly oblate and finally became unstable and broke up into two masses; and the corresponding problems of Astronomy, Physics of the Earth and Geology will have to be re-examined from the ground up.

§ 118. DARWIN'S *Graphical Method of Representing the Past History of the Earth and Moon Under the Secular Action of Tidal Friction.*

On account of the great importance of realizing fully the great strength of the celebrated graphical method which DARWIN developed at the suggestion of SIR WM. THOMSON (LORD KELVIN), as well as the weakness underlying the interpretation of it heretofore adopted, it becomes necessary to explain briefly the fundamental equations with the accompanying diagram.

Let M be the mass of the Earth, m that of the Moon, Ω the angular velocity of the two bodies about their common center of gravity, the orbit being supposed circular. Introduce a special system of units designed to reduce the analytical expressions to their simplest forms, and take the unit of mass to be $\frac{Mm}{M+m}$, the unit of length γ to be such a distance that the moment of inertia of the planet about its axis of rotation shall be equal to the moment of inertia of the Earth and Moon, treated as particles, about their center of inertia, when distant γ apart from each other. Then, if C be the Earth's moment of inertia about its axis of rotation, we shall have

$$M\left(\frac{m\gamma}{M+m}\right)^2 + m\left(\frac{M\gamma}{M+m}\right)^2 = C\,, \text{ or } \gamma = \left\{\frac{C\,(M+m)}{Mm}\right\}^{\frac{1}{2}}. \tag{289}$$

Take for the unit of time τ the interval in which the satellite revolves through 57°.3, when the satellite's radius vector is equal to γ; then $\frac{1}{\tau}$ is the orbital angular velocity, and by KEPLER'S law of periodic times,

$$\tau^{-2}\gamma^3 = \mu\,(M+m)\,, \tag{290}$$

where μ is the attraction between unit masses at unit distance. Substituting for γ its value in (289), we get

$$\tau = \left\{\frac{C^3\,(M+m)}{\mu^2\,(Mm)^3}\right\}^{\frac{1}{2}}. \tag{291}$$

This special system of units makes each of the following expressions unity: $\mu^{1/2}Mm\,(M+m)^{-\frac{1}{2}}$; μMm ; and C. The moment of momentum of orbital motion, in a circular orbit of radius r, is

$$M\left(\frac{mr}{M+m}\right)^2\Omega + m\left(\frac{Mr}{M+m}\right)^2\Omega = \frac{Mm}{M+m}r^2\Omega\,. \tag{292}$$

And KEPLER's law gives

$$\Omega^2 r^3 = \mu\,(M+m)\,, \text{ or } \Omega r^2 = \mu^{\frac{1}{2}}\,(M+m)^{\frac{1}{2}}r^{\frac{1}{2}}\,. \tag{293}$$

Therefore, by means of the special units, the moment of momentum of orbital motion in (292) becomes

$$\mu^{\frac{1}{2}}Mm\,(M+m)^{-\frac{1}{2}}\,r^{\frac{1}{2}} = r^{\frac{1}{2}}\,. \tag{294}$$

The moment of momentum of the Earth's rotation is Cn, where C is the moment of inertia and n the angular velocity of rotation. The total moment of momentum of the system is constant, and made up of two parts, one depending on the rotation of the Earth about its axis, the other on the orbital motion of the two bodies about their center of inertia; therefore if h be this constant, we have, in the special units,

$$h = n + r^{\frac{1}{2}}\,. \tag{295}$$

The kinetic energy of orbital motion is

$$\frac{1}{2}M\left(\frac{mr}{M+m}\right)^2\Omega^2 + \frac{1}{2}m\left(\frac{Mr}{M+m}\right)^2\Omega^2 = \frac{1}{2}\frac{Mm}{M+m}r^2\Omega^2 = \frac{1}{2}\mu\frac{Mm}{r}\cdot \tag{296}$$

The kinetic energy of the Earth's rotation is $\frac{1}{2}Cn^2$, and the potential energy of the system is $-\mu\frac{Mm}{r}\cdot$ The sum of these three energies, in the special units, becomes

$$2e = n^2 - \frac{1}{r}\cdot \tag{297}$$

Putting

$$x = r^{\frac{1}{2}}\quad,\quad y = n\quad,\quad Y = 2e\,, \tag{298}$$

DARWIN has illustrated these fundamental equations and another called rigidity, which gives the condition that the two bodies should revolve as parts of a rigid system:

Momentum, $$h = y + x\,. \tag{299}$$

Energy, $$Y = y^2 - \frac{1}{x^2} = (h-x)^2 - \frac{1}{x^2}\cdot \tag{300}$$

Rigidity, $$x^3 y = 1\,. \tag{301}$$

Equation (299) is the equation of conservation of moment of momentum; (300) the equation of energy; (301) that of rigidity. When the system is once started, h remains rigorously constant under any interaction between the two bodies, but Y degrades, and the curve of energy has maximum and minimum values defined by the condition $\frac{\partial Y}{\partial x} = 0$, or

$$x^4 - hx^3 + 1 = 0\,. \tag{302}$$

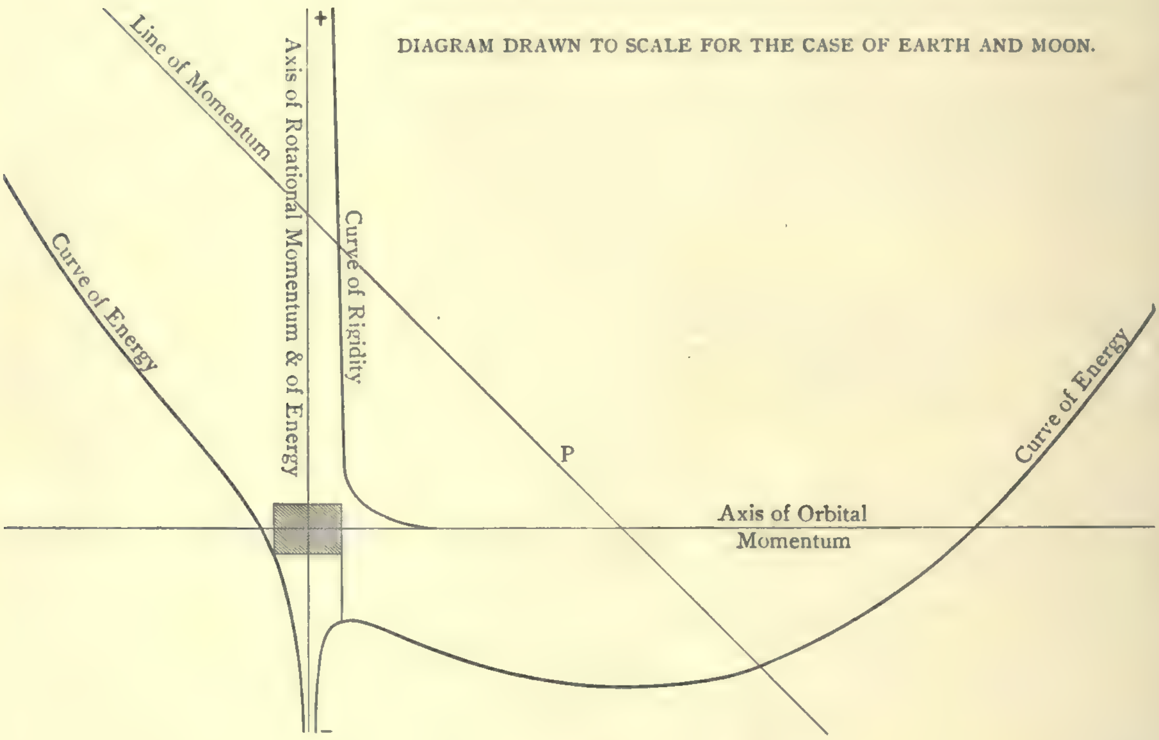


FIG. 33. DARWIN'S GRAPHICAL METHOD OF ILLUSTRATING THE SECULAR EFFECTS OF TIDAL FRICTION.

Taking the Moon's mass to be $\frac{1}{82}$ of the Earth's mass, and the Earth's moment of inertia as $\frac{1}{3}Ma^3$, DARWIN found the special unit of mass to be $\frac{1}{83}$ of the Earth's mass, the unit of length 5.26 radii of the Earth (33506 kilometers), and the unit of time $2^h\ 41^m$.

In these units the present angular velocity of the Earth's rotation becomes 0.7044 and the Moon's radius vector 11.454. This position of the Moon is indicated in the diagram by the point P, and the moment of momentum of its orbital motion is 3.384, and thus very large. This is DARWIN'S celebrated analysis of the interaction of the Earth and Moon (cf. *Proc., Roy. Soc.*, June 19, 1879; also THOMSON & TAIT'S *Nat. Philosophy*, Appendix G; or *Encyclopedia Britannica*, article "Tides").

As the energy curve has a maximum near the origin, corresponding to a small distance between the Earth and Moon, DARWIN inferred that they had once been a single mass, rotating temporarily as a rigid system; and that after the separation, the Moon had receded, according to the downward slope of the energy curve, till it reached its present distance. The time of the Earth's rotation was calculated to be about $2^h\ 41^m$, which would barely enable the equilibrium of the globe to maintain its stability under gravity (cf. *Phil. Trans.*, Part II, 1879, pp. 510, 537; and *Phil. Trans.*, Part II, 1880, pp. 835, 877). And as this pointed to the rupture of the globe from too rapid rotation, DARWIN inferred that it had actually occurred, and that the Moon had thus been detached from the Earth.

NOLAN, and others, pointed out the extreme difficulty the Moon would have in holding together under tidal strain within so small a distance of the Earth; and the inevitable disruption of such a satellite within 2.44 radii of the planet has been well established by the earlier researches of ROCHE and the subsequent investigations of DARWIN. So long as it was uncertain whether the Moon could hold together so near the Earth, it was, for a time, believed that the primeval satellite might have taken the form of a flock of meteorites when the separation first took place. The difficulty of making out how the Moon got started as a single mass so near the Earth, DARWIN has repeatedly acknowledged. As the result of NOLAN'S criticism, he found 6500 miles from the center of the Earth to be the minimum distance at which the Moon could revolve in its entirety (*Phil. Trans.*, Vol. CLXXVIII, 1887, p. 416); but this was not entirely satisfactory, and at the end of his important paper on the "Figures of Equilibrium of Rotating Masses of Fluid" (*Phil. Trans.*, Vol. CLXXVIII, 1887, p. 422), he concluded in some despair, that it is "necessary to suppose that, after the birth of a satellite, if it takes place at all in this way, a series of changes occur which are quite unknown."

Accordingly, we see that by tracing the Moon back towards the Earth, this supposedly reversed process brought them into close contiguity, one rotating and the other revolving in approximately the same time, and both not far from

the critical period of instability for the terrestrial spheroid. "Is this," asks DARWIN, "a mere coincidence, or does it not, rather, point to the break-up of the primeval planet into two masses in consequence of a too rapid rotation?"

In addition to the objections already advanced, another formidable one arises from the difficulty of finding any cause adequate to produce the supposed very rapid rotation of the primitive globe. This objection is now recognized to be much greater than it was supposed to be when DARWIN's work was finished, thirty years ago; for Laplacian conceptions were then universally current, and it was natural to think of the Moon as a part of the Earth, while such an idea as the capture of satellites would not have been entertained. In the views prevalent thirty years ago, the above question of DARWIN was naturally answered in the affirmative, in spite of outstanding difficulties of considerable magnitude. To-day with all the other satellites proved to be captured, the wonderful relations brought out by DARWIN's analysis must be declared to be only an accidental, but most deceptive, coincidence. It probably is the most remarkable result of this kind in the annals of science.

§119. STRATTON'S *Researches on Planetary Inversion.*

In the *Monthly Notices* of the Royal Astronomical Society for April, 1906 (Vol. LXVI, No. 6), MR. F. J. M. STRATTON, of Cambridge, England, has a scholarly discussion of the problem of planetary inversion, which had been suggested by PROFESSOR W. H. PICKERING's discovery of the retrograde motion of *Phœbe*, and the tacit assumption formerly adopted by all writers that the satellites have been detached from the planets about which they revolve.

In stating his problem MR. STRATTON says: "If, then, a satellite were thrown off in a very early stage of the planet's evolution, it would commence moving in a retrograde direction around the planet. If the oblateness of the planet were very small, or the satellite at a considerable distance from the planet's center, the plane of the orbit of the satellite would not follow the plane of the planet's equator as it tilted over, but would fall back into a stable position near the ecliptic — a term used in this paper for the plane of the planet's orbit. Such a satellite would remain of the retrograde type exemplified by *Phœbe*. If, however, the satellite were evolved in a later stage of the planet's development (after the planet had greatly contracted and become more oblate), the satellite would move in an orbit whose stable position was almost coincident with the planet's equator, and the satellite would follow the planet's equator. Most of the known satellites of the solar system fall into this class."

"PROFESSOR PICKERING urged in support of this view that the classical nebular hypothesis, according to which the planets were thrown off in the form of rings, required an initial retrograde rotation of the planet and not a direct one, as LAPLACE assumed. But of recent years SIR GEORGE DARWIN, PROFESSOR T. C. CHAMBERLIN, and DR. F. R. MOULTON, have adduced strong reasons for discarding the ring-theory, and it would seem that such confirmation as it would undoubtedly have given to this investigation must for the present be disregarded. Though, apparently, the classical form of the nebular hypothesis cannot now be accepted without considerable modifications, I have here followed it in general as regards the history of the planetary sub-systems, and have assumed a planet to be a gradually contracting body, which, from time to time, may pass through a form of instability, resulting in the evolution of a satellite."

MR. STRATTON found many difficulties and uncertainties in this work and has discussed them fully. On pp. 396–8 he has the following remarks: "There remains one other difficulty in connection with the time required for the working out of the theory, and that difficulty, though an almost necessary accompaniment of any such theory, would be alone sufficient to prevent one from urging its acceptance on dynamical grounds alone. It does not appear that, for such enormous periods of time as we are here concerned with, our ordinary dynamical equations are of sufficient exactitude to prevent the entrance of some unknown factors, which may profoundly modify the course of the evolution of the system. This difficulty must be regarded as an additional cause for receiving the theory with all reserve." . . . .

"The present small obliquity of *Jupiter*, requiring an almost impossibly great viscosity, if explained by solar tidal friction alone, had been regarded as a natural consequence of the tidal action of the satellites. And the large angle through which *Saturn** had tilted since the evolution of *Phœbe* had been looked upon as in great part due to the tidal action of its satellite." . . . .

"We may say, then, that the theory of planetary inversion suggests, but does not absolutely require as a condition for its truth, an annular stage in the history of the satellites of *Jupiter* and *Saturn*. More than this we do not care to state till a more detailed application of the tidal theory has been made to the case of a planet attended by a group of satellites. The very doubtful question whether perturbations in a ring of satellites could ultimately lead to the formation of one or several satellites must also be discussed before the difficulties considered in this section can be removed."

Again, in the summary of his results, on pp. 400–401, MR. STRATTON

* *Jupiter's* Eighth Satellite had not been discovered when MR. STRATTON'S paper was written.

continues: "*Jupiter* must have evolved its satellites after its obliquity had decreased below 90°; partly under their influence it has been driven down towards a stable position of small obliquity, which it has now nearly reached. *Saturn* shed *Phœbe*, and possibly also *Iapetus* and *Hyperion*, while its obliquity was greater than 90°; as under solar tidal influence it passed through the critical position, where its obliquity was 90°, *Phœbe* sank down into the ecliptic in a retrograde orbit, while *Iapetus* and *Hyperion* moved over with the planet's equator. Afterwards the inner satellites were evolved, and under their influence and the influence of the rings *Saturn's* obliquity has steadily diminished — and is still diminishing — towards a small stable value. As seems highly probable for a planet further removed from the Sun, and, therefore, less likely to have its increasing rotation checked by solar tidal friction, the satellites of *Uranus* were evolved in an earlier stage of its evolution, before its obliquity had decreased to 90°; they have stopped the decrease in obliquity, which would arise from the solar action, and they are now driving *Uranus* back to a stable position with an obliquity of 180°. *Neptune,* with its one satellite of extremely large tidal influence, is being driven towards an equilibrium position with an obliquity of 180°. I should add that uncertainty as to the data for the satellites of *Uranus* and *Neptune* leaves even the present direction of motion of their equators very doubtful, but that the results above given seem on the whole the most probable." . . . .

"I suggest as the easiest explanation of certain remaining difficulties that the satellites of *Jupiter* and *Saturn* have passed through an annular form at some previous stage in their history. This latter idea is not essential to the successful working out of the theory; at present it is only put forward very tentatively indeed, and as a subject for further research."

"Viewed broadly, then, the theory of planetary inversion, though it entails some difficulties of detail, remains a tenable hypothesis. As explained by Sir George Darwin's tidal theory it involves three main assumptions: (1) that the outer satellites of a planet were evolved before the inner ones; (2) that the determining factor producing secular alterations in a planet's obliquity has been tidal friction; and (3) that the time involved in the scheme is not so great as to invalidate the ordinary dynamical equations. A justification for these assumptions may, perhaps, lie in the satisfactory explanation which the theory affords, both of the large obliquities of *Uranus* and *Neptune* and of the presence of a satellite such as *Phœbe.* The secular motions with which the theory is concerned are so extremely slow that it can hardly yet be proved or disproved by reference to the gravitational theory of the motions of planets and their satellites; the theory would gain some support by the discovery of satellites to *Uranus* and

Neptune of the same type as *Phœbe*, if their motion were retrograde; it would be overthrown if their motion were direct. The theory remains then at present a speculative hypothesis, which is on the whole well supported by the theory of tidal friction, and which gives the only explanation so far offered for certain facts."

It is impossible to convey the contents of this lengthy and well prepared paper, even by quotations of such considerable length as are here given; but this seemed the only way of doing the author even moderate justice, because of the difficulty of condensing the results into smaller compass, without omitting some important considerations. The chief significance of MR. STRATTON's investigation lies in the continued adherence to Laplacian traditions, in spite of the negative and therefore unsatisfactory criticisms of MOULTON and CHAMBERLIN; and in the avoidance of any suggestion that the observed satellites might have been captured, though SIR GEORGE DARWIN, under whose inspiration MR. STRATTON's work was done, had eight years before published his celebrated memoir on "Periodic Orbits" (*Acta Mathematica*, Vol. XXI), and during the previous year had given valuable suggestions on cosmical evolution in his Presidential Address to the British Association at Capetown, 1905. One cannot but wonder to what extent MOULTON's misleading criticism of PROFESSOR W. H. PICKERING's suggestion of the possible origin of *Phœbe* by capture (*Astrophysical Journal*, October, 1905, pp. 177–180), with the accompanying fatal misinterpretation of JACOBI's Integral, may have been responsible for the rejection of the only idea which could simplify our theory of the observed satellites, and bring it into harmony with the purely mathematical results arrived at by PROFESSOR SIR G. H. DARWIN in his justly celebrated memoir on "Periodic Orbits."

§ 120. *On the True Physical Cause Which Determines the Direction of Planetary Rotation.*

It will be seen from the considerations already adduced, and examined with some care in Chapter X, on the "Dynamical Theory of the Capture of Satellites," that we explain the direction of rotation of the planets on the same principle by which we account for the direction of revolution of the satellites in their orbits. About each planet, within the HILL closed surface, and in the hour-glass surfaces which are not closed, waste matter from the nebulosity circulating about the Sun passes freely. As the hour-glass surface is not entirely closed for most of the particles, they naturally enter the region about the planet with a direct motion; and this same direction is naturally preserved when they fall down near the planet

so as to pass within the closed surfaces. Therefore in general the satellites have direct revolutions in their orbits and the planets have direct rotations on their axes. Only crossing satellites, or those of irregular foreign origin have retrograde revolution: and most of these are destroyed. Those which fall into the planet under the secular effects of resistance check its rotation but slightly.

Accordingly, while we admit MR. STRATTON's theory of planetary inversion under his postulated conditions, involving enormous duration of time, we deny that such history has been enacted in the solar system, unless possibly a slight effect of the kind has arisen in the systems of *Uranus* and *Neptune* which are so remote from the Sun. In our view the direct rotations of the planets are inevitable consequences of the capture of nebulosity in the sheltered regions enclosed within the HILL closed surfaces. These closed spaces are regions into which waste material drifts as inevitably as water runs down hill. In these sheltered and sequestered regions systems of satellites collect, because the nebular vortices arising there circulate incessantly, and the waste nebulosity finally goes to the building up of the planets or of their satellites. This conception of the sheltered vortex inside the HILL closed surfaces gives one a very clear idea of what takes place about the planets as they slowly develop in the vaster extent of nebulosity circulating about the Sun.

As the planets originate at much greater distance from the Sun than they now have, we cannot assume that their rotations may not be partly fixed before they reach their present positions. Even retrograde rotation might be started in remote planets; and may be this still partially survives in the systems of *Uranus* and *Neptune*. Accident has much to do with the rotations of remote bodies, but in the inner parts of the system a more orderly development prevails, because the retrograde motions are largely obliterated, as we see in the actual solar system. Various causes have modified the rotations and axial tilts of the planets, but direct rotation is natural; while planetary inversion seldom, if ever, takes place.

§ 121. *The Moon and Other Satellites, Being Small Captured Bodies, Probably Never Had Much Rotation, but Even This Has Been Destroyed by Resistance and by Tidal Friction.*

This proposition is almost obvious without elaborate analysis of the reasons why the smaller bodies have little rotational moment of momentum. For in coming together the elements of such a mass could hardly give it a rapid rotation about any axis, because the closed HILL surface about it is too small to give a large vortex for the collection of waste matter; and nothing but a large amount

of this gathered rubbish revolving under strong central force could produce a rapid rotation in the planet formed by the subsequent condensation of the material. Thus, owing to the small size of the HILL closed surface, and the feeble central attraction — both being due to the smallness of the mass — the rotation of a small body like the Moon can never be very rapid. Accordingly neither the Terrestrial Moon nor any of the other satellites of the solar system ever had rapid axial rotation, and the same remark applies to the planet *Mercury*. Yet what little rotations the Moon, the satellites of *Jupiter*, *Saturn*, and other planets, may have had, have been exhausted by subsequent resistance, and especially by the tidal friction of the planets about which they revolve. It is not surprising, therefore, that they show only one face towards their several planets. The result has long been regarded as probable, but previous writers, being unaware of the causes which determine the rotation, and not suspecting that the satellites were captured, have perhaps overrated the chances of primitive rapid rotation, and made the destruction of the axial rotations seem more important than it really is. For as the Earth has been thought to have rotated in about 2h 41m, according to DARWIN, it might naturally have been supposed that the rotation period of the Moon, also, was at one time comparatively short. If the present views are correct, this has never been the case, and, although tidal friction has been the main cause working to exhaust the rotations, there never was much rotation of the Moon to be destroyed. The force of this argument becomes more apparent by remembering that if the Moon is a captured body, there is no good reason to suppose that the Earth ever did rotate much more rapidly than it does at present.

Problems, such as the loss of the atmospheres of the Moon and of other satellites, also take on a new aspect; for we have no reason to believe any sensible atmosphere ever existed about these small captured bodies. Nor is it probable that there is snow or ice on the Moon's surface, as many writers have supposed. Whether the large craters can have been formed by the impact of small satellites upon a heated and molten surface, as the geologist G. K. GILBERT believed, will be carefully considered in Chapter XIV.

The Moon being in the present hypothesis a planet and not a portion of the Earth, we have to give up most of the supposed analogy between Terrestrial and Lunar volcanoes and mountains. The mountains on the Moon apparently were formed before it was captured by the Earth. And, therefore, while we lose by giving up the assumed analogy with the Earth, we gain by our new privilege of studying at close range a planet from the celestial spaces formed quite independently of the Earth. If this view be admissible, there will be considerable

advantage to science; for we never expected that this privilege of such close telescopic inspection of another planet of the solar system would be given to the inhabitants of our terrestrial globe.

In this connection, I may say, that on one or two occasions when the seeing was at its best during the observations of the planet *Mercury*, at Washington, in 1901 and 1902, I believed I obtained glimpses of the planet's surface of the same type as that of the Moon. It may well be that these brief glimpses, gained at moments of best seeing, supported as they are by the evidence of photometric measures, showing that the planet has a rough surface, rest on a more substantial basis than any one heretofore has ventured to believe. One gets the impression that the origin of the Moon and of the planet *Mercury* is essentially the same, and that in the remote past both revolved in the planetary spaces between the present orbits of *Mars* and *Jupiter*.

§ 122. *The Terrestrial Spheroid Itself Shows Little, if Any, Evidence of Having Had More Rapid Rotation in Former Times.*

The theory that the Moon is a captured body carries with it several important corollaries, which deserve careful consideration. Foremost among these is the question whether the Earth rotated much more rapidly in former times than it does now. It has long been believed that the Earth once had a much more rapid rotation than at present, and tables of the changes in the Earth's figure and physical constitution, arising from such supposed rapid rotation, have been calculated and published in various works on Geology and Physics.* But it is a remarkable fact that if we examine this work carefully, we shall find that it rests not on observed phenomena, but on DARWIN's celebrated papers on the "Origin of the Lunar-Terrestrial System," which have been analyzed above. On the other hand, the terrestrial spheroid itself gives little, if any, evidence of more rapid rotation in former times. No well established facts in Geology, Physics, or Geodesy support such a view.

It is true that the changes in the rate of rotation of our planet might be supposed to be so slow that all traces of the former state of the Earth would have been wholly obliterated by the transformations which have intervened; yet it is not at all certain that this would be so, and it seems more probable that the greater oblateness once existing would have left sensible traces of incomplete adjustment to modern conditions. So far as may be judged from accurate measurements of gravity, and from many trigonometric measurements carried out in all latitudes and in both hemispheres, by various Geodetic Surveys, no certain in-

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* cf. "The Rotation-Period of a Heterogeneous Spheroid," by CHARLES S SLICHTER, *Publication* 107 *of the Carnegie Institution*, 1909.

equalities pointing to a former rapid rotation of the Earth have been discovered. The inequalities found all seem to be local, and connected with the formation of the continents, which owe their elevation and outlines to the secular leakage of the oceans (cf. "Further Researches on the Physics of the Earth, and Especially on the Folding of Mountain Ranges and the Uplift of Plateaus and Continents Produced by Movements of Lava Beneath the Crust Arising from the Secular Leakage of the Ocean Bottoms," *Proc. Am. Philosophical Society*, Philadelphia, No. 189, 1908).

In his valuable work on "Tides and Kindred Phenomena in the Solar System," pp. 300–304, SIR GEORGE DARWIN discusses this question of the Earth's adjustment with some care. He admits that LORD KELVIN did not share his view that the Earth had adjusted its figure to suit its rate of rotation. He says LORD KELVIN held "that the fact that the average figure of the Earth corresponds with the actual length of the day proves that the planet was consolidated at a time when the rotation was but little more rapid than it is now." And adds: "The difference between us is, however, only one of degree, for he considers that the power of adjustment is slight, whilst I hold that it would be sufficient to bring about a considerable change of shape within the period comprised in geological history."

SIR GEORGE DARWIN then proceeds to analyze four classes of facts derived from observation — gravity, the ellipticity of the Earth, the lunar inequality depending on the Earth's figure, and the precession and nutation of the Earth's axis — and says that they are so intimately intertwined that one of them cannot be touched without affecting the others. In conclusion, he adds: "EDOUARD ROCHE, a French mathematician, has shown that if the Earth is perfectly plastic, so that each layer is exactly of the proper shape for the existing rotation, it is not possible to adjust the unknown law of internal density so as to make the values of all these elements accord with observation. If the density be assumed such as to fit one of the data, it will produce a disagreement with observation in others. If, however, the hypothesis be abandoned that the internal strata all have the proper shapes, and if it be granted that they are a little more flattened than is due to the present rate of rotation, the data are harmonized together; and this is just what would be expected according to the theory of tidal friction. But it would not be right to attach great weight to this argument, for the absence of harmony is so minute that it might be plausibly explained by errors in the numerical data of observation. I notice, however, that the most competent judges of this intricate subject are disposed to regard the discrepancy as a reality."

The views here expressed by DARWIN, who may be considered the highest authority on the subject, accord sufficiently well with those reached by the present

writer, on the theory that the Moon is captured, to justify the statement that the Earth itself shows little, if any, evidence of more rapid rotation in former times.

If the supposed greater tidal efficiency of the Moon in past ages is given up, various tidal and physical questions will be left unsettled, and most of the problems of the Physics of the Earth will have to be re-examined. The uniformitarian theories in Geology will gain some additional importance by changes in fundamental principles which exclude the Moon from a more active part in the past history of the Earth.

§ 123. *Light Thrown on the Earth's Primitive Rotation Period by the Observed Rotations of the Other Planets About Their Axes.*

Before finally dismissing this important subject it is worth while to remark that some further light on the question of the Earth's rotation in past ages may be gathered from the study of the other planets in space. If we consider attentively the present slow rotations of the other planets, we shall perceive how extremely improbable it is that the Earth once rotated rapidly enough to detach the Moon. The best determined rotation periods of the several planets seem to be the following (cf. *A.N.*, 4308):

Mercury,	88 days.	*Jupiter,*	9.928 hours.
Venus,	225 days, or 1 day.	*Saturn,*	10.641 hours.
The Earth,	24 hours.	*Uranus,*	10.1112 hours.
Mars,	24.62297 hours.	*Neptune,*	12.84817 hours.

In the case of *Venus,* I have given preference to the period found by LOWELL, though there is, perhaps, still a little doubt attached to the rotation period of this planet.* Working with the spectrograph at Poulkowa, BELOPOLSKI obtained apparently slight spectral displacements corresponding to a period of one day (cf. *A.N.*, 3641), but this result was not confirmed by LOWELL, who repeated the experiment at Flagstaff under favorable conditions. There are, however, two additional reasons for being very cautious about concluding what the period of *Venus* is: (1) From the mass of the planet, namely, 0.8153 of the Earth's mass (cf. *A.N.*, 3992, p. 118), one would expect an original rotation nearly as rapid as that of the Earth, owing to the physical cause which determines rotation, as set forth in the present Chapter. (2) If a rapid rotation once existed, in a period of about one day, the question arises whether it could have been destroyed by tidal friction. Heretofore we have been inclined to answer this question in

* This discussion is left as it was in *A.N.* 4343, but the rotation of *Venus* is further considered in Chapter XVI, where it is shown that the true period probably is $23^h\ 21^m$.

the affirmative, but it is not clear that we have been right. It is true that the tidal frictional resistance due to the Sun's action on *Venus* would be about 5.8 times what it is on the Earth; but DR. HECKER's recent observations at Potsdam indicate a yielding of the solid Earth under the action of the Moon of only about six inches, according to a statement by PROFESSOR SIR G. H. DARWIN in a public lecture at Cambridge, May 10, 1909. This corresponds to a solar tide in the solid Earth of three inches, and this would make the bodily tide in *Venus* not over seven inches. For in the paper on the "Rigidity of the Heavenly Bodies," *A.N.*, 4104, I have shown that the rigidity of *Venus* must be taken to be but little less than that of the Earth. If, then, the solid Earth yields to the Sun's attraction to the extent of about three inches, and the solid globe of the planet *Venus* not over seven inches, the question arises whether the frictional resistance against the rotation would not be excessively slow, and, in fact, almost insensible. If the moon has been captured, as set forth in this paper, it appears that we cannot point with certainty to any sensible retardation of the Earth's rotation, due to the action of the Sun and Moon; nor should we expect such a result from a tidal yielding of the Earth's mass of only about three and six inches, respectively, for these two disturbing bodies. Under the circumstances, it seems necessary to preserve an open mind about the rotation period of *Venus*.

However this question may be decided by future events, the period will in no case be appreciably less than a day, and this minimum value is sufficient for our present purposes. What is true of *Venus*, is even more certainly true of *Mercury*.

Now, the period of $2^h\,41^m$, or 2.7 hours, found by DARWIN, for the Earth when rotating as if rigidly connected with the Moon, is only about one-ninth of the present rotation period of the Earth; and even *Jupiter*, which has the largest mass and shortest period of any of the planets, rotates 3.7 times more slowly than our primitive Earth is supposed to have done. By dividing the primitive Earth's hypothetical period of 2.7 hours into the periods of the other planets, we obtain for the several planets the following minimum numbers, namely: *Mercury*, 9; *Venus*, 9; *Mars*, 9.1; *Jupiter*, 3.7; *Saturn*, 4.0; *Uranus*, 3.7; *Neptune*, 4.8; and we may calculate the probability that in seven different cases the observed periods would so much exceed that of the primitive Earth, or that the Earth's original period would have been so much shorter than that of any of the other planets. If the Earth, as an ordinary planet of very modest size, could really have attained to a rotation in so short a period as 2.7 hours, the chances that seven other planets would not all miss in the same direction, and by these amounts, the average being about 6.2, would be about as the continued products

of the above numbers, which is 193745. Thus the chances that the Earth could have had such a short period as 2.7 hours when calculated from the data furnished by the other planets scarcely exceeds 1 in 200000, or the chances are 200000 to 1 that no such short period as 2.7 hours ever existed. And if the known physical cause of the rotations, as established in this work, be introduced, the probability becomes practically infinity to one that such a rotation period as 2.7 hours never existed; and the probability remains enormous that the Earth never rotated much more rapidly than it does now. So far as one may judge, therefore, by the data furnished by the other planets, we are justified in rejecting once for all the hypothesis that the day was ever appreciably shorter than at present.

§ 124. *Summary of Results.*

These several considerations may be briefly summed up as follows:

(1) As all of the other satellites are proved to be captured bodies, the overwhelming presumption is that this is true, also, of the Moon, and this enormous probability is naturally increased by the demonstrated fact that all the planets, likewise, have been captured by the Sun, and not one of them detached from that central globe, as was formerly supposed by LAPLACE and other early writers on Cosmical Evolution.

(2) If we calculate the probability that the otherwise uniform rule of capturing companions has been broken in the single case of the planet Earth, we shall find the chances against it so overwhelmingly as to wholly exclude it from consideration.

(3) Thus the companions or satellites could originate in but one of two possible ways; namely, by capture, and by detachment. Let us make the case as favorable as possible to the theory of detachment, and put the probability of the two events each equal to $\frac{1}{2}$. Then, as we have eight principal planets, 25 satellites (besides our Moon), and over 660 asteroids — all certainly captured — the chances are at least $(2)^{693}$ to unity that the Moon has been captured. This number exceeds a decillion decillion (10^{66}) to the third power $(10^{66})^3$, and is so enormous that it passes all comprehension.

(4) Even a decillion decillion (10^{66}) is so large that we are compelled to resort to a method employed by ARCHIMEDES to illustrate it. Imagine sand so fine that 10,000 grains will be contained in the space occupied by a poppy seed, itself about the size of a pin's head; and then conceive a sphere described about our Sun with a radius of 200,000 astronomical units (α *Centauri* being at a distance of 275,000), entirely filled with this fine sand. The number of grains of sand in this sphere of the fixed stars would be a decillion decillion (10^{66}).

(5) But to correctly understand the actual probability of the origin of the Moon by capture, we must extend the method of ARCHIMEDES and conceive all the grains of sand included within this sphere with radius extending nearly to *a Centauri*, to be arranged in a continuous straight line as close together as possible (such a line will, of course, extend to infinity), and then imagine a cube erected on this infinite line as a base; and when this infinite cube is entirely filled with the finest sand, all the grains included within it against one is the probability that our Moon, also, has been captured, and that the Lunar-Terrestrial system forms no exception to the general rule of cosmical evolution by capture prevailing in the development of the solar system.

(6) As this mode of calculation by the theory of probability is entirely rigorous and not merely approximate, it, therefore, incontestibly follows that our Moon, too, has been captured and added to our terrestrial system from without, and, therefore, never has been nearer us than at present, but has come to Earth from heavenly space, as was announced by the author's cablegram in *A.N.*, 4325.

(7) Consequently we conclude that the events traced by DARWIN depend on occidental coincidences and do not represent the true physical history of Nature. Accordingly all our previous conceptions in Astronomy, Physics of the Earth, and Geology, as dependent on the Moon's supposed detachment from our planet,* must be wholly abandoned, and all the questions again re-examined, in the light of the new theory, from the ground up. This affords us an impressive illustration of the incompleteness of the Physical Sciences to-day.

(8) The present distance of the terrestrial Moon in the inner part of the closed HILL surface about the Earth corresponds with the theory that this body has been captured, in which case it could hardly have remained very near the outer portions of this space. When the Moon was first captured, however, its distance can hardly have been much less than twice what it is now; so that the distance probably has been greatly reduced with the lapse of ages.

(9) If this view be admissible, it follows that the mean distance has been reduced, principally by the secular action of the resisting medium; and the month has been shortened from some eighty days to 27.32166 days, as at present. The original month may have exceeded 100 days, but as DR. HILL has shown cannot have exceeded 204.896 days.

(10) If the mean distance has been so much reduced, it follows that the eccentricity of the orbit has also been correspondingly diminished. The present eccentricity of 0.05489972, therefore, agrees well with the capture theory. The

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* The theory that the Moon was thrown off from the Earth seems to date back to the Greek philosopher ANAXAGORAS, B.C. 500-428.

view that the present eccentricity is a survival of a larger value appears probable in itself; and is in harmony with the tendencies observed in other satellite systems, where the same cause has been at work.

(11) The inclination of the Lunar orbit to the ecliptic, $5° 8' 43''.35$, is about what would be expected from the capture theory, and naturally the orbital motion would be direct. For when a body is captured the chances of theory are much greater that it will move direct rather than retrograde, and we see this theory confirmed by what is observed in the other satellite systems. This follows naturally from the circumstances that a captured satellite has to cross the line of conjunctions before coming under the control of the planet, in order to give a retrograde motion, unless of course such satellite has come in at random and follows no law whatever.

(12) The great preponderance of the Moon's moment of momentum of orbital motion (3.384) over that of the Earth's axial rotation (0.7044) is of itself a suspicious circumstance, and difficult to account for, without introducing violent hypotheses. But if the Moon is captured, this unusual circumstance presents no difficulty.

(13) DARWIN's celebrated diagram does not show how the system of the Earth and Moon came to be started; but only shows what will follow from a given condition of the system. Now if the bodies were started to revolving in a perfect vacuum, they might separate as he supposed, but if the resisting medium is more effective than tidal friction, the bodies will approach one another in spite of the energy curve in the diagram; for this curve rests on dynamical equations which postulate no resistance. When the resisting medium is introduced the energy curve is no longer valid, but the outcome will depend on the relative importance of the two rival forces — tidal friction and the resisting medium, the secular effects of which are exactly opposite. In order to judge which is likely to predominate, it is sufficient to recall the circularity of the orbits of the planets and satellites noticed elsewhere in our system, and directly traceable to this latter cause and no other.

(14) HALLEY first suspected the existence of a secular acceleration of the Moon's mean motion in 1693. It was confirmed by DUNTHORNE, in 1749, and in the same year EULER advanced the view that all the heavenly bodies were subject to the secular effects of a resisting medium. Notwithstanding LAPLACE's celebrated discovery in 1787, that the secular decrease in the eccentricity of the Earth's orbit was responsible for most of the observed secular acceleration of the Moon, it continues to be an unsettled question. The correction of LAPLACE's process of calculation by ADAMS, in 1853, and the verification of the latter's pro-

cedure by Delaunay, Plana, Lubbock, Hansen, Cayley, and others, allows gravitational theory to account for only about two-thirds of the observed effect indicated by the most ancient observations, 6″.11 according to Delaunay, while the most ancient eclipses of the Sun make the observed secular acceleration about 8″.2. And, recently, Mr. Cowell has confirmed a secular acceleration of the Moon of some 9″ by new researches on eclipses, and, besides, found a sensible secular acceleration of the Sun, which could not be accounted for by any hitherto recognized cause. Why not go back to Euler's sagacious suggestion of the resisting medium, to explain both of these outstanding anomalies? If the resisting medium has shaped the orbits of the heavenly bodies, it has not yet entirely disappeared, but must produce small effects which are sensible to observations extending over long ages.

(15) And of all the bodies in our system adapted to disclosing the secular effects of this slowly acting cause, the Moon is by far the most sensitive, as was long ago remarked by Euler. It is like a delicately adjusted chronometer, and the slightest disturbance will at length become sensible to observation. The next most sensitive of the heavenly bodies is undoubtedly the Sun (or, rather, the Earth), because of the accuracy of our modern observations and the considerable period over which they have extended. And here it is that Mr. Cowell, of Greenwich, has recognized the anomalies which heretofore have been attributed to the secular effects of tidal friction in changing the length of the day.

(16) If the views set forth in this Chapter be admissible, they will tend to restore our confidence in ancient eclipse observations, and also in the steadiness of the Earth as a time-keeper, while they will give a severe shock to those who consider the heavenly spaces devoid of sensible resistance. And while the effects attributed to tidal friction seems to be less important than they have been supposed to be, on account of the present great distance of the Moon, and the indication that it has never been sensibly nearer the Earth; yet the importance of this cause will always be considerable, both in our own system, and in other systems observed in the immensity of space. The change in our point of view, of course, does not diminish the value of Professor Sir G. H. Darwin's celebrated work on this subject, but simply limits the scope of the results when applied to the systems nearest at hand. Even if inapplicable to the Moon, or application to but a limited extent, his beautiful analysis will always be the basis of future researches in this extensive subject, which deals with one of the most important physical causes effecting the figures and motions of the heavenly bodies.

§ 125. MOULTON'S *Latest Criticism of* DARWIN'S *Theory of the Terrestrial Origin of the Moon.*

Since the above discussion was completed and forwarded, in a somewhat abbreviated form, to the *Astronomische Nachrichten*, May 22, 1909, the author has had an opportunity of examining MOULTON'S recent criticism of DARWIN'S theory of the origin of the Moon, in publication No. 107, of the Carnegie Institution. Whilst the results there reached are negative, and consist in showing the untenability of the Fission Theory, they are of considerable interest as affording an additional confirmation, even if only an indirect one, of the new theory that the Moon is a captured body, and has come to the Earth from celestial space. The following are MOULTON'S chief results:

(1) He finds, by methods depending on equations of energy and moment of momentum, not greatly different from those of DARWIN, that the Moon cannot be traced back to close contact with the Earth; but that when they are brought into closest proximity deducible from these methods, an interval of 4201 miles separates the surfaces of the Earth and Moon, or a space 243 miles greater than the Earth's radius.

(2) It is not possible to bridge over this difficulty by any admissible hypothesis, such as supposing that the Earth once had a larger volume and greater oblateness and has since shrunk up. MOULTON considers the case in which the Earth is made so oblate as to reach the Moon, under conditions by which the volume and rotational moment of momentum are kept constant and LAPLACE'S law of density prevails; and finds that the polar radius would thus be 942 miles, and the equatorial radius 9194. "A scale drawing shows that this oblateness is out of the question, and a little consideration shows that the equatorial zone must have been so rare as to make it impossible to account for the mass of the Moon."

(3) By no variation of the data on the several admissible hypotheses considered by MOULTON could the Earth and Moon be traced back to closer proximity than 4201 miles between the surfaces. At this distance, corresponding to 9241 miles between the centers, the period of revolution could not be less than 4.93 hours. The accompanying diagram shows the Earth and Moon at their nearest approach; and he finds that there is no way in which such a separation can be reconciled with the fission theory. DARWIN encountered this same difficulty thirty years ago, but MOULTON'S researches have strongly emphasized it.

(4) Assuming that 4″ per century is the outstanding difference between observation and gravitational theory, in the matter of the secular acceleration

of the Moon's mean motion, MOULTON shows that it would take 30,000,000 years for the Moon to gain one revolution. If the physical condition of the Earth has been essentially constant, then it is shown that the length of the day was twenty of our present hours, and of the month twenty-four of our present days, not less than 220,000,000,000 years ago. "It is extremely improbable that the neglected factors, such as the eccentricity of the Moon's orbit, could change these figures enough to be of any consequence. This remarkable result has the great merit of resting upon but few assumptions and in depending for its quantitative character upon the actual observations. If it is accepted as being correct as to its

FIG. 34. NEAREST APPROACH OF MOON TO EARTH BY DARWIN'S THEORY.

general order, it shows that tidal evolution has not affected the rotation of the Earth much in the period during which the Earth has heretofore been supposed to have existed even by those who have been most extravagant in their demands for time. And if one does not accept these results as to their general quantitative order, he faces the embarrassing problem of bringing his ideas into harmony with these observations."

(5) MOULTON finds many results inconsistent with the fission theory of the origin of the Earth and Moon, and finally says that it will have to be rejected. "In a word, the quantitative results obtained in this paper are on the whole strongly adverse to the theory that the Earth and Moon have developed by fission from an original mass, and that tidal friction has been an important factor in their evolution. Indeed, they are so uniformly contradictory to its implications as to bring it into serious question, if not to compel us to cease to consider it as even a possibility."

(6) At the end of a second paper, on the fission of a contracting fluid mass, he again reaches negative results, both as regards the solar system, and the binary star systems, and finally expresses himself in despair as follows: "The results obtained by the computations above are quite adverse to the fission theory, in general, except if it is applied to masses in the nebulous state, and seem practically conclusive against it so far as the solar system is concerned, either in the future or past. Perhaps the hypothesis that stars are simply condensed nebulas, which has been stimulated by a century of belief in the Laplacian theory, should now be accepted with much greater reserve than formerly. Up to the present we have made it the basis not only for work in dynamical cosmogony but also in classifying the stars. It may be the time is ripe for a serious attempt to see if the opposite hypothesis of the disintegration of matter — because of enormous sub-atomic energies, which, perhaps, are released in the extremes of temperature and pressure existing in the interior of Suns, and of its dispersion in space along coronal streamers or otherwise — can not be made to satisfy equally well all known phenomena. The existence of such a definitely formulated hypothesis would have a very salutary effect in the interpretation of the results of astronomical observations. We should then more readily reach what is probably a more nearly correct conclusion, viz.: that both aggregation and dispersion of matter under certain conditions are important modes of evolution, and that possibly together they lead in some way to approximate cycles of an extent in time and space so far not contemplated."

§ 126. *Concluding Remarks on the Two Theories of the Origin of the Moon.*

It is unnecessary to comment on the unjustifiable character of MOULTON's last conclusion, above quoted, except to say that there is not the slightest foundation for the supposition that explosions of stars are in progress anywhere in the universe. Such a reckless suggestion is only an expression of a strong feeling of despair, the natural outcome of negative results. It is generally recognized that in destructive criticism MOULTON has had much better success than in any kind of constructive effort; but it is this failure, this ability to tear down, but not build up, which produces the feeling of despair and a tendency to take refuge in new and extraordinary hypotheses. If the new hypotheses had any observational or dynamical foundation they would be less objectionable, but being perfectly arbitrary and inconsistent with known laws, they must be considered illegitimate and beyond the domain of the recognized canons of philosophical speculation.

Notwithstanding this undeniable weakness in most of MOULTON's work on

Cosmical Evolution, it seems to the author that he has considerably improved our knowledge of the difficulties encountered in DARWIN's theory. It will become evident to all who study these papers carefully that the difficulties recognized both by DARWIN and by MOULTON cannot be overcome, except by abandoning the theory itself, and conceding that the Moon is no part of the Earth, but is a planet which came to us from the heavenly spaces.

Since these numerous difficulties are insuperable, on the fission hypothesis, while by the new theory it is shown on the one hand that the probability is infinity to one that the Moon originated by capture like the other planets and satellites, and on the other that the probability is infinity to one that the Earth could not have acquired a rotation sufficiently rapid to detach the Moon; and moreover, if disruption really took place, the scattered matter could not be gathered into a body such as our satellite is known to be; it follows that the new theory of the celestial origin of the Moon may now be regarded as thoroughly established by the most impressive evidence which Nature could offer to the natural philosopher. In a short time, when we come to realize fully the truly everlasting character of the foundation which underlies the new theory, it will appear to us exceedingly wonderful that any one should ever have entertained the absurd doctrine that any natural cause could have given our small planet a rotation sufficiently rapid to disrupt it by throwing off pieces which were supposed to have collected into a body like the Moon .

In view of the known cause of the rotations of the planets about their axes, as set forth in § 120 of this Chapter, it is absolutely impossible to explain such a supposed rapid rotation; and, of course, a rupture of our planet never really occurred, so that the whole theory is false and misleading. The proof of the correctness of this conclusion is furnished by the insuperable difficulties encountered by the fission theory, even if rupture could be admitted — such as the problem of how the fragments could be gathered into a single globe like the Moon; the impossibility of bridging over the separation shown to exist at the closest stage of fission; the 220,000,000,000 years required for a change of only four hours in the length of the day, etc. Besides we now know how the Moon could be captured, and the extreme hypothesis of a rupture of our primitive planet is not required to explain the genesis of the Lunar-Terrestrial system.

It seems advisable to put in exact mathematical form the different elements of probability which must be considered in dealing with the theory of the Moon's origin. There are shown to be three independent lines of proof that it never was detached from the Earth, as follows:

(1) The theory of capture, based upon the demonstrated capture of the planets by the Sun and of the other satellites by their respective planets.

(2) The dynamical theory of planetary rotation, given in § 120, which shows that a small body like the Earth could not acquire a motion of rotation sufficiently rapid to detach the Moon.

(3) The inconsistencies found in the theory that the Moon was detached (when our satellite is traced back towards its assumed original connection with the Earth), some of which were pointed out by DARWIN thirty years ago, while others have been brought to light by MOULTON's recent researches.

The whole evidence might, perhaps, be divided in a different manner, but as these divisions are essentially independent of each other, they form a suitable basis for calculation. Divisions (1) and (2) are natural, without regard to any hypothesis; while (3) results from the consideration of the LAPLACE-DARWIN hypothesis that the equilibrium of the primitive Earth broke down and the Moon was detached in the form of a ring, flock of meteorites, or in some other condition.

Let the concluded probability that the Moon was detached be denoted by P', then $P = 1 - P'$ will be the probability that our satellite was captured. But the compound probability P' is made up of three independent factors, namely,

$$p_1 = \frac{1}{1+i}, \quad p_2 = \frac{1}{1+j}, \quad p_3 = \frac{1}{1+k}. \tag{303}$$

These separate probabilities, $p_1, p_2, p_3,$ are based on the evidence supplied by the three groups of phenomena considered above, and their importance must be determined from the values assigned $i, j, k,$ by the judgment of the investigator. It thus follows that the total probability that the Moon was captured, as inferred from the compound probability resulting from the three groups of phenomena considered simultaneously, becomes

$$P = 1 - p_1 \cdot p_2 \cdot p_3 = 1 - \frac{1}{1+i} \cdot \frac{1}{1+j} \cdot \frac{1}{1+k} = \frac{(1+i)(1+j)(1+k) - 1}{(1+i)(1+j)(1+k)}. \tag{304}$$

The form of this expression is such that if $i, j,$ and k become very large and tend to approach infinity, P approaches unity with a degree of rapidity of the third order. For, with the increase of $i, j, k,$ the product of the independent probabilities

$$P' = p_1 \cdot p_2 \cdot p_3 = \frac{1}{(1+i)} \cdot \frac{1}{(1+j)} \cdot \frac{1}{(1+k)}, \tag{305}$$

becomes rapidly less than any assignable quantity, however small. The investigator would have to determine the limits within which this expression (305) should be taken; and there would be the most ample range for choice, the

total sum of the terms in the product involving the three variable parameters being given by the triple summation,

$$\pi = \sum_{i=0}^{i=i} \sum_{j=0}^{j=j} \sum_{k=0}^{k=k} (1 + i)(1 + j)(1 + k) . \tag{306}$$

If we agreed, as most mathematicians probably would do, that the upper limits should in each case be infinity, owing to the overwhelming strength of the evidence that the Moon is captured, the number of terms made by the product of all three elements connected with variable parameters would be equal to the number of points in space, namely:

$$\pi = \sum_{i=0}^{i=\infty} \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=\infty} (1 + i)(1 + j)(1 + k) = \infty^3 . \tag{307}$$

But, of course, in any one evaluation of probability only the extreme terms can be used. Under these circumstances the concluded probability that the Moon was captured, as given by (304), becomes

$$P = 1 - P' = \frac{(1 + i)(1 + j)(1 + k) - 1}{(1 + i)(1 + j)(1 + k)} = 1 , \tag{308}$$

because P' is reduced to zero with a degree of rapidity of the third order.

If this line of reasoning be admissible it is, therefore, an absolute certainty that the Moon is a captured planet which formerly moved in an independent elliptic orbit about the Sun. The satellites of such planets as *Jupiter* and *Saturn* pursued similar orbits before they were captured by their respective primaries; so that all our satellites originally were planets.

We conclude, therefore, that the Moon is like the goddess ATHÉNE, in HOMER'S *Iliad* (I, 194–195), and came to Earth from celestial space, (*ἦλθε δ' Ἀθήνη οὐρανόθεν*).

When the moon first left the sun's control and began to revolve about the earth, it must have been at something like twice its present distance, and have presented to the terrestrial observer about one-fourth its present angular dimensions, and given one-fourth as much moonlight as we now receive. How strange such a spectacle must have seemed if there were eyes here to witness the coming of this large star from the depths of space! Evidently this heaven-sent messenger was destined not only to give light at night, and thus cheer up the spirits of mortals, but also to adorn the blue sky by day. And, owing to the notable illumination of the human mind, through the study of the mathematical theory of the motion of our satellite, the Moon becomes indeed an everlasting palladium

for our safety in scientific research. The astronomer might, therefore, not inappropriately exclaim with ÆSCHYLUS:

Ναρθηκοπλήρωτον δὲ θηρῶμαι πυρὸς
Πηγὴν κλοπαίαν, ἣ διδάσκαλος τέχνης
Πάσης βροτοῖς πέφηνε καὶ μέγας πόρος.

— *Prom. Vinct.*, 109–111.

I brought to earth the spark of heavenly fire,
Concealed at first, and small, but spreading soon
Among the sons of men, and burning on,
Teacher of art and use, and fount of power.

CHAPTER XII.

On the Determination of the Secular Acceleration of the Mean Motion of the Moon from the Observations of Ancient Eclipses and on an Indicated Secular Acceleration of the Mean Motion of the Sun.

§ 127. *On the Nature of the Secular Acceleration of the Moon's Mean Motion.*

The secular acceleration of the Moon's mean motion is of so much historical importance that it is necessary to pause for a time in order to give an account of the results reached by various investigators who have attempted to evaluate it from the study of ancient eclipses. We shall later consider the secular acceleration found in the mean motion of the Sun, and the physical causes on which both of these inequalities depend, and then resume the consideration of the problems of cosmical evolution.

We have already remarked that the fact of the secular acceleration in the mean motion of the Moon was first recognized by the celebrated Dr. Edmund Halley in 1693, from an examination of the Lunar Observations of Albategnius compared to those given by Ptolemy in the *Almagest* (*Phil. Trans.*, 1693, p. 193; 1695, p. 174), and more fully confirmed by Dunthorne in 1749. The cause of this inequality, so far as it depends on gravitation, was sought for in vain by Euler and Lagrange, but finally discovered by Laplace in 1787.

From the study of the eclipses of the *Almagest*, Costard and Dunthorne had fixed the amount of the secular acceleration at 10″ (*Phil. Trans.*, 1749, p. 162; p. 412), while Tobias Mayer found the value to be 6″.7 (Mayer's *Lunar Tables*, 1752). In the second edition of these tables, published in 1770, after the death of the author, but from manuscripts left by him, the value of the secular acceleration was raised to 9″. In the *Memoirs* of the Paris Academy of Sciences for 1757, Lalande found the value to be 9″.886, and therefore adopted 10″ as the value to be used in lunar tables.

The deduction of the secular acceleration is made as follows: Let n_0 be the mean motion, t the time, so that $n_0 t$ is the mean longitude; then if the mean motion is constant, the actual longitude at any epoch becomes

$$L = C + n_0 t + \varpi \quad ; \quad \varpi = \sum_{i=0}^{i=i} B_i \sin(\beta_i t_i + \beta_i') \ , \tag{309}$$

where ϖ denotes the sum of the periodic terms, B_i, β_i, β_i' certain constants. At three epochs, t_0, t_1, t_2, we have

$$L_0 = C + n_0 t + \varpi_0 \quad ; \quad L_1 = C + n_0 t_1 + \varpi_1 \quad ; \quad L_2 = C + n_0 t_2 + \varpi_2 \ . \tag{310}$$

These three equations give two independent values of n_0, namely,

$$n_0 = \frac{L_1 - \varpi_1 - (L_0 - \varpi_0)}{t_1 - t_0} \quad ; \quad n_0 = \frac{L_2 - \varpi_2 - (L_1 - \varpi_1)}{t_2 - t_1} \ . \tag{311}$$

Now it is found by observation over long intervals that the value of n_0 is not constant, but varies slowly with the centuries, being greater in modern than in ancient times, as was first noticed by HALLEY in 1693. Hence we have for any epoch

$$L = C + n_0 t + \sigma \left(\frac{t}{100} \right)^2 + \sum_{i=0}^{i=i} B_i \sin(\beta_i t_i + \beta_i') \ , \tag{312}$$

where t is the time reckoned in Julian years, σ the secular acceleration, and $T = \frac{t}{100}$, the time reckoned in Julian centuries, which is the notation frequently employed by astronomers. The terms in the expression for the mean longitude (312) depending on the secular acceleration, is therefore proportional to the square of the time expressed in Julian centuries; but for very long intervals this is not quite accurate and it is necessary to introduce terms depending on higher powers of t.

Although EULER and LAGRANGE had searched diligently into every known perturbation, they failed to find the cause of the Moon's secular acceleration; the former holding that it could not be due to gravity, but must be ascribed to the resistance of the ether; and the latter that as it was not caused by gravity, it was questionable whether the somewhat uncertain observations of the ancients could be accepted as establishing the fact, and it was better to reject the phenomenon entirely.

§ 128. LAPLACE *Explains the Larger Part of the Observed Secular Acceleration by the Theory of Gravity.*

LAPLACE was more sagacious in his search for the cause of this inequality than EULER and LAGRANGE had proved to be. He first tried to explain it by the hypothesis that a finite duration of time might be required for the propaga-

tion of gravitation between the Earth and Moon. In this attempt to overcome the difficulty he did not succeed, but subsequently while working on the theory of *Jupiter's* satellites he noticed that a sensible term would be introduced into the mean longitude by a change in the eccentricity of the Sun's orbit; and then it occurred to him that in the same manner a decrease in the eccentricity of the Earth's orbit would produce the observed secular acceleration of the Moon's mean motion. LAPLACE first calculated the amount of the secular acceleration to be 11″.135, but later reduced it to 10″.18, owing to changes in the masses of *Mars* and *Venus*. TISSERAND says that with the masses now used LAPLACE would have obtained 10″.66.

This value 10″.18 was used by LAPLACE in his Theory of the Moon, *Mécanique Céleste*, Tome III, and agreed so well with the values found from observations by DUNTHORNE (10″), MAYER (9″), and LALANDE (10″), that the value of 10″ was introduced by BÜRG and BURKHARDT into their Tables of the Moon. PLANA afterwards recomputed the theoretical acceleration by LAPLACE'S method and found the value $\sigma = 10''.58$, while DEMOISEAU by the same process obtained 10″.72, and thus confirmed LAPLACE'S calculation, as well as that of TISSERAND, with the value of 10″.66.

§ 129. *Researches on Ancient Eclipses by* BAILY, AIRY *and* HANSEN.

During the lifetime of LAPLACE, in 1811, the English astronomer BAILY began the examination of the historical records of eclipses, with a view of improving chronology, and at the same time affording checks to the theories of the Sun and Moon in remote ages; and this method of inquiry was afterwards much extended by AIRY (*Phil. Trans.*, 1853; and *Mem. Roy. Astron. Soc.*, Vol. XXVI, 1857). In his first *Memoir* of 1853 AIRY found from the ancient observations of solar eclipses, $\sigma = 10''.72$. Using the new tables of HANSEN in the *Memoir* of 1857 he raised the value to $\sigma = 12''.18$, and remarked that 13″ would give a more satisfactory representation of two of the eclipses. As noticed below, however, AIRY misidentified the eclipse of Larissa, a city of the Medes, mentioned by XENOPHON as captured on the occasion of an eclipse of the Sun; and this error of date vitiated his calculation of the observed secular acceleration of the Moon. HANSEN had meanwhile made a number of determinations of the secular acceleration, which gave $\sigma = 11''.93$ (*A.N.* 443, 1843); $\sigma = 11''.47$ (*A.N.* 597, 1847); $\sigma = 12''.18$ (*Tables de la Lune*, 1857); and finally $\sigma = 12''.56$ (*Darlegung, Band*, II, 1864).

These results of AIRY and HANSEN, two of the most eminent mathematical

astronomers of their time, therefore, concurred in pointing to a theoretical and observed secular acceleration in good agreement and appreciably larger than that given by LAPLACE (10″.18).

Recent investigations of ancient solar eclipses by NEWCOMB, COWELL and FOTHERINGHAM have shown that AIRY and HANSEN erred in assigning so large a value to the secular acceleration of the Moon's mean motion. This was due partly to errors in HANSEN's tables of the Moon, which both of these older investigators had used, and partly to the fact that AIRY adopted the wrong date for the eclipse of Larissa. The three solar eclipses on which he mainly relied were those of THALES (− 584, May 28); Larissa (taken to have occurred − 547, May 19); and AGATHOCLES (− 309, Aug. 14). The eclipse of AGATHOCLES accords with any admissible value of the secular acceleration; while that of THALES is shown to have occurred near sunset, and was therefore so terrifying that the Medes and Persians stopped the battle, as mentioned by HERODOTUS. It harmonizes with the value of the secular acceleration of about 8″. The eclipse of Larissa really occurred either − 609, Sept. 30, or − 602, May 18, this latter date being preferred by COWELL, and not − 547, May 19, as supposed by AIRY and HANSEN (cf. COWELL's paper in *Monthly Notices*, supplementary number, Vol. LXVI, No. 9, 1906, pp. 530-541).

§ 130. ADAMS's *Correction of* LAPLACE's *Method of Calculating the Theoretical Amount of the Secular Acceleration.*

An important error in the gravitational theory of the Moon's secular acceleration was discovered by ADAMS in 1853, and it invalidated the procedure of LAPLACE, PLANA, and DEMOISEAU, as well as the conclusions of HANSEN and AIRY. The defect arose from the integration of the differential equations involving the variable eccentricity of the Earth's orbit, ADAMS holding that the actual variable eccentricity must be used, and that the constant value could be used only in the first approximations. For the coefficient of the integral

$$\int (e_0'^2 - e'^2)\, n_0 dt \,,$$

where

$$e' = e_0' - \alpha t \,, \quad e_0 = 0.016771 \quad , \quad \alpha = +\,0.000\,000\,4245 \,,$$

ADAMS by his method found

$$\frac{3}{2} m^2 - \frac{3771}{64} m^4 \,,$$

while PLANA obtained

$$\frac{3}{2}m^2 - \frac{2187}{128}m^4 .$$

ADAMS thus found the method of LAPLACE essentially incomplete. A controversy arose, with HANSEN, PLANA and DE PONTCÉOULANT contesting the claims of ADAMS; but the correctness of his procedure was confirmed by LUBBOCK, CAYLEY, and DELAUNAY, by means of independent methods of their own. PLANA recognized the error of his procedure, while HANSEN long clung to his theoretical values of between 12″ and 13″, but finally admitted that he had erred in the same way as PLANA, and that ADAMS's result was correct.

§ 131. *The Numerical Results of* ADAMS *and* DELAUNAY.

The theoretical secular acceleration was reduced by ADAMS to 5″.78 (*Phil. Trans.*, 1853, p. 397). DELAUNAY by his independent method confirmed all of ADAMS's coefficients, and by including terms in m^8 to the total number of 42, found the theoretical value to be σ = 6″.11. The significance of these terms as found by DELAUNAY may be judged by the following table (*Comptes Rendus*, Tome XLVIII, p. 825):

Term in m^2 =	+10″.659	Term in m^6 =	−0″.711
m^3 =		m^7 =	−0″.247
m^4 =	− 2″.343	m^8 =	−0″.062
m^5 =	− 1″.582		

The sum of these terms is 5″.714, in close agreement with the value found by ADAMS. DELAUNAY's final value 6″.18 may thus be regarded as proving that the theoretical value of σ is only about half of the larger values found from the ancient solar eclipses by AIRY and HANSEN, through the choice of a wrong date for the important eclipse of Larissa, on which they had so implicitly relied.

§ 132. HANSEN's *Tables of the Moon and the Earlier Researches of* NEWCOMB.

The publication of HANSEN's tables of the Moon at the expense of the British Government, in 1857, was expected to yield accurate places of our satellite, but this expectation has not been fulfilled, and the subject is still in a chaotic state, though the recent work of NEWCOMB, BROWN and COWELL have done much to clear up the situation. HANSEN's tables rest on observations from 1750 to 1850, and with the knowledge then available seemed to be well gotten up, but yet required empirical corrections to make them accord with observations. The errors

in longitude resulting from these tables amount to only about 1″ or 2″ between 1850 and 1860; but they became 5″ in 1870, 10″ in 1880, and 18″ in 1889. As they failed to represent the motion of the Moon after the epoch, it was natural to inquire how they represented the observations before 1750.

In 1878 PROFESSOR NEWCOMB published his *Researches on the Motion of the Moon*, in which he examined anew the records of ancient eclipses. He found the tables of HANSEN to be in error during the eight centuries preceding the Christian era by about 18′, so that the predicted longitude of the Moon at that epoch is on the average too large by more than half of a lunar diameter. Such a set of tables cannot be considered at all satisfactory. Here are NEWCOMB's results:

Epoch	Error = Obs. — Calc.
−687	−11′ ± 4′
−381	−27′ ± 5′
−189	−20′ ± 3′
−134	−16′ ± 4′
Average correction =	− 18′

It is the great magnitude of these corrections at remote epochs which has rendered the correct identification of ancient eclipses difficult. The position of the centre of the belt of totality becomes uncertain by at least 300 miles; and with such a great uncertainty in the path of totality, the vagueness of the descriptions of the eclipses often leaves us in doubt as to which date is to be adopted. By seeking new formulae that would establish harmonious relations among a number of ancient eclipses COWELL has reduced the uncertainty in the path of totality to about one-sixth part of its former value, the present uncertainty being about 50 miles. This is a very great improvement, and has aided in clearing up the identification of various eclipses.

§ 133. NEWCOMB'S *Latest Researches on the Unexplained Fluctuations in the Moon's Mean Motion.*

In the *Monthly Notices* of the Royal Astronomical Society, for January, 1909, the late PROFESSOR NEWCOMB has given a resumé of his recent researches on certain irregular movements of the Moon which he designates as *Fluctuations*, rather than *Inequalities*, because he believed they are irregular and not capable of being traced to any known cause. This paper is of great importance, because the results may be considered to embody the last word on the subject by this great mathematician who had devoted forty years to the subject. He remarks

that he has revised the results given in his *Researches on the Motion of the Moon*, published in 1878, and altogether covered a period of 2,600 years. The observations embrace:

1. The eclipses of the Moon found in PTOLEMY's *Almagest*, observed between 720 B. C. and 134 A. D.

2. Observations of eclipses by Arabian astronomers, extending from 829 to 1004.

3. Observations of eclipses of the Sun and of occultations of stars by the Moon made by GASSENDI, HEVELIUS, and others from 1620 to 1680.

4. Observations of occultations of stars from 1670 until the present time.

NEWCOMB remarks that after 1680 the observations are of a fair degree of precision, but there are frequent gaps during the last half of the 18th century. Since 1820 the observations are fairly continuous. This paper is so important that it is necessary to include almost the whole of it, in order to convey a correct impression of the present state of the subject:

"Taken in connection with the recent exhaustive researches of BROWN, which seem to be complete in determining with precision the action of every known mass of matter upon the Moon, the present study seems to prove beyond serious doubt the actuality of the large unexplained fluctuations in the Moon's mean motion to which I have called attention at various times during the past forty years. In the *Monthly Notices* for March, 1903, is found a comprehensive review of the whole problem so far as it had then been developed, so that I need not enter into details at present. Indeed, the general conclusions reached in the work of 1878 have only been slightly modified in the present one.

"The feature of most interest is the great fluctuation with a period of between 250 and 300 years. I call this a fluctuation rather than an inequality because, in the absence of any physical cause for its continuance, there is no reason to suppose that it will continue in the future in accordance with the law followed in the past. In the former paper I found it convenient to represent it as of the same period with the great Hansenian inequality due to the action of *Venus*. Singularly enough, the present research shows that the period which best represents it is still the same as that of the Hansenian inequality. In the former work I showed that the observed fluctuation could be represented by a mere change of sign of the constant term in the argument of this inequality.

"Putting

$$A = 18V - 16E - g,$$

HANSEN's value of this term is

$$15''.34 \sin (A + 30^\circ.2).$$

"The empirical term found by the writer in 1877 to best represent the observations was

$$-15''.5 \cos A.$$

"The sum of these gives a term having practically the same coefficient with an argument differing little from $A - 30°$. A natural suspicion would have been that an error of sign had crept into the theory. But this is disproved by the fact that the constant in question is a simple function of the longitude of the node of *Venus*, the relation of which to the inequality, in theory at least, admits of no doubt.

"The unaccounted-for fluctuation as now found is best represented by the term

$$12''.95 \sin [1°.31 (t - 1800) + 100°.6].$$

"The argument of the Hansenian term is

$$A = 1°.32 (t - 1800) + 183°.9.$$

"Practically, the annual motion 1°.32 will represent observations as well as 1°.31. We may therefore write the empirical term thus —

$$\Delta\lambda = 12''.95 \sin (A - 83°.3).$$

"The former empirical term was, as above,

$$15''.5 \sin (A - 90°).$$

"The following table shows the residual differences between the result of observations of the Moon since 1620 and pure gravitational theory. In deriving the elements of mean motion, it was necessary to divide the residual excess into two parts, one the great fluctuation just described, the other the smaller fluctuations which were superimposed upon it. In the table, the second column gives the minor fluctuations, which are in fact the residuals of the conditional equations. The third column shows the main fluctuation as computed from the expression given above. The sum of the two found in the fourth column is the total excess of the Moon's observed longitude over the result of gravitational theory. It is, however, to be remarked that in this theory is included the excess of the observed over the theoretical secular acceleration.

"The unit of weight, as the latter is given in the last column, corresponds to a probable error of about $\pm 0''.9$, and a mean error of about $\pm 1''.3$. Being in many cases partly a matter of judgment, round numbers are preferred where it is doubtful. The limiting value assigned is 60, it being judged that the actual probable error can never be below $\pm 0''.12$.

Mean Date	Minor Res.	Great Fluctuation	Total Fluctuation	Weight	Mean Date	Minor Res.	Great Fluctuation	Total Fluctuation	Weight
	″	″	″			″	″	″	
1621	+39	− 9.6	+29	.005	1866.5	+2.8	− 1.6	+ 1.2	6
1635	+13	−11.7	+ 1	.02	1867.5	+1.1	− 1.9	− 0.8	5
1639	−13	−12.1	−25	.04	1868.5	+1.0	− 2.2	− 1.2	10
1645	+ 5	−12.6	− 8	.03	1869.5	+1.6	− 2.5	− 0.9	9
1653	− 4	−12.9	−17	.03	1870.5	+0.7	− 2.8	− 2.1	6
1662	0	−12.7	−13	.06	1871.5	−1.3	− 3.1	− 4.4	10
1681	− 0.4	−10.5	−10.9	2.	1872.5	−1.8	− 3.3	− 5.1	16
1710	+ 0.5	− 3.7	− 3.2	6	1873.5	−1.8	− 3.6	− 5.4	12
1727	+ 0.1	+ 1.2	+ 1.3	3	1874.5	−2.2	− 3.9	− 6.1	8
1737	− 0.2	+ 4.1	+ 3.9	6	1875.5	−2.3	− 4.2	− 6.5	8
1747	0.0	+ 6.8	+ 6.8	5	1876.5	−2.1	− 4.4	− 6.5	30
1755	− 0.3	+ 8.7	+ 8.4	2	1878.5	−1.8	− 4.8	− 6.6	18
1771	+ 1.4	+11.2	+12.6	9	1879.5	−0.5	− 5.2	− 5.7	14
1784.7	+ 1.6	+12.7	+14.3	5	1880.5	−1.4	− 5.5	− 6.9	20
1792	+ 0.3	+12.9	+13.2	10	1881.5	−1.6	− 5.7	− 7.3	12
1801.5	− 0.6	+12.7	+12.1	12	1882.5	−1.4	− 6.0	− 7.4	8
1809.5	− 0.1	+11.9	+11.8	16	1883.5	−2.2	− 6.2	− 8.4	7
1813	− 1.2	+11.5	+10.3	16	1884.5	−2.1	− 6.5	− 8.6	30
1821	+ 0.1	+10.3	+10.4	14	1885.5	−2.5	− 6.7	− 9.2	50
1822.5	+ 0.6	+10.0	+10.6	10	1886.5	−2.8	− 7.0	− 9.8	18
1829.5	− 1.6	+ 8.6	+ 7.0	20	1887.5	−2.6	− 7.2	− 9.8	20
1833.5	− 1.6	+ 7.7	+ 6.1	10	1888.5	−3.5	− 7.5	−11.0	8
1838.5	− 0.6	+ 6.4	+ 5.8	30	1889.5	−3.5	− 7.7	−11.2	7
1843	+ 1.1	+ 5.2	+ 6.3	20	1890.5	−3.4	− 8.0	−11.4	10
1846.5	+ 1.1	+ 4.2	+ 5.3	10	1891.5	−3.1	− 8.2	−11.3	15
1848.5	+ 1.9	+ 3.7	+ 5.6	8	1892.5	−2.6	− 8.4	−11.0	17
1849.5	+ 0.1	+ 3.4	+ 3.5	15	1893.5	−2.7	− 8.6	−11.3	15
1850.5	+ 1.0	+ 3.2	+ 4.2	18	1894.5	−3.0	− 8.8	−11.8	30
1851.5	+ 0.8	+ 2.9	+ 3.7	12	1895.5	−2,2	− 9.0	−11.2	60
1852.5	+ 0.9	+ 2.6	+ 3.5	8	1896.5	−1.2	− 9.2	−10.4	60
1853.5	+ 0.7	+ 2.3	+ 3.0	7	1897.5	−2.0	− 9.5	−11.5	20
1854.5	+ 1.4	+ 2.0	+ 3.4	14	1898.5	−1.5	− 9.7	−11.2	28
1855.5	+ 2.1	+ 1.7	+ 3.8	6	1899.5	−1.4	− 9.9	−11.3	12
1856.5	+ 1.9	+ 1.4	+ 3.3	6	1900.5	0.0	−10.1	−10.1	15
1857.5	+ 2.1	+ 1.1	+ 3.2	9	1901.5	−0.1	−10.3	−10.4	16
1858.5	+ 3.5	+ 0.8	+ 4.3	8	1902.5	+0.3	−10.5	−10.2	18
1859.5	+ 3.9	+ 0.5	+ 4.4	6	1903.5	+0.6	−10.6	−10.0	12
1860.5	+ 3.9	+ 0.2	+ 4.1	12	1904.5	+1.1	−10.8	− 9.7	20
1861.5	+ 3.3	− 0.1	+ 3.2	5	1905.5	+1.5	−11.0	− 9.5	20
1862.5	+ 3.9	− 0.4	+ 3.5	7	1906.6	+1.3	−11.1	− 9.8	16
1863.5	+ 3.0	− 0.7	+ 2.3	6	1907.5	+2.0	−11.3	− 9.3	15
1864.5	+ 2.9	− 1.0	+ 1.9	10	1908.2	+2.1	−11.4	− 9.3	9
1865.5	+ 2.6	− 1.3	+ 1.3	5					

"The observed secular acceleration is now found to be less by 0″.37 than that which I derived in 1876. As for the theoretical value, I have added 0″.27 to the value found by Brown and myself, on account of the effect due to the combination of the Earth's oblateness with the secular diminution of the obliquity of the ecliptic. This carries the theoretical acceleration up to 6″.08. The value now found from all observations is

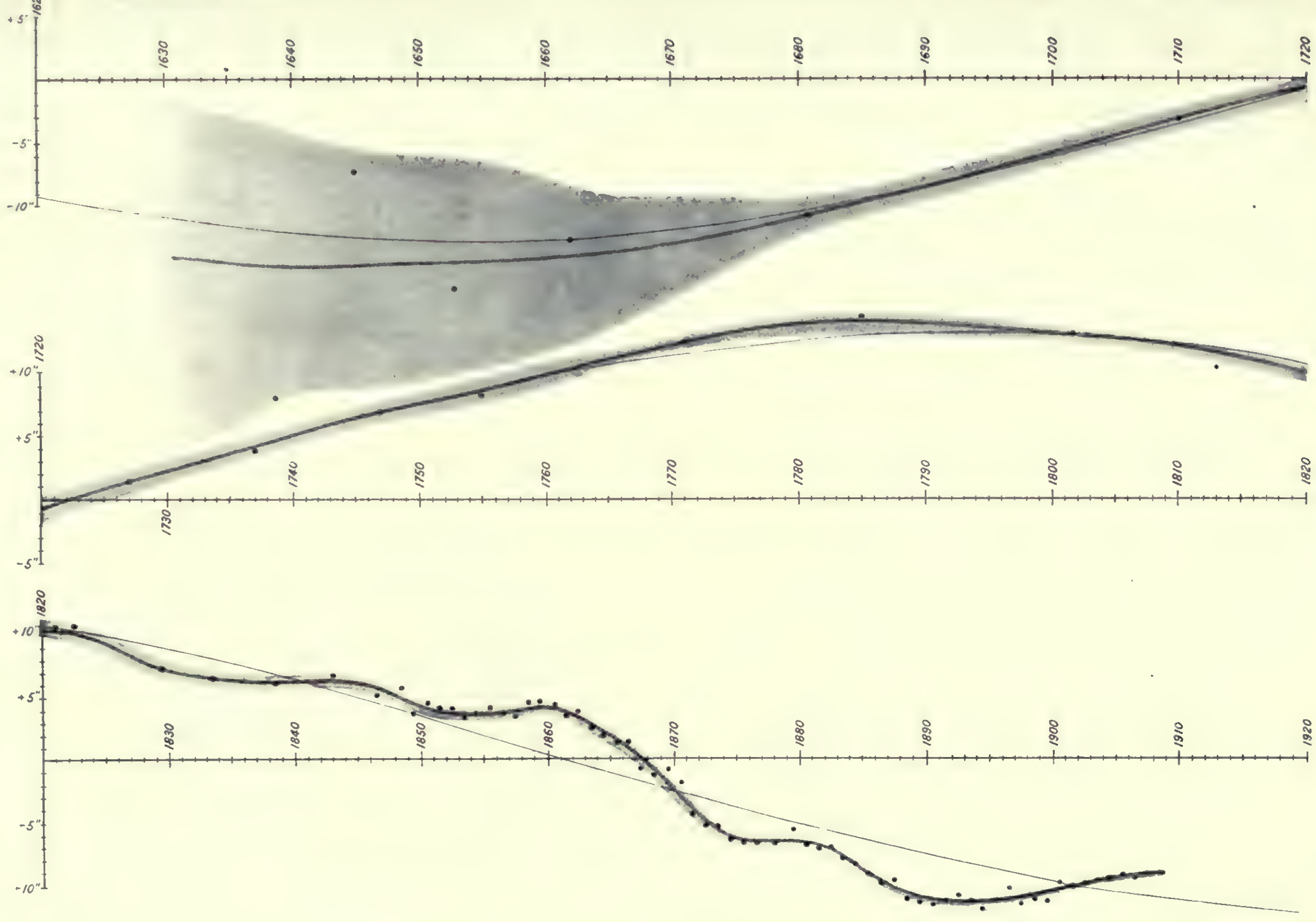
Sec. acc. from mean equinox	9″.07
Sidereal value	7″.96
Tidal excess	1″.88

"The accompanying plate (Plate II) gives a graphical representation in three sections of the residual deviations from pure theory, the motion derived from gravitational theory being represented by the straight medial lines. In order to show clearly the two parts into which the total fluctuation is divided the term of the great fluctuation is represented by a fine, sharp curve. The curve of actual longitude is bounded on each side by a shaded area showing the mean error at each point, which is nearly $\frac{3}{2}$ of the probable error. In this way not only the fluctuations as shown by observation are exhibited, but also the error to which the curve may be subject, the probability being $\frac{2}{3}$ that at any point the true curve lies inside the shaded area, and $\frac{1}{3}$ that it lies without it.

"We see by the curve as well as by the numbers, that before 1750 the observations are not sufficiently continuous, numerous, and accurate to show any fluctuation with certainty. The first minor fluctuation fairly well shown began about 1760. During the years 1765-1784 the Moon ran ahead by about 1″. Then the excess of motion ceased, and became temporarily reversed. Since 1820 the motion has been marked by rapid fluctuations, which can be so well traced on the plate that no description is necessary.

"Since what is actually observed is neither the acceleration nor the speed of motion, but changes of the longitude itself, of which these quantities are respectively the second and the first derivatives as to the time, it is not possible from the observations to make any approach to an accurate estimate of the accelerating or retarding forces. The most that we can say is that these varying forces are sufficient to bring about a change of annual motion amounting to between 0″.5 and 1″.0 by acting during a period of perhaps from 4 to 6 years.

"It must be remarked that the separation of the entire deviation into two parts, one the great fluctuation of long period, the other minor fluctuations superimposed upon the great one, is made merely for convenience in representing past observations, and does not imply a corresponding duplicity in the causes at play. We know that these causes have acted in a certain way in the past 250 years. but we cannot infer with confidence that they will act in the same way in the future. In other words, we cannot confidently predict a repetition of the great fluctuation through the next 250 or 300 years. Were there no minor fluctuations whatever, the belief that the great fluctuation was permanent might have some foundation, our conclusion then being that some natural cause was in action having the period in question. But, in the actual state of things, we have no



West, Newman lith.

UNEXPLAINED FLUCTUATIONS IN THE MOON'S MEAN LONGITUDE: 1630–1908.

PLATE VII. NEWCOMB'S INVESTIGATION OF THE FLUCTUATIONS IN THE MOON'S MEAN LONGITUDE, 1908.

reason to believe that the close correspondence between the observed motion and the great harmonic fluctuation is more than accidental. The fact is that the variations of accelerating force necessary to produce the minor fluctuations are much greater than the forces necessary to produce the great one, the measure of this force being, not the actual deviation, but the degree of curvature at each point of the line representing the path.

"The minor deviations during the past 100 years may be empirically represented by a trigonometric series, the principal term of which would have a period of 60 years, more or less, and an amplitude of perhaps 3″. But we have no reason to believe that, how accurate soever the representation may be by such a series, it will represent the future course of the Moon.

"It would be of interest to compare the present curve from occultations with the deviations derived by Mr. Cowell from the Greenwich meridian observations, which he has represented by a trigonometric series. But I deem it desirable that this interesting comparison, and the conclusions to be drawn from it, should be the work of someone else.

"I regard these fluctuations as the most enigmatical phenomenon presented by the celestial motions, being so difficult to account for by the action of any known causes, that we cannot but suspect them to arise from some action in nature hitherto unknown. A brief resumé of possible causes, and the difficulties in accepting them, may be attempted.

"Taking it for granted that the gravitation of all known bodies has been allowed for in the comparison, and that no unknown bodies exist, the first explanation to occur to us is that the inequalities are only apparent, being perhaps due to fluctuations in the Earth's speed of rotation, and therefore in our measure of time.

"I suggested this explanation in my earlier papers on the subject. It is open to the objection that it seems scarcely possible to account for changes so large as would be required through the action of known causes. But the explanation admits of an independent test from observation. If the fault is with our measure of time, it can be detected by the transits of *Mercury* and by the eclipses of the first satellite of *Jupiter*. As to the first, the discussions of the transits which I have already published, extending up to 1894, seem to preclude the possibility of any such changes in the measure of time as would account for the phenomena. The recent transit of 1907 November 14, which I have worked up in a preliminary way, seems conclusive on this point, since it would show our measure of time to be about 7 seconds slow, whereas to account for the observations of the Moon it should be more than this amount fast. I am now engaged in the

working up of observations of the first satellite of *Jupiter*, which may throw additional light on the subject,

"A tidal friction varying with ocean currents, ice, and meteorological conditions may suggest itself. To this there is a double objection. Accepting as complete the received theory, the only effect of tidal friction would be through a tidal couple acting between the Earth and Moon. Granting the completeness of the theory, this couple could only result in the doing of work upon the rotating Earth by the Moon, and never in the Earth doing work upon the Moon, because in this case the friction would have to be a negative quantity. But apart from this, the effect of any tidal couple would be to produce wider fluctuations in the Earth's rotation, and therefore in our measure of time, than those which would by themselves account for the fluctuations in the Moon's apparent motion. At the same time it is worthy of remark that the current theory of tidal friction, and the corresponding couple, is incomplete, in that it takes no account of the tide-producing action of the Sun, which it seems quite natural to consider as incapable of modifying the lunar couple. But this should be investigated, not assumed.

"The preceding suggestions seem to me to include every known cause of action. If we pass to unknown causes and inquire what is the simplest sort of action that would explain all the phenomena, the answer would be — a fluctuation in the attraction between the Earth and Moon. Accepting the law of the conservation of energy, such a fluctuation would involve an expenditure or absorption of energy somewhere in the solar system, which it seems difficult to admit. Precisely what changes of gravitation would be required I have not yet computed; but it seems quite likely that they would be below any that could be determined by experimental methods on the Earth. It would be natural to associate them with the Sun's varying magnetic activity and the varying magnetism of the Earth; but I cannot find that we have any data on this subject which would enable us to base any law upon varying magnetic action. At present I see nothing more to do than to invite the attention of investigators to this most curious subject.

"One general result of the present state of things is that we cannot draw any precise conclusions from a discussion of the Moon's motion in longitude, how refined soever we may make it. For example, it is impossible to derive from observation the accurate coefficient of the 18.6-year nodal inequality in longitude, owing to the varying fluctuation.

"It is also not possible to predict the future motion of the Moon with precision. If we require our ephemerides of the Moon's longitude to be as exact as possible, we must correct the tabular mean longitude from time to time by observations."

§ 134. *Remarks on* NEWCOMB'S *Latest Researches.*

Probably no more important contribution to the Lunar Theory than this has ever been made, because after 40 years' work PROFESSOR NEWCOMB definitely recognizes the existence of fluctuations not traceable to gravitational causes, and abandons hope of predicting the motion of the Moon except by supplementing mathematical theory by empirical corrections derived from observations. These fluctuations at the maximum exceed 13″, by which in 250 years the Moon oscillates from its mean position, having been behind prior to 1630, but gaining till 1722, then running ahead of the mean place till about 1785, when the displacement has a maximum value of about 14″, then again falling behind and reaching the mean position in 1867.5. Since this time the retardation has continued and has been rapid, for in 1893-4 it amounted to about 11″, but it has decreased a little during the past few years.

From this study of the observed motion it seems very clear that there are forces other than gravitational attraction acting on the Moon which produce very sensible fluctuations in its place. PROFESSOR NEWCOMB'S discussion of these motions is characteristically able and searching; but there is one cause which he has not taken account of, namely, resistance due to streams of cosmical dust, which is of variable intensity, owing to the non-periodic character of the forces at work. It cannot be assumed that the meteoric showers which are observed from the Earth are all that are encountered by the Moon; for the Moon is distant about sixty terrestrial radii, and besides departs very sensibly from the ecliptic. To recall the amount of this we need only remark that the Moon's parallax is 57′ 2″.55, or nearly a degree, and the inclination of the Moon's orbit to the ecliptic 5° 8′ 43″.35, so that it is above and below the mean path of the Earth by at least five times the Earth's radius either way, or moves through a total range of more than five times the Earth's diameter. It may therefore encounter very much more resistance than the Earth, owing to the wide range of its motion; and as the nodes are constantly shifting, the effects could not be uniform, even if the meteoric showers were regular, which they are not. We shall hereafter discuss the number of the meteorites swept up by the Earth and the effects which could thus arise. Without elaborate argument it is fairly clear that if this cause is not the true source of the fluctuations brought out by PROFESSOR NEWCOMB it will certainly produce effects which are sensible and thus prove to be one of the causes at work. Such a cause must be held to be much more probable than the other known causes which he has discussed and rejected as inadequate. If cosmical resistance of variable intensity is at work, acting

sometimes in one direction and then in another, owing to the various situations of the Moon with respect to the Earth and Sun, it may not be necessary to suppose any variation in the force of gravity. This is an hypothesis which should be adopted only as a last resort, since the resisting medium is a real cause which has exercised a great influence in the past and still known to be in operation.

§ 135. TISSERAND'S *Researches on the Secular Acceleration of the Moon's Mean Motion.*

TISSERAND has rediscussed the Arabian eclipses (*Comptes Rendus*, Tome CXIII, p. 669; *Annuaire du Bureau des Longitudes*, Paris, 1892; *Mécanique Céleste*, Vol. III, p. 419), and derived a value of 7″.1. He remarks that the departure from NEWCOMB'S value of 8″.8 found in 1878, is the outcome of a different method of calculation. He also points out that M. NEISON (*Monthly Notices*, 1878, Vol. XXXIX, p. 73) has concluded that the lunar eclipse given by PTOLEMY for − 687 is discordant, and should have smaller weight than NEWCOMB gave it; and that this would reduce the secular acceleration down to about 6″.3 or very nearly the theoretical value.

TISSERAND concludes his discussion as follows: "Thus we see that it is possible to represent the eclipses of PTOLEMY and those of the Arabians by the theoretical acceleration; we should then have no need to invoke tidal friction to produce a progressive retardation of the rotation of the Earth, giving the appearance of a secular acceleration of the Moon. Thus we should avoid the double inconvenience of disturbing the fundamental base in the measurement of time, and of introducing into the theory of the Moon an empirical member which cannot be determined by calculation, and would prevent the attainment of definitive results, even if all the other difficulties had been surmounted."

Assuming, however, the validity of the work of HANSEN and AIRY on the solar eclipses of Thales, Larissa, and Agathocles, which as treated half a century ago gave 12″, he thinks there is a grave objection to any mode of procedure by which the solar eclipses are not taken into account. As will be seen below these objections are now overcome, by the proof that AIRY took the wrong date for the solar eclipse of Larissa, and that the others harmonize with a value of about 8″ for the secular acceleration of the mean motion of the Moon. The values recently derived from the Lunar and Solar eclipses are therefore in good agreement.

§ 136. *Researches on the Eclipses Observed by the Arabians.*

In the time of LEMONNIER and EULER it was known that an Arabic manuscript containing observations by IBN JOUNIS existed at the University of Leyden, of which some extracts had appeared in the *Prolegomena* of TYCHO BRAHÉ. Towards the end of the eighteenth century it was entrusted to the Institute de France, at the request of the French Government, and in 1804 translated by PROFESSOR CAUSSIN, who held the chair of Arabic at the College de France, under the title "Le Livre de la Grande Table Hakemite." The greater part of the eclipses were published shortly afterward in Volume II of the *Mémoires de l'Institut.* They include twenty-eight eclipses of the Sun and Moon observed at Bagdad and at Cairo, between the years 829 and 1004.

What gives great importance to these eclipses of the Sun is the fact that at the moments of the first and second contact, the Arabians determined by observation the altitudes of the Sun or those of good stars, to a degree or half a degree. This gives much more accurate determinations of the time than in the eclipses recorded by PTOLEMY. And although the Arabian eclipses are only about half as far back as those observed by the Greeks, they have a final degree of accuracy almost as great. It is not known how these observations were made in all cases, but in some instances the Arabians observed the image of the Sun reflected from water. No doubt the instrument employed was some form of Astrolabe; for this instrument had been invented by the Greeks and described by PTOLEMY, and the Arabians always made great use of the *Almagest.* In 1877 NEWCOMB discussed these observations of the Arabians in relation to the Tables of HANSEN, with the following results, which are only slightly modified by his latest work already noticed in §133:

Epoch	Error = Obs. — Calc.	Number of Phases
850	$-3'.8 \pm 2'.4$	3
927	$-1'.6 \pm 1'.7$	7
986	$-4'.5 \pm 1'.3$	20

He describes his conclusions from the investigation of these eclipses, and of the Lunar Eclipses given in PTOLEMY's *Almagest,* as follows: "These historical accounts of total (solar) eclipses being uncertain, an attempt has been made to derive the Moon's acceleration from the eclipses of the Moon recorded by PTOLEMY in the *Almagest* and from the observations of the Arabian astronomers in the ninth and tenth centuries. These observations agree as fairly as could be expected in assigning to the Moon a secular acceleration of about $8''.4$, a little more than $2''$ greater than that computed from gravitation. On the other hand, if we accept the eclipse of Thales as total where the battle was fought, the total acceleration

will be about 12″, so that the two results are entirely incompatible. The question which is correct must be decided by future investigation; but the author believes the smaller value to be founded on the more trustworthy data" (*Popular Astronomy*, p. 100).

This statement regarding the eclipse of Thales now requires modification, for it is shown to be reconcilable with a secular acceleration of only about 8″. In a paper on the eclipse of Agathocles in the *Monthly Notices* for December, 1904, NEWCOMB concludes, on the basis of CELORIA's identification of it with an eclipse described by CLEOMEDES as being total at the Hellespont, but leaving one-fifth of the Sun's diameter still visible at Alexandria, that the secular acceleration formerly adopted by him (8″.4) might be diminished by about 1″.5. He refers to a paper by NEVILL in *Monthly Notices*, Vol. XXXIX, p. 72, where it is suggested that if the large weight assigned to one of PTOLEMY's eclipses be reduced, the eclipses of the *Almagest* as discussed in his *Researches* of 1877 would give a value smaller than was found by 1″ or perhaps 1″.5.

NEWCOMB concludes his discussion in the following prophetic language: "The important point is that this reduction will carry the observed acceleration down almost to the theoretical value, in which no allowance is made for tidal retardation. In other words, the conclusion to which the new evidence points is that the actual retardation of the Earth's rotation is almost evanescent. Although no numerical determination of the probable amount of retardation, as given by theory, has, so far as I know, ever been made, I think any estimate must make probable a value larger even than that corresponding to my former result. It therefore seems likely that neutralization of the effect of tidal friction is produced by some cause not yet fully investigated."

§ 137. *Results Recently Reached by* COWELL, FOTHERINGHAM, *and* NEWCOMB *from their Researches on Ancient Eclipses.*

(1) In the *Monthly Notices* for 1905, supplementary number, pp. 861-869, MR. COWELL discusses the solar eclipses of − 1062, − 762, − 647, − 430, and + 197, and concludes that the records made at Babylon and Nineveh, as well as those of ARCHILOCHUS at Thasos, THUCYDIDES at Athens, and TERTULLIAN at Utica are all quite safe. He uses a secular acceleration of 7″, taking account of the accelerations of longitude and node, and finds corrections − 0″.18 and − 0″.05 which are smaller than the probable errors. Thus he says these eclipses are well represented by a secular acceleration of 7″.

(2) In the *Monthly Notices* for November, 1905, MR. COWELL, taking into

account some suggestions of NEWCOMB, finds the sidereal acceleration of the Moon to be $+ 10''.9$ and of the Sun $+ 4''.1$. Then, on examining the observations of the transits of *Mercury*, he found:

Secular acceleration of *Mercury*	$0''.0$
Secular acceleration of Earth	$3''.0$

In spite of PROFESSOR NEWCOMB'S criticisms in the *Monthly Notices* for December, 1905, MR. COWELL maintains his ground that the Sun has a secular acceleration.

(3) In the *Monthly Notices* for May, 1906, MR. NEVILL has a lengthy and well-written review of the principal ancient eclipses, in which he shows that while COWELL'S work is a great improvement on that of HANSEN and AIRY, it is still not entirely conclusive. MR. NEVILL says (p. 418): "But it will be seen that this new hypothesis as to the origin of the correction required by the Moon's argument of latitude removes one theoretical difficulty only at the expense of restoring another; for though the assumed secular acceleration of the Moon's node is brought into accord with theory, it is only at the cost of reinstating the discordance between $6''.0$, the theoretical value of the secular acceleration in mean longitude, and that of $10''.9$, the value necessitated by MR. COWELL'S new hypothesis. It is true that such a discordance conceivably might arise from tides in the body of the Earth, regarded as a thin rigid shell covering a viscous fluid interior, or in an ocean entirely covering the surface of the Earth, though not from tides in smaller oceans having the physical configuration of those actually existing on the terrestrial surface; but there is no real evidence in support of a tidal retardation in the rotation of the Earth of a nature which could be held to account for a material discordance between the observed value of the secular acceleration of the Moon's mean longitude and that derived by theory from the diminution in the eccentricity of the Earth's orbit."

In the supplementary number of the *Monthly Notices* for 1906, pp. 523-542, MR. COWELL replies at length, and seems to justify his results. He shows that the important eclipse of Thales accords with his formulae: "Equations of condition are given for -584 and -556; for the former date my formulae make the central line pass very slightly to the south of Iconium at about $7^h\ 4^m$, or about three minutes before sunset. There is of course an uncertainty as to the exact position of the battlefield; but this is compensated by the fact that formulae which throw the central line much further south make it miss the mainland altogether.

"If an eclipse be total just before sunset, the parts of the Sun that are first uncovered by the Moon are roughly the same as those that first disappear below

the horizon. The phenomenon may thus be somewhat prolonged and to that extent more terrifying." He had already described his method of procedure:

"In exhibiting more eclipse calculations, I have had to decide whether to adopt a secular acceleration for the Sun in my formulae or a secular acceleration of the node contrary to gravitational theory. I have preferred the former course, but I wish it to be clearly understood that the ancient eclipses only afford evidence of the relative movements of the Sun, Moon, and node, and not of the position of the equinox" (p. 524).

TABLE OF THE LUNAR ECLIPSES OF THE ALMAGEST,

The Data being taken from COWELL'S Paper, *Monthly Notices*, 1906, Supplementary Number, Vol. LXVI, No. 9, p. 526.*

No.	Date	G.M.T. of Middle of Eclipse (Tabular)	Semi-Duration in Minutes	Observed — Tabular Time: Beginning	Middle	End	Tabular Magnitude	Observed Magnitude
		h m	m	m	m	m		
1	−720, Mar. 19	7 35.2	111.3	[−85]	.....	....	1.42	Total
2	−719, Mar. 8	9 52.4	15.9	......	+34	....	0.02	3 digits = 0.25
3	−719, Sept. 1	5 53.8	66.8	−49	.....	....	0.42	More than half
4	−620, Apr. 21	14 58.0	54.1	−23	.....	....	0.20	0.25
5	−522, July 16	9 25.5	86.4	......	+14	....	0.59	0.5
6	−501, Nov. 19	9 50.8	59.9	−29	.....	....	0.25	0.25
7	−490, Apr. 25	8 21.2	42.4	......	+ 6	....	0.14	2 digits = 0.17
8	−382, Dec. 22	17 36.0	55.1	?	.....	....	0.26	Small
9	−381, June 18	6 52.6	83.7	−21	.....	− 8	0.54	
10	−381, Dec. 12	8 35.3	103.4	+ 4	.....	....	1.54	Total
11	−200, Sept. 22	5 24.7	93.8	−28	.....	−34	0.77	Partial
12	−199, Mar. 19	11 26.3	104.4	−13	.....	....	1.30	Total
13	−199, Sept. 11	12 55.9	107.6	−30	−34	....	1.52	Total
14	−173, Apr. 30	12 6.4	73.7	− 5	.....	+11	0.53	7 digits = 0.58
15	−140, Jan. 27	8 22.6	42.4	+27	.....	....	0.16	3 digits = 0.25
16	+125, Apr. 5	7 9.4	46.1	+ 3	.....	....	0.17	0.17
17	+133, May 6	9 13.9	106.0	......	− 6	....	1.07	Total
18	+134, Oct. 20	9 14.6	93.8	......	−29	....	0.78	[0.33]
19	+136, Mar. 5	13 54.4	71.0	......	+10	....	0.48	0.50

Again: "As we cannot replace $4'' T^2$ in the formula by $-100T$, because such a correction is incompatible with modern observations, it follows that there is no geometrical alternative to correcting the secular terms until the errors of the present tables have been reduced from the order $4'' T^2$ to, say, the order $0''.4T^2$. When this degree of accuracy is reached small corrections may be applied, either to the secular terms or mean motions; in fact we get as many arbitrary quantities as equations of condition; and hence I consider that my formulae will only give eclipse tracks correctly to within fifty miles. I maintain that they are free

* In the *Monthly Notices* for November, 1906, p. 2, will be found a valuable table by MR. NEVILL, giving the errors of HANSEN'S Tables for each of these eclipses. The errors range from —13 to —83 minutes, with an average value of about —43 minutes of time. Of course no such errors as these could have been made by the Greek astronomers.

from errors of 200 to 300 miles, and that the present tables are not free from such errors." (p. 540) . . . "AIRY, however, only uses three eclipses. One of these three is the eclipse of Agathocles, where the uncertainty in the position of Agathocles causes the eclipse to satisfy nearly any formulae; another eclipse is that of Thales, which is well known to be satisfied by HANSEN'S tables; and the third eclipse is that of Larissa, to which AIRY assigns an impossible date. Inferences from eclipses, in view of the want of absolute precision in the records, depends upon the production of an overwhelming degree of coincidence. AIRY'S investigations, in reality, deduce a result from Thales, and show, as they were nearly certain to do, that the central line for Agathocles comes within one day's journey of Syracuse; and the whole degree of confirmation lies in the fact of the supposed agreement of the eclipse of Larissa, for which he has taken a wrong date. In no previous attempt to explain ancient eclipses has it ever happened that confirmation has been obtained from other eclipses. But in my case the eclipse at Babylon, the lunar eclipses, the eclipses of Thales and Larissa have all supported the explanation at which I had previously arrived from other evidence, viz., the eclipses of Nineveh, Archilochus, Thucydides, Agathocles, and Tertullian."

(4) In the *Monthly Notices* for November, 1906, p. 4-5, MR. NEVILL shows that the nineteen Lunar Eclipses of the *Almagest* cannot be reconciled with any value of the secular acceleration less than 7″.40. And he shows (p.10) that the Arabian eclipses can be represented by a secular acceleration of 6″.20, while those recorded by PTOLEMY require a value of at least 7″.40; both are not reconcilable to any one value, as 6″.80 fails to represent either. NEVILL also says that he had often found evidence of an unexplained apparent secular acceleration in the motion of the Moon's argument of latitude, but he doubted its reality; COWELL'S explanation, that it is due to a secular acceleration of the Earth, not having occurred to him.

In the *Monthly Notices* for December, 1907 (p. 110), COWELL shows that OPPOLZER'S results agree well with his, and says he thinks OPPOLZER obtained his formulae from lunar eclipses. He adds that because the mean motions seemed impossible, OPPOLZER'S formulae were long rejected; but as his numerical accuracy is beyond reproach, COWELL says he feels sure there is a solid foundation beneath the superficial blemishes of impossible mean motions. He concludes that the solar eclipses are consistent with a secular acceleration of the Sun of 2″.

In the *Monthly Notices* for Nov., 1908, p. 25, MR. FOTHERINGHAM shows that GINZEL'S elements of ancient eclipses represent modern observations, if the secular acceleration be taken to be 9″.7. FOTHERINGHAM is uncertain whether

there is a secular acceleration of the Sun, though so far as he has gone COWELL'S values give better results. In the *Monthly Notices* for March, 1909, COWELL states that his formulae makes the outstanding residuals in ancient eclipses only about ⅓ as large as the formulae of other investigators.

He concluded that it is not likely that the resultant correction of the secular acceleration of the Moon would exceed $\pm 0''.5$. In general, the eclipse of HIPPARCHUS confirms NEWCOMB'S values, but is somewhat less consistent with COWELL'S work, while HANSEN'S and GINZEL'S results are still more discordant. Using NEWCOMB'S values, this eclipse gives little indication of a secular acceleration of the Sun, but is consistent with a small value of $1''$ or less.

(5) In the *Monthly Notices* for January, 1909, MR. J. K. FOTHERINGHAM has identified the eclipse mentioned by CLEOMEDES as one observed by HIPPARCHUS in $-$ 128, Nov. 20, and not that of AGATHOCLES, as NEWCOMB and CELORIA believed. Undoubtedly FOTHERINGHAM is right. In working up the eclipse* he uses a secular acceleration of $8''.012$, communicated to him by NEWCOMB on the basis of his studies of all the eclipses between those mentioned by PTOLEMY and 1908. FOTHERINGHAM concludes that NEWCOMB'S value ($8''.012$) admits of but small corrections between $+ 0''.4$ and $- 0''.1$. "It would appear altogether that if we do not revise the centennial motions, the eclipse can only be satisfied by a secular acceleration of the Moon amounting to $8''.15 \pm 0''.55 + n''$, accompanied by a secular acceleration of the Sun amounting to $\frac{4}{7}n''$. Here n'' is any quality which may be added to harmonize the observations.

(6) In the *Monthly Notices* for March, 1909, NEWCOMB sums up the evidence to date. He begins thus: "The passages in the writings of ancient authors supposed to refer to total eclipses of the Sun have been so fully discussed during the last few years, especially by COWELL and NEVILL in the *Monthly Notices*, and quite recently by MR. FOTHERINGHAM, that the subject is fairly well thrashed out so far as the question of interpretation is concerned. Most of the supposed eclipses on which stress was laid by AIRY, HANSEN, and other older authorities, in testing the lunar tables, have been nearly eliminated from consideration by doubts and inconsistencies of various kinds. The only one of these I need mention is the eclipse of Thales. The questions associated with this eclipse may now be considered as well cleared up. From the corrections which I have applied to the lunar elements, it would appear that the Sun set upon the combatants only a short time, perhaps fifteen minutes, before the total phase commenced.

*In the *Monthly Notices* for November, 1908, p. 29, MR. FOTHERINGHAM gives a list of twenty-one solar eclipses not yet worked up, between -393, when the first eclipse was observed at Cheronea, and $+590$, when the last was observed at Constantinople. Some of these eclipses may yet give us much additional data on the secular acceleration, and until all of these are fully discussed our knowledge of the subject will remain somewhat incomplete.

Thus the accuracy of the phraseology used by HERODOTUS, and indeed the whole story as he narrates it, seemed to be confirmed in a remarkable way. But the eclipse still remains useless for any astronomical purpose."

He then explains his methods of calculation, and treats of the seven solar eclipses of value, giving among the data the following results:

		G.M.T.	Radius of Shadow	Ancient Authority
Eclipse of Babylon,	−1062,	July 30.7548	+ 94′	Babylonian inscription.*
Eclipse of Nineveh,	− 762,	June 14.8691	+151′	Assyrian cuneiform tablet. Inscription at Nineveh.†
Eclipse of Thasos,	− 647,	Apr. 5.8765	+178′	ARCHILOCHUS.
Eclipse of Athens,	− 430,	Aug. 3.1327	− 55′	THUCYDIDES.
Eclipse of Syracuse,	− 309,	Aug. 14.8505	+145′	AGATHOCLES (Diodorus Siculus).
Eclipse of Hellespont,	− 128,	Nov. 20.0565	+ 15′	HIPPARCHUS (Cleomedes).
Eclipse of Utica,	+ 197,	June 2.9958	− 6′	TERTULLIAN.

"The preceding eclipses, that of — 128 excepted, were discussed by MR. COWELL. He shows that five of the eclipses in question can all be represented in two ways, either by a change in the secular acceleration of the node, or by a hitherto unsuspected acceleration of the Sun's longitude, combined with an equal correction to the Moon's secular acceleration. It will be seen that the preceding equations are confirmatory of MR. COWELL's results." . . . . "It may also be remarked that an increase of 1″ in the secular acceleration of the mean longitude, coupled with a diminution of 1″ in that of the node, would suffice to make the eclipses of — 1062 and — 647 total, without throwing that of — 128 quite away from the Hellespont. But this change in $\Delta^2\Omega$ is outside the limit of theoretical uncertainty, and in $\Delta^2\lambda$ is difficult to admit." . . . . "Equally difficult of explanation on any theory is a secular acceleration of the Earth's orbital motion. Moreover, while modern observations, so far as I have discussed them, do not exclude the possibility of such an acceleration, they do render it improbable."

MR. FOTHERINGHAM adds a note, in which he reaches conclusions in close accord with those of NEWCOMB. "The resultant corrections to PROFESSOR NEWCOMB's secular terms are therefore quite small. The value obtained for $S_F — S_D$ suggests that there is in existence either a very small acceleration of the Sun, or some very small, hitherto unexplained, retardation of the node; but when we allow for the possibility of errors in the centennial motions, and for the slight disturbance in the results that might be occasioned by an error in the position of the perigee, it seems impossible to affirm such small corrections

* "On the 26th day of the month Sivan, in the 7th year, the day was turned into night, and fire in the midst of heaven."

† "In the month Sivan the Sun underwent an eclipse." (cf. Paper by COWELL, *Monthly Notices*, Vol. LXV, Supplementary Number, 1905, p. 861.)

on the basis of so few eclipses. It would appear, therefore, that these eclipses do not by themselves prove the existence of a secular acceleration of the Sun, though they are consistent with an acceleration of about 1″ a century" (*Monthly Notices*, March, 1909, p. 469).

§ 138. *Concluded Secular Accelerations of the Moon and Sun.*

It will be seen from the foregoing discussion that there is still a small element of uncertainty in the total amount of the secular acceleration of the Moon. But there is no essential difference of opinion, as to the facts, among those best qualified to judge; and the conclusion is that the secular acceleration of the Moon is in excess of the theoretical amount by about 2″; which is 25 per cent. of the whole observed secular acceleration, and this outstanding inequality cannot be due to errors of observation.

To appreciate this clearly, we need only remark that at any past epoch the outstanding inequality in the secular acceleration is given by the expression $2''T^2$, where T is the time which has elapsed expressed in Julian centuries. Thus at the epochs indicated we have the following results:

HIPPARCHUS, − 128 B.C., $2''\ (20.37)^2 = 830''$, corresponding to an error in the obs. of an eclipse of 28^m.

MARDOCEMPAD, −720 B.C., $2''\ (26.29)^2 = 1382''$, corresponding to an error in the obs. of an eclipse of 46^m.

Solar eclipse of Babylon, −1062 B.C., $2''\ (30)^2 = 1765''$, corresponding to an error in the obs. of an eclipse of 59^m.

The errors in the times of eclipses of the Moon would never exceed some ten minutes, even when all sources of error are considered. Consequently the outstanding inequality in the secular acceleration of the Moon is a well established fact, and no difference of opinion on this point exists among the recent investigators of the subject. This result is also consistent, as MR. FOTHERINGHAM points out, with a small secular acceleration of the Sun; for this latter is not contradicted by any modern observations. We therefore adopt a secular acceleration of the Moon of 8″.00, or about 2″ larger than DELAUNAY's last value (6″.18) of the part depending on gravitation as corrected by NEWCOMB (6″.08), in the *Monthly Notices* for January, 1909. And for the Sun, we adopt the value 0″.75 as the most probable amount of the secular acceleration. It seems to be fairly well established that a small secular acceleration of the Sun really exists. Its exact amount may remain uncertain for a good many years, but in view of all the evidence it is not likely to exceed 1″.25, nor fall short of 0″.25. The difficulty of explaining these discrepancies from gravitational theory are by no means

so great as has been heretofore imagined. An explanation is readily found in the resistance arising from cosmical dust in space, and in the slight increase in the mass of the Sun and other bodies of the solar system.

Both the Earth and Moon pass through streams of meteorites at certain times, and the number of meteors swept up by the Earth amount to at least 1,200,000,000 daily (cf. *A.N.*, 3618). In the *Astrophysical Journal* for June, 1909, p. 379, PROFESSOR W. H. PICKERING has estimated that the mass of these bodies is much larger than we have heretofore supposed. He gives grounds for holding the diameter of the average naked-eye meteor to be 6 or 7 inches. If this reasoning be admissible, we shall have to augment the calculated effects of meteoric resistance in the ratio of a grain, or one-fifteenth of a gramme, to seven kilograms. This increase in number and in mass multiplies the effects of resistance by at least 1,000,000. As we shall see in the next chapter, the late PROFESSOR THEODORE VON OPPOLZER found by calculation that the effects of meteoric resistance are adequate to account for the observed outstanding inequality in the lunar acceleration. In YOUNG'S *General Astronomy* (edition of 1904, Article 778, p. 475), however, it is stated that the effects are too small to account for this outstanding inequality. But if the mass of each meteor has been much underrated, and the total number of these masses correspondingly underestimated, it is highly probable that our present more exact data would show that meteoric matter is the cause of the outstanding inequality in the Moon's secular acceleration. This brings the Moon nearer to us by about a quarter of an inch per annum.

§139. *Examination of the Influences Which Might Tend to Change the Period of the Earth's Rotation.*

The principal causes which might change the time of the Earth's rotation are the following:

(1) Oceanic and Bodily Tidal Friction, tending to lengthen the day. At the present time the effect of this cause probably is very slight, but perhaps not wholly insensible.

(2) The secular expansion of the terrestrial globe, due to leakage of the oceans, which produces earthquakes, the uplift of mountains, plateaus, etc.; together with the expulsion of lava, volcanic dust, etc.; all tending to lengthen the day. The displacements thus arising are often very considerable, as at Yakutat Bay, Alaska, 1899 — where the coast was raised for 100 miles with a maximum elevation of of 47½ feet. Against this is set the secular shrinkage of the globe

due to cooling, which, however, is so minute as to be infinitesimal, the change in radius not exceeding 1.5 inches in 2000 years.

(3). The accumulation of meteoric dust, tending to lengthen the day by increasing the dimensions of the Earth; but here again the effect is slight, and more than counterbalanced by a considerable eastward whirl of the cosmical vortex about the Earth, and therefore tending to accelerate the Earth's rotation. This last cause probably is one of the most powerful of the various influences now at work, since the Earth's axial rotation, as well as that of the other planets, has arisen in that way. Though at one time much more powerful than at present, it still seems to have considerable importance, and may easily counteract all the other causes combined. So far as observations go, the indications are indeed that it nearly or quite nullifies the secular effects of tidal friction. The net result of all these forces is that the day remains of exceedingly constant length. Accordingly it would appear that the rotatory motion of a planet such as the Earth about its axis probably is by far the best of all the possible timekeepers which might be selected from among the heavenly bodies yet known to us. For it happens that the algebraic sum of all the changes now going on is nearly evanescent, and the length of the day therefore undergoes no sensible change. It seems almost certain that the change in the length of the day has not amounted to $\frac{1}{1000}$ of a second in 2000 years, for this would in that time accumulate to over 12m: it is not probable therefore that the period of the Earth's rotation has been altered from all causes combined by more than 0.001 of a second, since the days of HIPPARCHUS.

CHAPTER XIII.

Investigation of the Several Physical Causes Which May Contribute to the Production of the Observed Secular Acceleration in the Mean Motion of the Moon and of the Indicated Secular Acceleration in the Mean Motion of the Sun.

§ 140. Oppolzer's *Theory of the Outstanding Inequality in the Secular Acceleration of the Moon's Mean Motion.*

As we have seen in Chapter IX, the late Professor Theodore von Oppolzer, of Vienna, was among the earliest investigators of the relationship shown to exist between comets and meteoric showers. In this way he came to look upon the cosmical dust pervading the celestial spaces as a source of disturbance to the purely gravitational motions of the heavenly bodies. In the *Astronomische Nachrichten*, Nos. 2314 and 2319 (1880), he discusses the motions of certain comets, but more especially those of Encke and Winnecke, taking the resistance to be proportional to the surface, or as the square of the radius, and inversely as the mass; and therefore for two unequal spheres of the same density inversely as their radii.

For when the body is supposed to exert little or no attraction on neighboring particles and to be resisted in the simple proportion to the matter encountered in the cylindrical path traced out by a section perpendicular to the tangent to the instantaneous orbit, we have

$$F = K\frac{S}{m} = K\frac{4\pi r^2}{\frac{4}{3}\pi\sigma r^3} = K\frac{3}{\sigma r}\cdot \tag{313}$$

And for two spherical masses of the same mean density $F : F' = r' : r$, or inversely as the radii. Thus, if the radii be very unequal, as of a planet and a particle, the resistance suffered by the particle will be relatively very large, while that of the planet will be extremely slight. Under these circumstances the particle will rapidly drop nearer and nearer the Sun, while the mean distance of the planet will not be sensibly diminished.

In discussing the motions of comets OPPOLZER remarks that if electric forces are assumed to be at work, they must be supposed to be proportional to the surfaces on which the charges are distributed, whereas gravitation always is proportional to the mass or amount of solid matter in a body. The evidence derived from the motions of comets seemed to him to point to the existence of a resisting medium which is discontinuous in character, as if dependent on the action of streams of cosmical dust in space.

In the *Astronomische Nachrichten* No. 2573 (Feb. 14, 1884), OPPOLZER applies his theory to the Secular Acceleration of the Moon, and reaches results nearly identical with those independently obtained by the author, in June and July, 1909, before the latter became acquainted with the little-known results, published by the former a quarter of a century ago. OPPOLZER begins his paper by some general remarks, in the course of which he says that DELAUNAY invoked oceanic tidal friction to explain the difference between the Moon's theoretical secular acceleration of 6″ and the observed secular acceleration, which he thinks may be taken at 11″.

It is the outstanding difference of 5″ which OPPOLZER seeks to explain. He adds that if continents run from the north to the south pole, so as to interrupt the movement of the waters of the oceans, the circumstances will not be very favorable to DELAUNAY's hypothesis. And since this is very nearly true on the actual Earth, OPPOLZER remarks that DELAUNAY's hypothesis will hardly suffice, even if it be not altogether denied; and that a second cause may be suggested, among others which will not be discussed, to afford the desired explanation of the outstanding inequality in the Moon's secular acceleration.

There is no justifiable doubt, he says, that space is filled with finely divided cosmical dust, of which the larger constituents flying into our atmosphere are observed as shooting stars. If one admits the existence of such fine dust in space, its presence will, under very plausible assumptions, explain the actual acceleration of the Moon. He adds that it is obvious that all the phenomena of the motions of our solar system will be influenced by this cosmical material; but as it is only in the case of the Moon that the effects will become sensible, owing to the remoteness of the other bodies, he restricts his inquiry to the secular acceleration of the Moon.

He assumes that the particles have no elasticity of rebound, so that when they strike a heavenly body they remain united with it; and moreover that the direction of motion of the individual particles, on the average, when referred to the center of the Sun, is equal to zero; which, he says, holds true with great accuracy for the shooting stars. Under these conditions the cosmical dust will make itself noticeable in the motion of the Moon in three ways:

(1) The mass of the Earth and Moon increases through the addition of dust particles; and, notwithstanding the fact that the major axis of the Moon's orbital motion, in its mean value, does not appear to be altered, there arises, besides insensible periodic perturbations, a term in the Moon's mean longitude multiplied by t^2.

(2) A part of the material swept up by the Moon will diminish its tangential velocity, and thereby give rise to a term in the mean longitude of the Moon likewise multiplied by t^2.

(3) The material swept up by the Earth, which must be set rotating by our planet's motion about its axis, will change the rotational velocity, and thereby alter in a secular manner the assumed uniform measure of time; the principal influence here considered indeed is also friction; but if we transfer this variation in the measure of time to the motion of the Moon, the rotational velocity of the Earth being erroneously taken as constant, there will thus come to light in the mean longitude of the Moon a term multiplied by t^2.

Assuming that the mean density of the cosmical dust does not change with the time, OPPOLZER proceeds to evaluate the influence of the assigned cause.

§ 141. *Determination of the Secular Terms in the Moon's Mean Longitude Depending on the Increase in the Mass of the Earth and Moon, and on the Retardation of the Earth's Rotation Arising from the Deposit of a Non-rotating Layer of Cosmical Dust.*

If we designate by k^2 the attraction of unit mass at unit distance, in the unit of time, for which the Julian century will be taken; and by m and n the increase in the mass of the Earth and Moon in a century, and by r the distance of the Moon from the Earth; then the perturbation in the radius vector due to the increase in the mass becomes

$$R_0 = \frac{-k^2(m+n)}{r^2}t. \tag{314}$$

If we neglect, as here seems allowable, the eccentricity of the Moon's orbit, and the disturbing influence of the Sun and designate by a the mean distance of the Moon from the Earth, we get for the differential equation of the perturbation in the mean longitude

$$\frac{d\Delta L_1}{dt} = \frac{2k(m+n)}{a^{3/2}}t.$$

And the integral is

$$\Delta L_1 = \frac{k(m+n)}{a^{3/2}}t^2. \tag{315}$$

OPPOLZER remarks that there also arises an analogous term in the longitude of the perigee of the form

$$\Delta\pi = -\tfrac{1}{2}\Delta L_1 \tag{316}$$

If we denote the mass of the Earth by M and of the Moon by N, and choose as the unit of mass $M + N = 1$; then $\frac{k}{a^{3/2}}$ will represent the mean motion of the Moon in a century.

Postulating that the cosmical dust which falls upon the Earth in a century is uniformly distributed over the whole globe, we may consider it replaced by a layer of the mean density of the Earth, of thickness h. Before combining with the Earth this layer is conceived as quiescent, but thereby set in rotation, and the Earth's rotation therefore correspondingly retarded. If ω denote the magnitude of the Earth's rotation in a century, ϱ the radius of the Earth regarded as a sphere; then the retardation, when the globe is assumed to be homogeneous, becomes

$$-5\omega\frac{h}{\varrho}t\,;$$

the coefficient 5 being the assumed observed outstanding inequality in the Moon's secular acceleration. And the retardation of the meridian with respect to its undisturbed place will be

$$\Delta l = -\frac{5}{2}\frac{h}{\varrho}\omega t^2\,.$$

If, however, we take the time as progressing uniformly with the Earth's rotation, and transfer this correction of the measure of the time to the orbital motion of the Moon, multiplying the same by the relation of the angular velocity of the Moon's motion to that of the Earth's rotation, here designated by $f = \frac{1}{27.32166}$, we shall have

$$\Delta L_3 = \frac{5}{2}\frac{h}{\varrho}\omega f t^2 = \frac{5}{2}\frac{h}{\varrho}\frac{k}{a^{3/2}}t^2\,. \tag{317}$$

Let the mean distance of the Sun be denoted by R; then the volume swept over by the Earth in a century becomes

$$V = 2\pi R\,.\,100\,.\,\pi\varrho^2\,. \tag{318}$$

The mass m added to the Earth in sweeping through this volume of space in a Julian century is

$$m = 4\pi\varrho^2\,.\,h\,; \tag{319}$$

the density of the layer h being taken the same as the mean density of the Earth. Therefore the density of the resisting medium in space referred to the mean density of the Earth as the unit is

$$\delta = \frac{m}{V} = \frac{h}{50\pi R}. \tag{320}$$

If ϵ denote the radius of the Moon, the volume swept over by the Moon in a century becomes

$$V\frac{\epsilon^2}{\rho^2};$$

and

$$m + n = m\left(1 + \frac{\epsilon^2}{\rho^2}\right). \tag{321}$$

If we express m in units of the chosen unit of mass, $M + N = 1$, taking $80\,N = M$, we get

$$M + m = \tfrac{4}{3}\pi\,(\rho + h)^3 = \tfrac{4}{3}\pi\,\{\rho^3 + 3h\rho^2 + 3h^2\rho + h^3\},$$

of which the second term represents m, the terms in h^2 and h^3 being neglected as insensible; and therefore $\frac{m}{M} = \frac{3h}{\rho}$,

$$\text{or} \quad m = \frac{80}{81}\frac{3h}{\rho}, \quad \text{since} \quad M = \frac{80}{81}, \quad \text{and}$$

$$m + n = \frac{80}{27}\frac{h}{\rho}\left(1 + \frac{\epsilon^2}{\rho^2}\right). \tag{322}$$

Accordingly equation (315) may be written

$$\varDelta L_1 = \frac{80}{27}\frac{h}{\rho}\left(1 + \frac{\epsilon^2}{\rho^2}\right)\frac{k}{a^{3/2}}t^2, \tag{323}$$

which makes $\varDelta L_1$, and $\varDelta L_3$ dependent on the same factor h.

§ 142. *Determination of the Term in the Moon's Mean Longitude Depending on the Tangential Resistance Arising from the Cosmical Dust in the Volume Swept Over by the Moon in Its Orbital Motion About the Earth.*

It now remains to consider $\varDelta L_2$, which arises from the sweeping up of relatively quiescent particles through the orbital motion of the Moon about the Earth. Here we have to consider not the whole mass of dust falling upon the Moon, but only a fraction of this augmentation n, which we shall call ν; this latter represents the matter included in the volume swept over by the Moon in a century:

$$\nu = 2\pi a \,.\, \pi\epsilon^2 \,.\, f \,.\, 36525\,; \tag{324}$$

and therefore

$$\left.\begin{aligned}\nu = v\delta = 2\pi a \cdot \pi\epsilon^2 \cdot f \cdot 36525 \frac{h}{50\pi R} &= 4a\pi\epsilon^2 f \frac{365.25\, h}{R} \\ &= 1461\pi \frac{a}{R}\epsilon^2 fh\,.\end{aligned}\right\} \tag{325}$$

The relative loss of tangential velocity will be $\frac{\nu}{N}$, and if the density of the Moon be $\sigma' = \frac{\varrho^3}{80\epsilon^3}$,

we get

$$N = \frac{4}{3}\pi\left(\frac{1}{80}\frac{\varrho^3}{\epsilon^3}\right)\epsilon^3 = \frac{\pi\varrho^3}{60}, \tag{326}$$

and

$$\frac{\nu}{N} = 1461\frac{a}{R}\epsilon^2 \cdot f \cdot \frac{60\,h}{\varrho^3}. \tag{327}$$

With Oppolzer we may put $R = 400a$ and neglect the eccentricity of the Moon's orbit. The formula for the perturbation of the mean motion thus becomes simply

$$\frac{d\mu}{dt} = -\frac{3}{2}\frac{\mu}{a}\frac{da}{dt} = -\frac{3}{2}\frac{k}{a^{3/2}} \cdot \frac{1}{a} \cdot \frac{2a^2}{k\sqrt{a}}S_0 = -\frac{3}{a}S_0\,. \tag{328}$$

And our expression for S_0 itself is

$$\left.\begin{aligned}S_0 &= -1461 \cdot \frac{1}{400} \cdot \frac{\epsilon^2}{\varrho^3} \cdot f \cdot h \cdot 60 \cdot \frac{k}{a^{3/2}} \cdot at\,, \\ &= -\frac{4383}{20}\frac{\epsilon^2}{\varrho^3} \cdot f \cdot h \cdot \frac{k}{\sqrt{a}}t\,.\end{aligned}\right\} \tag{329}$$

Therefore for the differential equation of the term in mean longitude we have

$$\frac{d\varDelta L_2}{dt} = -\left(\frac{3}{a}S_0\right) = +\frac{3}{a}\left[\frac{4383}{20}\frac{\epsilon^2}{\varrho^3} \cdot \frac{k}{\sqrt{a}}f \cdot h\right]t\,; \tag{330}$$

and integration with respect to the time gives

$$\varDelta L_2 = \frac{13149}{40}\frac{\epsilon^2}{\varrho^3}\frac{k}{a^{3/2}}f \cdot h \cdot t^2\,. \tag{331}$$

§ 143. *Numerical Evaluation of the Three Terms.*

The equations (323), (331), and (317) give the desired solution as soon as we make a definite assumption with regard to h, since all the other coefficients are known. Oppolzer chose as the unit of h the millimeter, and used the following approximate values of the other constants:

$$
\begin{aligned}
&\text{Log. } \frac{k}{a^{3/2}} = 9.239 \text{ (unit of the second of arc)},\\
&\text{Log. } \omega = 10.675 \text{ (unit of the second of arc)},\\
&\text{Log. } \rho = 6.804 \text{ (unit of the meter)},\\
&\text{Log. } R = 11.477 \text{ (unit of the meter)},\\
&\text{Log. } \epsilon = 6.255 \text{ (unit of the meter)},\\
&\text{Log. } f = \log. \frac{k}{a^{3/2}} - \log. \omega = 8.563 - 10.
\end{aligned}
$$

With these values he found

$$
\begin{aligned}
\Delta L_1 &= +0''.87ht^2,\\
\Delta L_2 &= +0.26ht^2,\\
\Delta L_3 &= +0.68ht^2.\\
\hline
\Delta L\ &= +1.81ht^2.
\end{aligned}
\tag{332}
$$

Remarking that the observed lunar acceleration is about 11″, and the theoretical value 6″, leaving the residual $\Delta L = 5''$, he says this sum of the three terms will make $h = 2.8$mm.; which he considers not too large. Taking the density of the Earth to be 5.6 that of water, and water 800 times that of air, he adds that the density of planetary space is equivalent to

$$\frac{\text{Density of Air}}{3{,}760{,}000{,}000{,}000}.$$

Of the terms entering into equation (332), only one, namely ΔL_3, was derived on the supposition that the outstanding inequality in the secular acceleration is 5″. For an outstanding inequality of 2″, we should therefore multiply the coefficient 0″.68 by the factor $\frac{2}{5}$, and this gives 0″.27. Now if we put the sum $\Delta L = 2''$, as found by recent researches, and we shall get

$$
\begin{aligned}
\Delta L_1 &= +0''.87ht^2,\\
\Delta L_2 &= +0.26ht^2,\\
\Delta L_3 &= +0.27ht^2,\\
\hline
\Delta L\ &= +1.40ht^2;
\end{aligned}
\tag{333}
$$

which gives the value $h = 1.43$ mm., or almost exactly one-half the value found by Oppolzer twenty-five years ago. A meteoric fall of 1.43 mm. in a century certainly could not be recognized by any means at present known to Science. For Leconte (*Elements of Geology*, p. 11) estimates the average rate of erosion of the basin of the Mississippi to be 1 foot (304.8 mm.) in 5,000 years, or 1 mm. in 16.4 years, or 6 mm. in a century. The rainfalls of other river systems wear away their basins at a rate from two to four times faster than this.

We may take the average rate of erosion for the whole Earth to be 12 mm. in a century, or one foot in 2,500 years. If erosion is 12 mm. in a century, and the fall of cosmical dust goes on at the rate of 1.43 mm. per century; in other words, if erosion is 8.4 times as fast as the fall of cosmical dust, it is evident that as the cosmical dust washed away would always be combined with eight times as much terrestrial dust, we could not perceive it under the conditions existing in nature.

§ 144. *Criticisms of* OPPOLZER'S *Investigation by* BRAUN *and* KLEIBER, *with Answer to the Same by the Present Author.*

In the *Astronomische Nachrichten*, No. 2582, DR. C. BRAUN, of Kalocsa, Hungary, has published a criticism of OPPOLZER'S paper, the contents of which may be briefly noticed. BRAUN makes three points:

1. He admits the correctness of OPPOLZER'S analysis, as given above, but holds that the other sciences will not permit us to admit so large a downfall of meteoric dust. He says that PIAZZI SMYTH, in *Nature*, of January 30, 1884, estimates the fall of cosmical dust to be not much less than 100 tons daily, and not much over 1,000 tons. But if OPPOLZER is right, the downfall would amount to 218 million tons daily, or about a million times that calculated by competent investigators.

2. In a period of forty million years he says the Earth would be covered with a layer one kilometer deep. He considers that the bed of the ocean is but little disturbed, and ought to give evidence of this deposit of dust; but deep sea explorations disclose chiefly chalk, volcanic, and other dust, mostly of terrestrial origin.

3. Referring to the Krakatoa disaster, he says 5,000 million tons were blown into the air, and caused the afterglow, or red sunsets which were observed to continue for some months; and concludes that if 218 million tons fall daily, we should have a *permanent afterglow* like that following the Krakatoa eruption.

We may answer these objections as follows:

(1) PROFESSOR H. A. NEWTON estimated the downfall of meteors to number 15,000,000 daily; but in *A.N.* 3618, the writer has found, by observations taken at an altitude of 7,000 feet in the dry climate of Arizona, that the number of telescopic meteors is about 100 times larger than was calculated by NEWTON. The previous estimate of the average mass of a shooting star was one grain; but in the *Astrophysical Journal* for June, 1909, PROFESSOR W. H. PICKERING gives grounds for estimating the mass of the more conspicuous meteors as high as seven kilograms; this is about 105,000 times one grain, and the mass of such meteors

could therefore be multiplied by 100,000. And if 100 times as many meteors fall as NEWTON estimated, with an average mass of even one-tenth of that suggested by PICKERING, the total mass would be 1,000,000 times that previously adopted. The writer's estimate of the number of telescopic meteors is certainly not too large, but it may easily be too small. It seems therefore certain that the amount of meteoric dust falling upon the Earth has been greatly underrated.

(2) BRAUN'S argument regarding the deposits in the deep sea is not valid, because the terrestrial and the celestial dust cannot be separated; and the amount of terrestrial deposit is known to be considerable, probably averaging ten times that falling from the sky. As the Earth has been built up by gradual accretion of cosmical dust, and not at all by any condensation of masses detached from the Sun, chemical and geological changes, radio-activity, etc., are the only causes which could make the surface substances different from those still coming to us from celestial space.

(3) BRAUN'S inference from the Krakatoa eruption is not justifiable, because we are unable to judge how the air would look if all the dust were entirely removed from it. The sudden injection of an extra load of dust would of course increase the sunset glow, as was observed in 1883; and that is all that we can infer. This does not preclude the possibility that a large amount of cosmical dust is constantly falling to the Earth.

BRAUN also remarks that OPPOLZER took no account of the contraction due to the secular cooling of the Earth, and says this might fully offset the effect of the layer of cosmical dust deposited on the surface. Since BRAUN'S criticism appeared, twenty-five years ago, the writer has shown (*Proc. Am. Philos. Soc.*, Phila., 1907-1908) that the contraction long held to occur from the cooling of the globe is a deception, and that the Earth is really expanding about one hundred times more rapidly than it is contracting. BRAUN adds in conclusion that he spoke with PROFESSOR WEISS in Vienna some fifteen years before (that is, about 1869), and communicated to him his result, that 0.88 of a German cubic mile of cosmical matter of density 5.5 fell upon the Moon, and 9.2 upon the Earth in a century. He says this would explain a secular acceleration of 6″ in the Moon's motion, without regard to the retardation of the Earth's rotation. PROFESSOR WEISS, he says, found his idea plausible, and the required quantity of dust not altogether too large. Nevertheless BRAUN could not bring himself to believe in it, and allowed the idea to drop as not corresponding to the truth.

In *A.N.*, No. 2657, and No. 2664, the late M. J. KLEIBER, of St. Petersburg, discusses cosmical dust, and finds OPPOLZER'S density of space, $3 \cdot 10^{-16}$, giving a layer 2.8 mm. on the Earth in a century, equivalent to 300 times his own

concluded maximum density. But KLEIBER'S work rests on H. A. NEWTON'S conclusion that 450,000 meteors strike the Earth in an hour, and therefore is too small. Taking the average weight to be five grammes, KLEIBER says 2,250 kilograms of matter are added to the Earth in an hour. He estimates the density of planetary space to lie between 10^{-22} and 10^{-18} that of water (*A.N.*, 2657, p. 264). In *A.N.*, No. 2664, KLEIBER calculates that the mean distance of the Earth diminishes yearly by not more than three millimeters, and not less than 0.0003 millimeter; and thence infers that our planet will be brought down to graze the Sun's surface in not less than 5.10^{13} years, and not more than 5.10^{17} years. But of course even the shortest period of fifty trillion years is too long to have any present interest for us.

§145. *Retardation of the Earth's Rotation by Tidal Friction as Evaluated by* ADAMS *and* DARWIN.

In § 830 of THOMSON and TAIT'S *Natural Philosophy* will be found a discussion of the retardation of the Earth's rotation by tidal friction. The numerical estimate is by ADAMS, who took the Earth to be homogeneous, and the retardation as the square of the respective tide-generating forces depending on the Sun and Moon. He found twenty-two seconds as the error by which the Earth, regarded as a time-keeper, would in a century get behind a perfect clock rated at the beginning of the century. At the end of a century a meridian would get behind its undisturbed place by 330″, which is a sensible amount. Part of the discussion given in THOMSON and TAIT'S *Natural Philosophy*, § 830, is so important that it must be quoted in full:

"Besides the secular contraction of the Earth in cooling, referred to above, which counteracts the tidal retardation of the Earth's rotation to a very minute degree, there exists another counteracting influence, as has been pointed out by SIR WILLIAM THOMSON, which, though much more considerable, is still but small in the amount of its accelerative effect, compared with the actual retardation as estimated by ADAMS. It is an observed fact that the barometer indicates variations of pressure during the day and night, and it is found that when these variations are analyzed into their diurnal and semi-diurnal harmonic constituents, the semi-diurnal constituent rises to its maximum about 10 A.M. and 10 P.M. The crest of the nearer atmospheric tidal protuberance is thus directed to a point in the heavens westward of the Sun, and the solar attraction on these protuberances causes a couple about the Earth's axis by which the rotation is accelerated. As the barometric oscillations are due to solar radiation, it follows that the Earth and Sun together constitute a thermodynamic engine. SIR WILLIAM THOMSON

computes, as a rough approximation, that from this cause the Earth gains 2.7 seconds in a century on a perfect chronometer set and rated at the beginning of the century. On the other hand the fall of meteoric dust on the Earth must cause a small retardation of the Earth's rotation, although to an amount probably quite insensible in a century.

"Whatever be the value of the retardation of the Earth's rotation, it is necessarily the result of several causes, of which tidal friction is almost certainly preponderant. If we accept ADAMS's estimate (according to which the Earth would in a century get twenty-two seconds behind a perfect clock rated at the beginning of the century) as applicable to the outcome of the various concurring causes, then if the rate of retardation giving the integral effect were uniform, the Earth as a time-keeper would be going slower by .22 of a second per year in the middle, and by .44 of a second per year at the end, than at the beginning of the century.

"The latter is $\frac{1}{71.7 \times 10^6}$ of the present angular velocity; and if the rate of retardation had been uniform during ten million centuries past, the Earth must have been rotating faster by about one-seventh than at present, and the centrifugal force must have been greater in the proportion of 817^2 to 717^2, or of sixty-seven to fifty-one. If the consolidation took place then or earlier, the ellipticity of the upper layers must have been $\frac{1}{230}$ instead of about $\frac{1}{300}$, as it is at present. It must necessarily remain uncertain whether the Earth would from time to time adjust itself completely to a figure of equilibrium adapted to the rotation. But it is clear that a want of complete adjustment would leave traces in a preponderance of land in equatorial regions. The existence of large continents (§ 832′), and the great effective rigidity of the Earth's mass (§ 848), render it improbable that the adjustments, if any, to the appropriate figure of equilibrium would be complete. The fact then that the continents are arranged along meridians, rather than in an equatorial belt, affords some degree of proof that the consolidation of the Earth took place at a time when the diurnal rotation differed but little from its present value. It is probable therefore that the date of consolidation is considerably more recent than a thousand million years ago. It is proper however to add that ADAMS lays but little stress on the actual numerical values which have been used in this computation, and is of opinion that the amount of tidal retardation of the Earth's rotation is quite uncertain."

These remarks include LORD KELVIN's important argument that the diurnal rotation of the Earth has not sensibly changed since the consolidation of the globe, to which allusion has been made in Chapter XI. The subject is further treated by PROFESSOR SIR G. H. DARWIN, in his paper on the "Precession of a Viscous Spheroid," and reproduced in Appendix (G) to the *Natural Philosophy.*

ADAMS had taken the Earth to be homogeneous, and supposed the tides to consist of a bodily deformation of the mass: DARWIN therefore adapts his work to oceanic tides on a heterogeneous Earth, and works out the results with some care. After discussing the equations obtained in the paper on the "Precession of a Viscous Spheroid," taking account of the factor $\frac{2}{11}$, which is the ratio of the density of the water to the mean density of the Earth, and adopting equilibrium heights for the tides, he confirms the result of ADAMS that 6″ in the lunar acceleration corresponds to twenty-two seconds of time in a century. Including, however, the effects of the obliquity of the ecliptic and of the diurnal tide, DARWIN finds that 1″ in the Moon's acceleration would correspond to 3.6274 seconds at the end of a century. Taking NEWCOMB's observed value 8″.4, with DELAUNAY's theoretical value of 6″.1, he finds $2''.3 \times 3.6274 = 8.3$ seconds. According to this evaluation the meridian at the end of a century would be behind its undisturbed place by 124″.8. The great difference between this result and that found from the revision of the work of ADAMS, which DARWIN makes $23^s.4 = 351''$, in a century, is due to the small difference, 2″.3, in NEWCOMB's value of the outstanding inequality, compared to HANSEN's secular acceleration of 12″.56, leaving outstanding 6″.46. If the outstanding inequality in the secular acceleration were 2″, the result would be 108″.822. This retardation of the meridian would cause the undisturbed Moon to have an apparent gain in mean longitude of

$$\frac{108''.822}{27.32166} = 3''.9829.$$

In concluding his discourse DARWIN remarks that his result would be only slightly vitiated by the incorrectness of the hypothesis as to the heights of the tides, and the retardation of their phases. "Hence the result is not sensibly affected by some inexactness in the hypothesis, nor by the fact that the oceans in reality only cover a portion of the Earth's surface."

If instead of 2″, as the outstanding inequality in the Moon's secular acceleration, we had adopted the smaller value of 1″.88 given by NEWCOMB in the *Monthly Notices* of the Royal Astronomical Society for January, 1909, p. 167, the above results would have been still further decreased.

§ 146. *Further Considerations on* DARWIN'S *Evaluation of the Effects of Tidal Friction on the Apparent Motion of the Moon.*

The above estimates deal simply with the effect of the retardation of the Earth's rotation, without considering the effect of the reaction on the orbital motion of the Moon. This latter causes our satellite to recede from the Earth

and thereby gives rise to an apparent retardation of the mean motion; the total effect of tidal friction on the Earth's rotation and of tidal reaction on the Moon's orbital motion is, therefore, the difference of these separate effects.

WILLIAM FERREL, the American mathematician and meteorologist, was the first to apply the theory of tidal friction to the Earth's rotation, as giving the physical cause of the outstanding inequality in the secular acceleration of the Moon's mean motion (*Proceedings of American Academy of Arts and Sciences*, Vol. VI, p. 379), having presented his explanation December 13, 1864, while DELAUNAY's explanation was presented to the Paris Academy of Sciences, December 11, 1865. In his collected *Tidal Researches*, published in the *U.S. Coast Survey Report* for 1874, FERREL takes account of the effect of Tidal Friction on the Earth's rotation and of the tidal reaction on the Moon's orbital motion; and makes the two tendencies as 141″ to 54″, with a difference of 87″, which goes to produce an apparent acceleration of the Moon's mean motion. This subject has since been treated by DARWIN in his paper on the "Precession of a Viscous Spheroid" (*Phil. Trans.*, 1879, Part II, § 14, pp. 477-485). Under different hypotheses as to the nature and behavior of the tides and the physical properties of the Earth, SIR GEORGE DARWIN finds different results. He establishes the equation,

$$412.6\ E \sin 2\epsilon + 123.9\ E \sin \epsilon' + 7.042\ E'' \sin 2\epsilon'' , \tag{334}$$

as giving the number of seconds of arc by which the Moon should be accelerated in a century; and using 4″ as the observed outstanding inequality due to tidal friction and reaction, and introducing other simplifications, he finds that the lag of the tide due to friction is quite small, less than half a degree, when the Earth is highly rigid. On this hypothesis, apparently considered the most probable, he made the effect of friction 7″.1, and of the reaction 3″.1, giving a difference of 4″ in a century, as found by observation. On the elastico-viscous hypothesis DARWIN finds a true secular acceleration of 3″.521 per century. The ratio of the effect of tidal friction to that of tidal reaction by the above figures is about 2.3, while FERREL had found a ratio of 2.61, which is a fairly good agreement.

With our values we should use 2″ instead of 4″, in DARWIN's equation (63); and with his constants and half his assumed acceleration this would lead to the cubic

$$\frac{1320.7}{27.32}x + 7.042\frac{2x}{x^2+1} = 4 ,$$

or

$$x^3 - 0.08275x^2 + 1.2921x - 0.08275 = 0. \tag{335}$$

20

The real root of this cubic is $x = 0.064083$. Working out the corresponding quantities for the phase retardations, we find $2\epsilon'' = \frac{\pi}{2} - 3° 40'$; $2\epsilon = \frac{\pi}{2} - 8'$; $\epsilon' = \frac{\pi}{2} - 16'$. And DARWIN's equation (63)

$$412.6 \sin 4\epsilon + 123.9 \sin 2\epsilon' + 7.042 \sin 4\epsilon'' = 4\,, \tag{336}$$

is very nearly satisfied, the constant being 3.97 instead of 4. Thus our values in DARWIN's formulae give for the effect of friction 3″.51, and of the reaction 1″.51, leaving the outstanding difference 2″, as shown by the latest discussion of ancient eclipse observations. At the close of his discussion (*Phil. Trans.*, 1879, Part II, p. 483), DARWIN adds: "The conclusion to be drawn from all these calculations is that, at the present time, the bodily tides in the Earth, except perhaps the fortnightly tide, must be exceedingly small in amount; that it is uncertain how much of the observed 4″ of acceleration of the Moon's motion must be referred to the Moon itself, and how much to the tidal friction, and accordingly that it is equally uncertain at what rate the day is at present being lengthened."

It will be seen from this whole discussion that the effects of tidal friction and reaction probably could not produce an outstanding inequality greater than 2″. And if most or all of the observed residual inequality of 2″ in the motion of the Moon is due to the effects of cosmical dust, we naturally ask if there is not some other cause counteracting the influence of tidal friction. This is an important question to which we shall hereafter give attention, but it seems advisable to first consider briefly the analysis of oceanic tidal friction developed by FERREL, to which allusion has been made above.

§ 147. FERREL's *Method of Analysis in the Evaluation of the Secular Effects of Oceanic Tidal Friction.*

The part of the tide-generating forces of the Moon and Sun resolved in the direction of the tangent to the rotatory motion of any particle of the Earth's mass has a tendency to accelerate or retard the motion, according as it is in the direction of motion or contrary to it. To get the final effect on the Earth's rotation, we must integrate so as to take the algebraic sum of the moments of these forces with respect to the Earth's axis of rotation, each force being multiplied by the mass and length of arm on which it acts. If the result of this integration is zero, the Earth's rotation remains unchanged; but if not, the rotation is increased or diminished, according as the moments of the tangential forces in the direction of the Earth's rotation are greater or less than those in the contrary direction.

Unsymmetrical distribution of the fluid matter acted on by the tide-generating forces, in relation to the meridian of the disturbing body, is the circumstance which gives rise to forces which are not compensated by any acting in the contrary direction; and hence arise uncompensated moments about the Earth's axis of rotation. Irregularities in the shape and situation of the continents and in the contours and depths of the sea make it inconceivable that the moments about the Earth's axis should entirely vanish, but it remains difficult if not impossible to obtain an accurate evaluation of the mean effects of these irregular movements. To obtain a precise evaluation we should have to resort to a process of double integration over the Earth's surface, similar to that used by DARWIN in his treatment of the fortnightly tide (THOMSON and TAIT's *Nat. Phil.*, § 848). We may, however, obtain a fair approximation to the mean effect, by the following analysis due to FERREL.

§ 148. *Evaluation of the Retardation of the Earth's Rotation by Oceanic Tidal Friction.*

Let

ϱ = the distance of any particle dm from the Earth's centre;
ϱ' = the minor semi-axis of the tidal spheroid;
$\varrho' + \alpha$ = the major semi-axis;
$\varrho' + \beta$ = any radius of the spheroid;
$\varrho' k$ = the Earth's principal radius of gyration;
σ = the Earth's mean density;
$nt + u$ = the Earth's angular velocity of rotation.

Then it may be shown that the tide-generating potential is

$$V = \sum_{s=0}^{s=s} N_s \cos s\,(nt + \varpi - \psi)\;; \tag{337}$$

the coefficients N_s being functions of the polar distance θ of the undisturbed particle of the sea, and of the declination δ of the tide-raising body (cf. FERREL's *Tidal Researches, Coast Survey Report*, 1874, p. 26).

The tangential force acting upon any particle,

$$dm = \varrho^2 d\varrho \sin\theta\, d\theta\, d\varpi\,, \tag{338}$$

and tending to accelerate the rotation, is evidently

$$\frac{\partial V}{\partial \varpi}\,\frac{1}{\varrho \sin\theta}\;; \tag{339}$$

and if we multiply the mass of each particle by this force and into the distance $\varrho \sin\theta$ from the Earth's axis of rotation, we have for the integral of the sum of the moments

$$\sum_{i=0}^{i=i} H_i = \int_{\varrho'}^{\varrho'+a} \int_0^{\pi} \int_0^{2\pi} \frac{\partial V}{\partial \varpi} \frac{\varrho \sin\theta}{\varrho \sin\theta} \varrho^2 \sin\theta \, d\varrho \, d\theta \, d\varpi \,. \tag{340}$$

Dividing this expression by the mass of the sphere multiplied into the square of the principal radius of gyration, we get for the rate of the displacement of the meridian:

$$\frac{d^2u}{dt^2} = \frac{\displaystyle\int_{\varrho'}^{\varrho'+a} \int_0^{\pi} \int_0^{2\pi} \frac{\partial V}{\partial \varpi} \varrho^2 \sin\theta \, d\varrho \, d\theta \, d\varpi}{\frac{4}{3}\pi\sigma\varrho^3 \,.\, \varrho^2 k^2} \,. \tag{341}$$

For the expression under the integral, we use the principal term depending on the semi-diurnal tide, which FERREL shows to have the form

$$\left.\begin{aligned} V &= 0.9208 \sin^2\theta Z \cos 2\,(nt-\psi+\varpi)\,; \\ \text{and} \quad \frac{\partial V}{\partial \varpi} &= -1.8416 \sin^2\theta Z \sin 2\,(nt-\psi+\varpi)\,. \end{aligned}\right\} \tag{342}$$

Here the coefficient Z depends on constants in the tide-generating forces, such as the masses, distances, and eccentricities of the orbit of the tide-raising body. Accordingly we have finally

$$\frac{d^2u}{dt^2} = \frac{\displaystyle -1.8416 \int_{\varrho'}^{\varrho'+a} \int_0^{\pi} \int_0^{2\pi} \varrho^2 d\varrho \, d\theta \, d\varpi \sin^3\theta Z \sin 2\,(nt-\psi+\varpi)}{\frac{4}{3}\pi\sigma k^2 \varrho^5} \,. \tag{343}$$

In the theory of oceanic tides we suppose the tides lag and that the vertex of the tidal spheroid is east of the meridian of the disturbing body by the angle l, depending on friction. If therefore we regard ϱ as constant for the small range of β, we get

$$\left.\begin{aligned} \int \varrho^2 d\varrho &= \varrho^2 \,.\, \beta = \varrho^2 \alpha \sin^2\theta \cos^2(nt-\psi+\varpi-l) \\ &= \tfrac{1}{2}\varrho^2\alpha \sin^2\theta + \tfrac{1}{2}\varrho^2\alpha\sin^2\theta \{\cos 2\,(nt-\psi+\varpi)\cos 2l + \sin 2\,(nt-\psi+\varpi)\sin 2l\} \end{aligned}\right\} \,. \tag{344}$$

The triple integral now reduces to a double integral, and we have

$$\frac{d^2u}{dt^2} = \frac{-0.9208\, a \int_0^{\pi} \int_0^{2\pi} d\theta\, d\varpi \sin^5\theta\, Z \sin^2 2(nt - \psi + \varpi) \sin 2l}{\frac{4}{3}\pi\sigma k^2 \varrho^8}. \tag{345}$$

Here the terms which vanish between the limits of the integrals are omitted, leaving only the last term depending on $\sin\ 2l$ to be considered. If we now effect the integration with respect to θ and ϖ, we get

$$\frac{d^2u}{dt^2} = 0.7366 \frac{aZ}{\varrho^8 \sigma k^2} \sin 2l\ . \tag{346}$$

Integrating this with respect to t, we have

$$u - u_0 = -0.3683 \frac{aZ}{\varrho^8 \sigma k^2} t^2 \sin 2l\ , \tag{347}$$

where u_0 is the value of u corresponding to $t = o$.

It is found that $Z = g \times 0.8957$ foot, and $\sigma = 5.5$; also by LAPLACE'S law for the density of the Earth $k^2 = 0.331278$, or simply $\frac{1}{3}$. And therefore

$$u - u_0 = \frac{-0.181}{\varrho^8} a g t^2 \sin 2l. \tag{348}$$

Putting $g = 32.2$ feet, $a = 2$ feet, and $\varrho = 20{,}926{,}062$ feet, $t = 36525 \times 86164$ mean solar seconds, 86,164 being the number of seconds in a sidereal day, we get

$$u - u_0 = -2630'' \sin 2l. \tag{349}$$

This is the coefficient of the expression for the secular acceleration, and

$$u - u_0 = -2630''\, t^2 \sin 2l, \tag{350}$$

is the expression for the secular acceleration, due to the retardation of the meridian, when t is expressed in centuries. As the motion of the Earth on its axis is 27.32 times faster than that of the Moon in its orbit, we have for the effect on the Moon's longitude

$$\lambda - \lambda_0 = \frac{2630''}{27.32} t^2 \sin 2l = 96.3 t^2 \sin 2l\ . \tag{351}$$

The solar tides, if proportional to the forces producing them, are about 0.4516 of the lunar tides, and the combined effect of the two disturbing bodies therefor becomes

$$\lambda - \lambda_0 = 96''.3\ (1 + 0.4516)\ t^2 \sin 2l = 141''\ t^2 \sin 2l. \tag{352}$$

§ 149. *Evaluation of the Reaction on the Moon's Motion, Due to Oceanic Tidal Friction.*

We shall now find the effect of the tidal reaction on the Moon's motion in its orbit. Put

λ = the Moon's mean longitude in its orbit;
n = its mean velocity;
a = its mean distance, in terms of the Earth's radius;
μ = its mass in terms of the Earth's mass;
o = the mean obliquity of the ecliptic.

Then from the constancy of the moment of momentum in the system of the Earth and Moon, we have

$$\frac{d\,(a^2n)}{dt} = \frac{2630''\,k^2 \cos o \sin 2l}{\mu}. \tag{353}$$

But by KEPLER's law of planetary motion, $a^3\,n^2$ is a constant, and therefore $a^{6/3}n^{4/3} = a^2n^{4/3} = C$, and hence $\frac{d}{dt}(a^2n^{4/3}) = 0$.

Accordingly, from the preceding equation, we get

$$\frac{d}{dt}(a^2n) = \frac{d}{dt}(a^2n^{4/3}\,.\,n^{-1/3}) = -\frac{1}{3}\,\frac{a^2dn}{dt} = \frac{2630''\,k^2 \cos o \sin 2l}{\mu}.$$

But since $\frac{dn}{dt} = \frac{d^2\lambda}{dt^2}$, we find

$$\frac{d^2\lambda}{dt^2} = -\frac{3 \times 2630''\,k^2 \cos o \sin 2l}{\mu a^2}. \tag{354}$$

In this expression a may be regarded as constant, and hence if we put $a = 60$, $k^2 = \frac{1}{3}$, $\mu = \frac{1}{80}$, and integrate, we get

$$\lambda - \lambda_0 = -\,54''\,t^2 \sin 2l. \tag{355}$$

This tidal reaction on the Moon is negative, corresponding to an increase in the Moon's distance, and produces a retardation of the Moon's mean motion. *Hence the total effect of the apparent acceleration of the Moon's orbital motion, due to the tidal retardation of the Earth's rotation, and of the real retardation of the Moon's orbital motion, due to the tidal reaction on the Moon, is the difference of the two separate effects,* or

$$\lambda - \lambda_0 = (141'' - 54'')\,t^2 \sin 2l = 87''\,t^2 \sin 2l. \tag{356}$$

This is all that can be inferred from pure tidal theory. In order to find the lag l, or phase retardation due to friction, we have to equate this expression to

$2''$, the outstanding inequality in the Moon's secular acceleration. Therefore $2'' = 87'' t^2 \sin 2l$, gives $l = 39'.5$, or $l = 89°—20'.5$, according as the tides are direct or inverted. These changes of phase due to friction are thus very small, corresponding to a retardation of the time of high water of only 2.56 minutes, or an acceleration of the same amount, in the case of inverted tides.

Ferrel remarks that with a canal extending east and west around the Earth, the part of the tidal force used in overcoming friction is to the whole force as $\sin 2l$ to unity, or, with the above value of l, as 1 to 43.5.

§ 150. *Criticism of the Theory of Tidal Friction by* Ferrel *and* Newcomb.

If these values be admissible, it follows that for the Earth as a whole the friction of the ocean is quite small, which is conformable to the results of experience and investigation. In Section 262, of his *Tidal Researches* (p. 267), Ferrel comments on these results as follows:

"The action of the Moon and Sun upon the tides tends to produce a westward current with a motion gradually accelerated until the amount of friction between the fluid and the nucleus becomes equal to the sum of the tangential forces tending to decrease the rotation of the fluid part, and then the whole effect is transferred through friction to the nucleus. This westward motion, however, is a very small part of what is known as the equatorial current, which is due almost entirely to other causes, and has its counterpart in higher latitudes so far as these causes are concerned.

"The doctrine has been advanced and pretty generally entertained that all the heat arising from tidal friction and the force which turns the wheel of a tide-mill is at the expense of the Earth's rotation on its axis; but from what has been shown the whole of the tangential force tending to retard the Earth's rotation is the action of this force upon the part by which the tidal spheroid differs from the inscribed sphere, while the whole of tidal friction is equal to this same force acting upon the whole mass of the ocean. The former, therefore, is to the latter as the amplitude of the tide is to the whole depth of the ocean, and hence the former is a very small part of the latter. If, therefore, there is heat arising from tidal friction, a very small part of it only is at the expense of the Earth's rotation, and the balance must be continually increasing in proportion to the time; and as the force of attraction of the Moon and Sun upon which it depends, and the status of the matter acted upon with reference to the Earth's centre, remain the same, there is a continually increasing amount of energy in that form to which there seems to be no corresponding loss in any form

anywhere else, and which consequently seems to be irreconcilable with the doctrine of a constant and unalterable amount of energy in the universe."

His reasoning after the words "If, therefore, there is heat," etc., seems to the present writer to be based on a misconception; and a similar slip appears to vitiate the remarks of PROFESSOR NEWCOMB on the same topic in the *Monthly Notices*, of the Royal Astronomical Society, for January, 1909. For the heat produced by the oscillations of the sea must necessarily be proportional to the friction, or resistance to the motion of the fluid, and therefore also as the square of the velocity of the fluid relatively to the Earth; but it is to be observed that only a very small part of the motion is lost by friction, while the rest is restored to the Earth's rotation by the returning oscillation of the sea. *On the other hand, according to the accepted theory of tidal friction, a far greater part of the moment of momentum of the Earth about its axis, lost through the effect of tidal friction in raising tides which are unsymmetrically situated with respect to the meridian of the Moon, is transferred mechanically to the orbital motion of the Moon, as an indirect effect of the friction of the tides due to the modification of the attraction, and without the development of any heat whatever. The motion converted into heat by the oscillations of the sea is thus but a very small part of that mechanically transferred to the Moon through the modification of the Earth's attraction arising from the displacement of the tidal apex incident to friction.* The theory of tidal friction is therefore consistent with all known mechanical laws, and there remains no difficulty in this subject.

But in addition to the oceanic tidal friction, there is a minute effect due to atmospheric tides, and still another due to bodily tides in the solid globe of the Earth. These latter have been experimentally observed by DR. O. HECKER, at Potsdam, and estimated by SIR G. H. DARWIN to have a vertical movement of about six inches. This enables DARWIN to confirm the earlier work of LORD KELVIN and himself, that the solid Earth as a whole has about the effective rigidity of steel. Bodily tides in so rigid a globe will exert but a very slight influence on the motion of the Moon, and we need not further consider the matter at present.

§ 151. *On the Vortex of Cosmical Dust Revolving About the Earth and Proved to Exist by the Phenomena of the Gegenschein.*

Among the possible influences which could modify the rotation of the Earth, there is none so probable as the downfall of cosmical dust upon the surface; especially if there be a vortex of this dust revolving about it in the direction in which

the Earth rotates. We shall now consider this question. In the theory of satellites, Chapter X, we have seen that there are vortices of nebulosity developed about each planet, by the particles leaving the control of the Sun and revolving through the neck of the hour-glass space extending around the planet. And when moving within this closed space about the planet we have seen how the particles may be captured and abide there permanently as satellites. There is thus developed about each planet a vortex, which revolves in the direct sense, corresponding to the planet's rotation; in fact, it was the continuation of such vorticose motion and the precipitation of the particles upon the surface of the planets which finally developed their rotation. If this be true, is it probable that the action has yet ceased? Or shall we look to this cause for modifications of the Earth's rotation? The capture of the satellities and the reduction in the size of their orbits, as well as the decrease in the eccentricity, shows not only what has gone on in the past, but also what is going on now in a lesser degree. The longer a vortex revolves, the smaller and more circular become the orbits of the elements of which it is composed. This clearly follows from the theory of a resisting medium, and is verified by the observed orbits of our actual satellites. As the vortex gets smaller and smaller, the inner particles eventually come within ROCHE's limit, as in the case of *Saturn's* rings, and finally the nearest ones are consumed as meteors in the atmsophere. *Saturn's* rings, in fact, exhibit to us a cosmical vortex, about one of the largest planets, so dense that it is plainly visible from the Earth; and we know from the spectroscopic observations of KEELER, CAMPBELL, FROST and BELOPOLSKI, that it revolves as here indicated. If we could sufficiently remove from our telescopes the glare about *Jupiter*, no doubt we should perceive also a faint ring about that great planet; at least it would be surrounded by a lenticular mass of particles analogous to those constituting the zodiacal light about the Sun. In the same way each planet has a swarm of particles about it, of which the satellites alone are large enough to be observed in our telescopes. The Earth is no exception to this rule, for the *Gegenschein*, independently discovered by BRORSEN (1855), BACKHOUSE (1868), and BARNARD (1875), and since carefully observed by many astronomers, gives unmistakable evidence of cosmical dust revolving about the Earth.

§ 152. GYLDÈN'S *Explanation of the Gegenschein.*

In a memoir entitled *Sur un Cas Particulier du Problème des Trois Corps*, published in the first volume of the *Bulletin Astronomique* of the Paris Observatory, the celebrated Swedish astronomer GYLDÈN, first explained the *Gegenschein* by

the motion of particles about the point opposite to the Sun, which gives one of the straight line solutions in the problem of three bodies. GYLDÈN showed that meteors passing this point might pursue one or more oscillating orbits about it in ellipses of various sizes, before resuming their circuits about the Earth and Sun. If the number of such particles was very large they would exert no sensible attraction on one another, yet present to us the aspect of a hazy patch of light exactly opposite to the Sun. According to this theory the centre should be at the anti-sun, and the outline should be elliptical, with the longer axis along the ecliptic. This is exactly what is observed, but it may be noted that the axes of the elliptical patch of light are not very unequal; which seems to show that some meteors move about the Earth in all directions, with only a sensible increase near the plane of the ecliptic.

As observed by the writer in the dark pine forests about the Lowell Observatory in Arizona, in 1897, the appearance of the *Gegenschein* was similar to that of the zodiacal light, but the outline less oblate than that of the lenticular mass of particles revolving about the Sun. The only question of doubt heretofore has been whether there are enough particles with initial conditions of motion enabling them to oscillate about the anti-sun to explain the observed counterglow; but as the writer showed by observations taken in Arizona that the telescopic meteors are about 100 times more numerous than NEWTON had estimated (15 million), giving us at least 1,200,000,000 meteors daily, it seems certain that there would be enough of these fine particles of cosmical dust to give the feeble light characteristic of the *Gegenschein*. The following discussion will show why the dust constituting the *Gegenschein* ought not to be closely confined to the plane of the ecliptic, but somewhat spread out in latitude. This distribution follows from the part played by the Moon in capturing much of the dust revolving in the vortex about the Earth.

We may throw some additional light upon the nature of the swarm about the Earth by the analogy of *Jupiter's* action upon Periodic Comets. In Chapter VIII we have seen that *Jupiter* has thrown the comets of his family and the Asteroids within his own orbit. The motion of the Asteroids was originally direct, and of course that feature remains the same; but in case of the Periodic Comets their motions have been made to conform to the same general law, nearly all now being direct, though originally more of them may have been retrograde.

In the problem of the meteors we may replace *Jupiter* by the Moon, and consider what will happen to bodies passing near the Earth. The Sun is a powerful disturbing body, and controls the space at something over twice the distance of the Moon. We need consider only the meteors which come within HILL'S

closed surface about the Earth. It is evident that within this space the Earth corresponds to the Sun in the *Jupiter* comet-problem, while the Moon corresponds to *Jupiter*, but the mass is relatively about thirteen times larger. Accordingly just as *Jupiter* gathers comets within his orbit about the Sun, so also even more powerfully the Moon gathers meteors within her orbit about the Earth. And this process applies to all meteors which come within the sphere of the Earth's control.

Moreover in revolving about the Earth the Moon carries an hour-glass shaped space with it, and certain particles circulating about the Earth can make circuits about the Moon, and afterwards return to the Earth's control. Also the closed surface about the Earth is somewhat extended in the direction of the Moon, and as the Moon revolves does not remain quite symmetrical with respect to the Sun. All these causes tend to gather in more and more cosmical dust, much of which is finally thrown within the Moon's orbit, and made to revolve in the same direction in which the Moon revolves in her elliptic path about the Earth.

§ 153. *Analysis of the Total Mass of Meteorites Falling Upon the Earth.*

The vortex of cosmical dust revolving about the Earth is therefore made up of particles which have left the Sun's control by passing through the neck of the hour-glass shaped space, and of others which the Moon has aided the Earth in capturing, together with those captured by the Earth itself. In moving originally about the Sun some of these particles have had their orbits transformed by the attraction of the Earth into ellipses with aphelia near the Earth, just as *Jupiter's* comets have their perihelia near his orbit, and have again been perturbed by him. This tendency of the Earth to gather meteors within the terrestrial orbit facilitates the capture of meteors from the Sun; for on returning near the Earth, they may leave the Sun and revolve about the Earth, in the same direction as the Moon. From these causes it follows that there is a very considerable vortex of cosmical dust revolving about the Earth, some within the orbit of the Moon, and the rest beyond. According to the analogy of the satellites, most of these particles, perhaps nine-tenths of the whole, move direct, but others move retrograde. It was the operation of this vortex during past ages which shaped the rotation of the Earth about its axis, after our planet had begun to form on the outer parts of the solar system.

It now remains to inquire what effect the terrestrial vortex will have on the Earth's rotation. By the tendency of the other satellites of the solar system we may infer that about 0.9 of the dust revolves direct, the other tenth retrograde.

This will leave 0.8 of the particles in the vortex tending to accelerate the Earth's rotation. Near the surface the free velocity is seventeen times that of the Earth's rotation, and hence the fall of such particles will communicate an impulse proportional to $0.8 \times 17h$, where h is the thickness of the layer deposited. Since, however, the total mass of meteors falling upon the Earth is made up of two parts, namely, those moving in parabolic and elliptic orbits about the Sun, and those confined to the vortex about the Earth, we must multiply the above expression by the factor p, corresponding to the ratio of the dust falling from the vortex to the total amount falling, the rest being swept up from the orbits about the Sun directly.

Bodies entering the atmosphere from planetary space will descend at all angles; while those which are regularly attached to the vortex about our planet will have paths so nearly horizontal that they may be burnt up in the higher regions of the atmosphere, and seldom or never become visible from the Earth, unless it be through telescopic observation. In our ordinary experience, therefore, we see chiefly the meteors which come down more or less vertically from celestial space, as the *Leonids*, *Andromids*, *Perseids*, etc., while those burnt up in the higher regions of the atmosphere, on the whole, escape notice. It is these unseen meteors striking the higher layers of the atmosphere with large velocity, and nearly all moving in an easterly direction, which accelerate the rotation of the Earth by an amount proportional to

$$\frac{4}{5} \cdot \frac{17 \cdot f \cdot \omega \cdot pht}{\varrho}.$$

§ 154. *Numerical Calculation of the Three Terms in the Moon's Mean Longitude Depending on Changes in the Earth's Rotation.*

By integration this gives in a Julian century an acceleration of the meridian with respect to its undisturbed position amounting to

$$\varDelta L''_3 = -\frac{s''}{2} \cdot \frac{4}{5} \cdot 17 \cdot \frac{\omega f p h t^2}{\varrho} = -\frac{s''}{2} \cdot \frac{68}{5\varrho} \frac{k p h t^2}{a^{3/2}}. \qquad (357)$$

Here, as in OPPOLZER's work, s'' is the number of seconds of arc outstanding in the lunar acceleration, or counteracted by tidal friction. Since the fraction $\frac{4 \cdot 17}{5}$ is large, this term $\varDelta L_3''$ will counteract $\varDelta L_3'''$ (tidal friction), and $\varDelta L_3'$, even if p be no very large fraction. From the foregoing considerations we see that the three terms of $\varDelta L_3$ are as follows:

(1) $$\varDelta L'_3 = +\frac{s''}{2} \frac{(1-p)}{\varrho} h \omega f t^2 = +\frac{s''}{2} \frac{(1-p)}{\varrho} h \frac{k}{a^{3/2}} t^2,$$

depending on the fall of the $(1-p)$th part of the cosmical dust, as imagined by OPPOLZER.

(2) $$\varDelta L_3'' = -\frac{s''}{2}\frac{68}{5}\frac{k}{a^{3/2}}\frac{ph}{\varrho}t^2,$$

depending on the fall of the pth part of the dust from the vortex, and accelerating the rotation of the Earth.

(3) $$\varDelta L_3''' = +\frac{s''}{2}q\left(1-\frac{\gamma}{s}\right)t^2,$$

depending on tidal friction. This retards the rotation of the Earth, because $\frac{\gamma}{s}$ is the ratio of the tidal reaction on the Moon's orbital motion to the tidal frictional retardation of the meridian, when transferred to the orbital motion of the Moon, and thus about $1:2.5$, so that $\left(1-\frac{\gamma}{s}\right)=\frac{3}{5}$, nearly, while q is a multiplier which may be put equal to $\frac{20}{3}$. Thus the whole expression of $\varDelta L_3$ becomes

$$\varDelta L_3 = \frac{s''}{2}\left\{\frac{(1-p)}{\varrho}\frac{k}{a^{3/2}} - \frac{68}{5}\frac{k}{a^{3/2}}\frac{p}{\varrho} + \frac{q}{h}\left(1-\frac{\gamma}{s}\right)\right\}ht^2. \tag{358}$$

To evaluate this expression, we may safely take $p=\frac{1}{3}$; then the three terms reduce to the following numerical values:

$$\begin{aligned} \varDelta L_3' &= +\overset{\prime}{0}.18ht^2,\\ \varDelta L_3'' &= -1.23ht^2,\\ \varDelta L_3''' &= +1.92ht^2,\\ \hline \varDelta L_3 &= +0.87ht^2. \end{aligned} \tag{359}$$

It will be seen that with these values the effect of the acceleration due to the fall of cosmical dust from the vortex about the Earth nearly compensates for the retardation due to tidal friction, but not quite. Of course other values of $\varDelta L_3$ could be found from the use of other constants in the coefficients; but I believe these values to be as satisfactory as any which can be obtained in the present state of the subject.

PROFESSOR NEWCOMB and others have expressed the opinion that some unknown cause is at work counteracting the secular effects of tidal friction. (cf. *Monthly Notices*, Royal Astronomical Society, December, 1904, p. 183; and January 1909). This seems a sufficient ground for assigning values to these two principal terms, making them nearly equal but of opposite sign. Then, too, tidal friction alone has been supposed to be about $2''$, and moreover it seemed desirable to limit h to about 1^{mm}. Whilst the system of values adopted is thus arbitrary,

the terms evidently have the right signs, and also the correct relative importance. The residual difference of 2″ alone does not enable us to adjust so many terms except so as to satisfy the equation. Accordingly, we get finally

$$\begin{aligned} \Delta L_1 &= +\,0.87ht^2, \\ \Delta L_2 &= +\,0.26ht^2, \\ \Delta L_3 &= +\,0.87ht^2, \\ \hline \Delta L &= +\,2.00ht^2 = 2'', \end{aligned} \tag{360}$$

which gives $h = 1^{mm}$ in a century.

§ 155. *Remarks on the Theory Depending on the Downfall of Cosmical Dust.*

It thus appears that the outstanding inequality in the secular acceleration of the mean motion of the Moon may be satisfactorily explained by this simple theory based on real causes, and without any extravagant assumptions. Tidal friction is a real cause, but it is not the only cause; on the contrary, there is both acceleration and retardation of the Earth's rotation depending on the fall of cosmical dust. The reality of this latter cause is beyond doubt, and it largely counteracts tidal friction, in the ways already explained. All four of the terms considered to arise from the fall of cosmical dust are real; but the largest single term has a sign contrary to the rest. As the planetary rotations have arisen from this cause we cannot suppose it to have entirely disappeared even now. That one-third of all our meteoric dust should come from the vortex about the Earth is highly probable. Such meteors would fall in the upper part of the atmosphere, and scarcely be perceived from the surface of the Earth.

As to the supply of meteors, Braun objected to Oppolzer's estimate requiring 218 million tons daily, and cited Piazzi Smyth's estimate of from 100 to 1,000 tons; but in view of the fact that the telescopic meteors are about 100 times more numerous than the naked-eye meteors, and the additional fact that Prof. W. H. Pickering has given good grounds for increasing the average mass by a factor of the order of 100,000, we see that a downfall of seventy-eight million tons daily required by our estimate of $h = 1^{mm}$ in a century, is not at all improbable. The quantity of dust falling cannot well fall short of this amount. So much of the Earth is uninhabited and covered by the vast extent of the oceans, while our observations are usually embarrassed by our location near the sea level, which leaves us in the dark regarding the combustion of meteors in the upper atmosphere, that we cannot make a good estimate of the extent of the meteoric downpour, except that the amount is quite large, and certainly much greater than we have heretofore considered probable.

§ 156. *On the Secular Acceleration of the Sun.*

We have seen that an acceleration of the Sun was suspected by the great Swiss mathematician EULER, in 1749, and has been more fully confirmed by the recent researches of MR. COWELL on ancient eclipses. The exact amount of this secular acceleration is not yet known with accuracy, but it seems unlikely to exceed $1''.25$, and it may be no larger than $0''.5$ or $0''.25$ per century. It is therefore advisable to consider what influences might tend to modify the orbital motion of the Earth. The following are the main causes:

(1) An increase in the mass of the Sun and Earth, owing to the fall of cosmical dust. This has been shown to be the main cause at work on the Moon, and the effect on the Sun is similiar. Moreover the equatorial acceleration of the Sun's globe is due mainly to the fall of dust upon that surface from the vortex of matter revolving about it. The increase in the Sun's mass from this source must be considerable.

(2) Resistance to the Earth's orbital motion, directed along the tangent to the orbit, and proportional to the amount of dust included in the volume swept over by the Earth in a century.

(3) Retardation of the Earth's rotation by the downfall of cosmical dust and tidal friction. This effect seems to be very minute, and will be less on the Sun than on the Moon in the ratio of their mean motions; so that the coefficient $0''.87$ in the motion of the Moon must be divided by 13.369, giving a coefficient of $0''.065$ for the Sun.

(4) Repulsion of the Sun's light, tending to lengthen the year, and causing a retardation of the Earth's orbital motion. The secular equations for these changes were given by LAPLACE in the *Mécanique Céleste*, Liv. X, Chap. VII, §20.

(5) Closely connected with the repulsion of the Sun's light is another similar repulsive influence depending on the electrically charged particles driven out of the Sun in streams of varying density. In the *Monthly Notices* for 1904, MR. MAUNDER has shown that when the Earth passes through the varying magnetic field thus resulting, the condition of the Earth's magnetism is disturbed and Auroral displays occur.

(6) A certain amount of electric repulsion constantly exerted by the Sun against the Earth, owing to the similar electric charges borne on the surfaces of the two bodies. If this force were constant it would be equivalent to a slight decrease in the Earth's mass and thus produce no change in the length of the year; but as it is slightly variable, depending on the state of the corona and of the charge on the surface of the Earth, it may produce very slight variations in the Earth's motion.

§ 157. *Calculation of the Terms in the Sun's Mean Longitude Depending on the Downfall of Cosmical Dust Upon the Solar Surface.*

Now the first three causes here mentioned tend to produce a secular acceleration of the Sun; while the last three tend to produce a retardation. It is not therefore remarkable that these two opposite effects should nearly balance, leaving but a very slight secular acceleration. For all these effects are minute, except the first, which seems likely to be considerably larger than any of the others. We shall now calculate the effects of the two first causes. Making use of the same formulae as were used in the case of the Moon, but with notation adapted to the Sun and Earth, we have

$$\Delta L_1 = \frac{328714}{328715} \cdot \frac{3H}{R_0}\left(1 + \frac{\varrho^2}{R_0{}^2}\right)\frac{k}{a^{3/2}}t^2 . \tag{361}$$

where R_0 is the radius of the Sun, $\frac{k}{a^{3/2}}$ the mean motion of the Earth in a Julian century, the combined masses of the Sun and Earth being unity, and H the thickness of the layer of dust falling on the Sun from the regions of celestial space beyond the Earth's orbit.

To get the expression for ΔL_2 we proceed thus:

$$V = 2\pi R \;.\; \pi\varrho^2 \,.\, f' \,.\, 36525 = 200\pi^2\varrho^2 R \;; \tag{362}$$

where $f' = \frac{1}{365.25}$ is the ratio of the mean motion of the Sun to that of the Earth's rotation.

And

$$V\delta = 200\pi^2\varrho^2 R \cdot \frac{h}{50\pi R} = 4\pi\varrho^2 h . \tag{363}$$

The ratio, $\frac{\nu'}{M}$ is found by noticing that $\nu' = V\delta = 4\pi\varrho^2 h$;

$$\frac{\nu'}{M} = \frac{4\pi\varrho^2 h}{\frac{4}{3}\pi\sigma\varrho^3} = \frac{3h}{\varrho} . \tag{364}$$

$$S_0 = -\frac{3h}{\varrho}\left(\frac{k}{a^{3/2}}\right)at = -3\frac{h}{\varrho}\frac{k}{\sqrt{a}}t . \tag{365}$$

$$\Delta L_2 = -\frac{3}{2a}\left[\frac{-3h}{\varrho}\frac{k}{\sqrt{a}}\right]t^2 = \frac{9}{2}\frac{k}{a^{3/2}}\frac{ht^2}{\varrho} . \tag{366}$$

Now the mean motion of the Sun in a Julian century is $\frac{k}{a^{3/2}} = 1296\,00000''$; and $\varrho = 6370\,000\,000^{\mathrm{mm}}$;

therefore

$$\Delta L_2 = +\frac{9}{2}\frac{129600000}{6370000000}ht^2 = +\frac{9}{2}\cdot\frac{1296}{63700}ht^2 = +0''.0916\,ht^2; \qquad (367)$$

where h is expressed in millimeters.

If $h = 1^{mm}$, then $\Delta L_2 = +0''.0916t^2$.

If we evaluate (361) we get $1 + \frac{\rho^2}{R^2_0} = 1.0001$, which is nearly unity;

$\frac{k}{a^{3/2}} = 129600000''$; $R_0 = 696098000000^{mm}$; and $\Delta L_1 = \frac{388800000}{696098000000}Ht^2 = \frac{3888}{6960980}Ht^2$
$= 0''.0005584\,Ht^2$.

Using the factor 1.0001, and $H = 1{,}000^{mm} = 1^m$, we get $\Delta L_1 = 0''.5590t^2$; or a secular acceleration of the Sun amounting to 0″.5590 in a century. The three causes (1), (2) and (3) would, on the hypotheses here adopted, give

$$\begin{aligned}\Delta L^\circ_1 &= +\,0''.5590\,t^2\\ \Delta L^\circ_2 &= +\,0.0916\,t^2\\ \Delta L^\circ_3 &= +\,0.0650\,t^2\\ \hline \Delta L^\circ &= +\,0.7156\,t^2\end{aligned} \qquad (368)$$

A downfall of cosmical dust which would produce a secular acceleration of the Sun of 0″.71 per century, must be considered most probable.

§ 158. *On the Amount of Matter Falling Into the Sun and on the Cause of the Equatorial Accelerations of the Sun, Jupiter and Saturn.*

A layer of matter of the mean density of the Earth one meter deep all over the Sun's surface would amount to 1 : 177.81 of the Earth's mass; or a little less than half the mass of the Moon. This is about $\frac{1}{100}$ part of the meteoric downfall postulated by J. R. Mayer in 1848 for explaining the Sun's heat; and $\frac{1}{378}$ part of the meteoric increase calculated by Lord Kelvin in his paper on the "Mechanical Energies of the Solar System," *Trans. Roy. Soc.*, Edinb., April, 1854; reprinted in *Mathematical and Physical Papers*, Vol. II, Art. LXVI. Here the estimated increase of the Sun's mass was $\frac{1}{47}$ of the Earth's mass annually. Lord Kelvin's estimate was nearly four times larger than Mayer's; and Newcomb therefore adopted an approximate mean between these extremes (*Popular Astronomy*, p. 515), namely, an amount equal to the Earth's mass in a century, which is 177.81 times the value calculated above, and adequate to make a layer over the Sun's surface 177.81 meters deep.

In assuming such a downpour of meteorites Mayer realized that the motions

of the planets would be sensibly accelerated, but he assumed a simultaneous wastage of solar substance analogous to that imagined by SIR ISAAC NEWTON, under the emission theory of light. Without assuming any wastage of matter due to radiation, LORD KELVIN calculated that a downfall of matter equivalent to the mass of the Earth in forty-seven years, if it came from beyond the Earth's orbit, would shorten the year by six weeks in 2,000 years. This value seems to be somewhat too large; for by recalculating the effect I find the decrease to be twenty-six days instead of forty-two; but of course all these estimates are much too large. If our value of a layer of matter 1^m deep over the Sun's globe,* or $\frac{1}{177.81}$ of the Earth's mass in a century, be accepted, and all the matter be derived from sources extraneous to the Earth's orbit, the secular acceleration of the Sun becomes $0''.71T^2 = 284''$ in 2,000 years, corresponding to $1^h\ 48^m$ in time since the epoch of HIPPARCHUS. It seems highly probable that a layer of solid matter from ten to fifty metres deep falls into the Sun in a century; but that most of it comes from within the Earth's orbit, and only a layer of something like 1^m deep comes from beyond the Earth's orbit, and gives rise to a secular acceleration of the Sun of say $0''.75$ per century.

This small downfall of extraneous cosmical dust therefore gives us the true cause of the Sun's secular acceleration, while the larger internal downfall produces the observed equatorial acceleration of the Sun's rotation. Corresponding downfalls of cosmical dust give rise to the equatorial accelerations observed in *Jupiter* and *Saturn*, and enable us to conclude that the observed equatorial acceleration is but an indication of the general process by which cosmical rotation is established and augmented. This throws an interesting light upon the past history of our system; for these survivals in the way of equatorial accelerations have not been clearly understood heretofore.

§159. *Summary of the Results of the Present Investigation.*

It now remains to consider what results may be considered established by the present investigation. Before entering upon this discussion, however, we may remark that electric or magnetic forces sometimes imagined to be exerted by the heavenly bodies, such as the Sun and Moon, upon the Earth can hardly give rise to sensible effects, for two reasons:

(1) The forces are small, and repulsive in character, and therefore would always tend to develop a retardation in the mean motions, while the observed effect is an acceleration. These forces are feeble, because confined essentially to the surface layers of the bodies on which they exist; and moreover obey the

*This should be understood to be the *excess* over that wasted by radiation, which may be a considerable amount.

same law of inverse squares as the attractive force of gravitation; and therefore if these forces are constant, the total effect is equivalent to a slight diminution of the masses of the two bodies.

(2) The electric charges borne by the Sun and Moon are shown by observation to be fairly constant, though slight fluctuations in the Earth's magnetic field arise when our planet moves through the varying field of charged particles emitted by certain local areas on the Sun's surface. These additional forces, however, in the long run will be equally distributed over all parts of the orbits of the bodies concerned, and therefore produce no secular effect, except an additional retardation in the mean motions.

Accordingly we may have no hesitation in dismissing the electric and magnetic forces as exerting no considerable secular effect. It follows also that tidal friction, as a real physical cause, has been much overrated; and while it still has to be considered, it is of secondary importance compared to the accelerative or retardative effects of cosmical dust. This latter cause has diminished the major axes and eccentricities of the orbits of all revolving bodies in our system, and is still the most powerful influence at work on the motions of the heavenly bodies.

Any supposition that gravitation itself varies is not to be seriously entertained, except as a last resort; and fortunately the known effects of cosmical dust relieve us of the necessity of introducing this inadmissible hypothesis, while at the same time it enables us to connect the movements now going on with those which have been so powerful in the past history of the solar system. The explanations here put forward in regard to the secular accelerations of the Sun and Moon, therefore, leave little to be desired; but additional observations and researches will no doubt be required to improve the details of the theory.

§ 160. *The Outstanding Difficulty Connected with the Fluctuations in the Moon's Motion Probably Not Insurmountable.*

But whilst it is certain that the phenomena connected with the secular accelerations are easily and naturally accounted for by the above simple theory based on true physical causes, it is not yet certain that the theory will account for the irregular fluctuations recently discussed by NEWCOMB. We are not yet able to find from observation the magnitude of the forces required to produce the observed effects. NEWCOMB remarks: "Since what is actually observed is neither the acceleration nor the speed of motion, but changes of the longitude itself, of which these quantities are respectively the second and the first derivatives as to the time, it is not possible from the observations to make any approach to an accurate

estimate of the accelerating or retarding forces. The most that we can say is that these varying forces are sufficient to bring about a change of annual motion amounting to between $0''.5$ and $1''$, by acting during a period of perhaps from four to six years" (*Monthly Notices*, January, 1909, p. 167).

It appears from these considerations that the forces on which the fluctuations depend are decidedly larger than those producing the outstanding $2''$ per century in the secular acceleration. It is the irregular character of the fluctuations and their considerable magnitude that renders any attempt to explain them very difficult; though in the long run these irregularities are less important than the secular acceleration, the effect of which accumulates with the lapse of ages and may finally become quite large. For reasons already pointed out, it would be natural to refer the fluctuations to irregular resistance due to swarms of cosmical dust. But the intensity of the forces would have to be greater than those invoked to explain the secular acceleration. At present it is uncertain whether the amount of cosmical dust encountered in space is sufficient to account for the fluctuations; we must also remember that some hitherto unrecognized gravitational cause may after all produce the large fluctuation with a period of about 250 years. Under the circumstances it is necessary to wait for further light on the subject.

It seems certain that a considerable increase in the amount of cosmical dust, beyond that postulated to explain the secular acceleration, might be admitted without contradicting any known phenomenon. For example the vortex of dust revolving about the Earth might be much more extensive than we have imagined; and there might thus result an increased acceleration of the Earth's rotation on its axis; but this could not well be introduced without augmenting the retardative effects of tidal friction, which are believed to be small. The terms $\Delta L_3'' = -1.23ht^2$ $\Delta L_3''' = +1''.92ht^2$, in § 151, might each be augmented by several times their present values, without introducing quantities which are wholly inadmissible. Then again sensible fluctuations in the motion of the Moon might result from the temporary movement of streams of dust about the Earth which afterwards return to the control of the Sun.

Accordingly whilst the present explanation is not demonstrated to be adequate to account for the fluctuations, it is not wholly excluded by any known phenomenon; and it is therefore natural to believe that the secular acceleration is only the more lasting effect of a general cause affecting the Moon's motion, and that this cause rests on the effects of streams of dust through which the Lunar-Terrestrial System is moving. As the Moon is a captured planet which has been brought nearer and nearer the Earth, this movement of approach must still be going on; and it could not well take place without sensible fluctuations, which in the course of

many centuries disappear and leave outstanding only a residual acceleration of 2″ per century. At least until some other real cause can be brought forward, this may be held to be the most probable explanation.

Some additional reasons may be given for inclining to this view, but the subject is involved in so much obscurity that the following considerations are tentative, yet not likely to be without some value. FERREL's discussion of tidal friction in §§ 149, 150 assumes that for the Earth as a whole there is some average tidal ellipsoid with phase retardation l. When we come to compare the High Water Interval (H.W.I.) by which the high tide follows the Moon, as given in the *Tide Tables of the U. S. Coast and Geodetic Survey*, for the various points on the Earth's surface, we are struck by the fact that all intervals exist from $0^h\ 0^m$ to $12^h\ 0^m$. And if the whole Earth could be studied it seems likely that the mean interval for the entire globe would be not far from six hours. This would make the tidal apex, so far as such a thing may be said to exist, lie in a position at right angles to the Moon's radius vector; and consequently as the semi-diurnal tides are nearly equal, no couple of sensible magnitude would be produced about the Earth's axis of rotation, by the change in the figure of the sea due to the tides in the oceans.

If any residual couple remained when the whole surface of the Earth is carefully integrated to get the mean effect, the outstanding result certainly would be one of extreme minuteness. Therefore it seems certain that if there is any tidal reaction on the Moon's motion, it must be excessively slight. Accordingly while tidal friction is at work against the Earth's rotation, it is by no means certain that an appreciable reaction against the Moon's orbital motion takes place.

This is a question of actual integration over the surface of the Earth requiring further investigation, and will have to be left to the future; but it seemed so doubtful whether there is a sensible couple retarding the rotation of the actual Earth, that attention should be drawn to the problem, in the hope that it may receive analytical and numerical treatment.

Should it turn out that the average interval of high water after the Moon's transit is about six hours, the tidal reaction on the Moon would be reduced to insensible magnitude. If it exceeds six hours, as seems possible, then the effect would be to bring the Moon nearer the Earth by a minute quantity. The annulling of the retardative effect heretofore ascribed to tidal friction, would have the effect of apparently accelerating the Earth's rotation, and thereby apparently retarding by that amount the mean motion of the Moon in its orbit.

If this condition really exists it may well be that the outstanding inequality in the secular acceleration of the Moon's mean motion is greater than is indicated

by observation. For just as a retardation of the Earth's rotation would have the effect of producing an apparent acceleration of the Moon's motion, so likewise an apparent acceleration of the rotation by the abandonment of the supposed retardative effects of tidal friction would give rise to a corresponding apparent retardation of the Moon's motion, and thus diminish the apparent lunar acceleration which would result from the uniform rotation of the Earth about its axis.

At present one cannot feel entirely confident as to what is the actual state of fact. It seems certain that the effects of oceanic and bodily tidal friction are very minute, and it might easily have a sign contrary to that heretofore imagined. In this case the Moon would be nearing the Earth even more rapidly than the observed secular acceleration would indicate; and this abandonment of tidal friction would more than ever confirm the theory that the Moon is a planet which came to the Earth by capture. The evidence at present available is not decisive, and the settlement of this question must be left to the future; but so far as one may now judge, this last view is not at all improbable.

We have treated the problem of the Moon's secular acceleration in a somewhat detailed manner, in order to bring out the difficulty and uncertainty attaching to the subject, and in the hope of opening the question to further research. It may be a long time before the question can be definitely settled, but nothing is more certain than that our theories on this subject have been very incomplete; and we have deemed it better to consider new causes in the hope of winning new truth than to merely follow the beaten paths heretofore so generally preferred by previous investigators.

Ἔχει δ' ὑπέρφρον σῆμ' ἐπ' ἀσπίδος τόδε,
φλέγονθ' ὑπ' ἄστροις οὐρανὸν τετυγμένον·
λαμπρὰ δὲ πανσέληνος ἐν μέσῳ σάκει,
πρέσβιστον ἄστρων, νυκτὸς ὀφθαλμὸς, πρέπει.

And on his shield he bears this proud device,—
A firmament enchased, all bright with stars;
And in the midst the full Moon's glittering orb,
Sovran of stars and eye of Night, shines forth.

—Aeschylus, Ἕπτα ἐπι Θήβας, 387-390, *Translated by* Plumptre.

CHAPTER XIV.

On the Craters, Mountains, Maria and Other Phenomena Observed on the Surface of the Moon, and on the Indicated Processes of Planetary Growth.

§ 161. *Historical Resumé of the Theories of the Origin of the Lunar Craters.*

Heretofore most investigators have studied the mountains of the Earth, and then attempted to approach those of the Moon, in the hope of discovering clearer indications of a common mode of formation.

In passing beyond the mountains of the Earth they seem to have adopted the suggestion of Aeschylus: *ἀστρογείτονας δε χρὴ κορυφὺς ὑπερβάλλουσαν* — "Thou must cross those summits near the stars" (*Prom. Vinct.*, 721–2). But it is far from certain that this is a correct procedure, for as we shall see in the course of this chapter the origin of the Terrestrial mountains have generally been referred the wrong physical cause. And if we have not known the cause of Terrestrial mountain formation, it is clear that we have no grounds for asserting that there is any analogy between the mountains of the Earth and those of the Moon.

The true mode of origin of the so-called craters on the Moon has long been a matter of discussion, but no theory heretofore developed has proved to be entirely satisfactory. Nevertheless it has been generally believed that these craters are of volcanic origin, and such a view is still held at the present time. This theory is the traditional one which has come down from the time of Galileo. It is true that it was neither suggested nor formulated by the great Florentine astronomer, but was first proposed by Hooke (*Micrographia*, 1667, Obs. LX. pp. 242-246*).

*"These seem to me to have been the effects of some motions within the body of the Moon, analogous to our earthquakes, by the eruption of which, as it has thrown up a brim or ridge round about, higher than the ambient surface of the Moon, so has it left a hole or depression in the middle, proportionately lower." Hooke says of his experiment with boiling alabaster, that "presently ceasing to boyl, the whole surface will appear all over covered with small pits, exactly shaped like those of the Moon. The earthy part of the Moon has been undermined, or heaved up by eruptions of vapours, and thrown into the same kind of figured holes as the powder of alabaster. It is not improbable also, that there may be generated within the body of the Moon, divers such kind of internal fires and heats as may produce exhalations" (Robert Hooke, *Micrographia*, 1667, Obs. LX, pp. 242-246; cf. Humboldt's *Cosmos* IV, p. 496).

The volcanic theory was, however, definitely asserted by such authorities as SIR JOHN HERSCHEL, and is therefore adopted in all our hand-books of Astronomy.

It was accepted with greater reserve and hesitation by HUMBOLDT, who has the following remarks on the subject: "The progressive perfection of our acquaintance with the formation of the surface of the Moon as derived from numerous observers, from TOBIAS MAYER down to LOHRMANN, MÄDLER and JULIUS SCHMIDT, has tended on the whole rather to diminish than to strengthen our belief in great analogies between the volcanic structures of the Earth and those of the Moon; not so much on account of the conditions of dimension and the early recognized ranging of so many ring-shaped mountains, as on account of the nature of the *rills* and of the system of rays which cast no shadows (radiations of light) of more than 400 miles in length and from 2 to 16 miles in breadth, as in *Tycho*, *Copernicus*, *Kepler* and *Aristarchus*" (cf. *Cosmos*, Vol. V, p. 448, BOHN's Translation). He recalls also that GALILEO in a letter to FATHER CHRISTOPH GRIENBERGER, *Sulle Montuosita della Luna*, compared the annular mountains on the Moon to the circumvallated districts of Bohemia, and that HOOKE attributed the circular type of formation to the *reaction of the interior of the Moon's body on the exterior*.

HUMBOLDT remarks with surprise that the central mountains have been found by SCHMIDT always to lie below the walls by which they are surrounded. According to this astronomer's investigations, begun at Olmütz and afterwards continued at Athens, it appears that no single central-mountain attains the height of the wall of its crater, but that in all cases it probably even lies together with its summit considerably below that surface of the Moon from which the crater is supposed to have been erupted.

This certainly is a remarkable state of fact, and we shall recur to it again when we come to deal with the origin of these fortress-like structures.

But whilst the volcanic theory of the Lunar craters is still very generally taught in the universities and other schools of Europe and America, a few sagacious thinkers have always hesitated to accept this orthodox interpretation of the most remarkable phenomena presented by the Moon's surface. In his *Popular Astronomy*, edition of 1878, p. 320, NEWCOMB hints at a new theory of the Lunar craters, as follows: "The mountains consist, for the most part, of round saucer-shaped elevations, the interiors being flat, with small conical mounds rising here and there. Sometimes there is a single mound in the centre. It is very curious that the figures of these inequalities in the Lunar surface can be closely imitated by throwing pebbles upon the surface of some smooth plastic mass, as mud or mortar."

§ 162. *Theory of Impact Elaborated by* GILBERT, *but Not Accepted by Contemporary Investigators.*

In his Presidential Address to the Philosophical Society of Washington for 1892 (*Bulletin,* Vol. XII, pp. 241–292), the eminent geologist DR. G. K. GILBERT took as his topic "The Moon's Face: A Study of the Origin of Its Features;" and thus developed the hint thrown out by NEWCOMB and others into a more elaborate theory. In examining the Lunar surface he made use of the great Equatorial Telescope of the U.S. Naval Observatory at Washington, and thus dealt directly with the telescopic aspects of the Moon as shown in powerful instruments.

GILBERT's theory was that the Lunar pits may be indentations in a globe yielding to the action of forces produced by infalling meteorites or planetoids; and he showed by experiment that under certain conditions pits of a similar type, with similar central cones, may be produced by impact. In this way he accounted for many of the phenomena presented by the Lunar surface, but not all. The numerous flat-bottomed craters remained somewhat of an enigma, or required the theory of fusion and floods; nor could any satisfactory explanation be found of the almost total obliteration of many craters, the walls of which may now be traced with extreme difficulty. The great smoothness of certain large areas like the plains and seas was equally difficult to account for, except by the violent hypothesis that a deluge of soft material had spread over a large part of the Lunar surface.

GILBERT's theory of the impact of moonlets against the Lunar surface attracted some attention at the time of its publication, but has not been accepted even by Geologists. Thus in their new work on *Geology* (Vol. I, p. 598) CHAMBERLIN and SALISBURY mention GILBERT's paper in a footnote, but in the text adhere to the orthodox volcanic theory. Astronomers therefore naturally have continued to hold the volcanic theory, though it certainly is erroneous, as will more fully appear from a number of considerations adduced below. The following references to the recent discussions of astronomers must suffice.

In the *Atlas Photographique de la Lune*, made from photographs taken at the Paris Observatory, Part I, 1896, LOEWY and PUISEUX have adhered to the volcanic theory as sufficient to explain the phenomena of the Lunar surface; and during June and July of the present year, 1909, two papers bearing on this theory have been presented by M. PUISEUX to the Paris Academy of Sciences. He has treated of the Lunar surface also in the *Revue Scientifique* for May 8, 1909, and concluded that the craters of the Moon present many points of resemblance to the volcanoes of the Earth.

In *A.N.*, 4348, HERR PAUL FUCHS has published a paper on the origin of the Lunar craters, the purport of which is to lend support to the well-known experiments of PROFESSOR H. EBERT, tending to show that the form of the craters is due to the alternate extrusion and retraction of a suitable magma through an orifice in the crust. PROFESSOR EBERT's more recent experiments have led him to the conclusion that the lunar maria are composed of a kind of natural volcanic glass, such as vitrophyr or obsidian; and he has explained the systems of bright rays radiating from some of the craters as due to lines of fracture in the glassy surface, when viewed under a high Sun (cf. *Journal of British Astronomical Association*, Vol. XIX, No. 9, July 30, 1909, p. 378).

After the present theory of impact was completed and ready for publication, the author finally succeeded in obtaining a copy of GILBERT's paper of 1892; it was then found that his treatment covered so many points that a more detailed summary of this early work was advisable. Accordingly we add here the main conclusions reached by GILBERT, without changing any of the rest of the discussion previously prepared, in the belief that the somewhat different views thus developed may not be wholly without value to others who may study the subject.

§ 163. *Summary of* GILBERT'S *Conclusions.*

(1) GILBERT describes the types of craters and classifies them, saying that the majority of writers consider them of volcanic origin. But he points out the fundamental difference between Terrestrial and Lunar craters thus: "Ninety-nine times in one hundred the bottom of the Lunar crater lies lower than the outer plain; ninety-nine times in a hundred the bottom of the Vesuvian crater (taken as the type of Terrestrial volcano) lies higher than the outer plain. Ordinarily the height of the Lunar crater rim is more than double its outer height; ordinarily the outer height of the Vesuvian crater rim is more than double its inner height. The Lunar crater is sunk in the Lunar plain; the Vesuvian is perched on a mountain top. The rim of the Vesuvian crater is not developed, like the Lunar, in a complex wreath, but slopes outward and inward from a simple crest-line. If the Vesuvian crater has a central hill, that hill bears a crater at the summit and is a miniature reproduction of the outer cone; the central hill of the Lunar crater is entire, and is distinct in topographic character from the circling rim.

"The inner cone of a Vesuvian volcano may rise far higher than the outer; the central hill of the Lunar crater never rises to the height of the rim and rarely to the level of the outer plain. The smooth inner plain characteristic of so many

Lunar craters is either rare or unknown in craters of Vesuvian type. Thus, through the expression of every feature the Lunar crater emphatically denies kinship with the ordinary volcanoes of the Earth."

(2) He recalls most of the theories advanced prior to 1892, and finally rejects them all, except the theory of impact, which was suggested by PROCTOR as far back as 1873. Among the theories reviewed is the well known theory of PROFESSOR H. EBERT, which has found considerable favor in Germany. "That a circular ridge may be built up by the alternate extrusion and retraction of a suitable substance through an orifice, has been demonstrated by EBERT, who devised apparatus and conducted a series of experiments (on metallic magmas). The crater rims he achieved sloped regularly outward and were steep and rudely terraced inward, thus reproducing the more important features of the Lunar rims, with the exception of the wreath, and by special manipulation he was able to approach the character of the wreath."

Having disposed of these several theories, GILBERT adopts the impact theory, and discusses it as follows: "If a pebble be dropped into a pool of pasty mud, if a raindrop fall upon the slimy surface of a sea marsh when the tide is low, or if any projectile be made to strike any plastic body with suitable velocity, the scar produced by the impact has the form of a crater. This crater has a raised rim, suggestive of the wreath of the Lunar craters. With proper adjustment of material, size of projectile, and velocity of impact, such a crater scar may be made to have a central hill. Thus scars of impact may simulate in many ways the scars of the Moon's face, and a number of theories have accordingly been broached which agree in regarding the craters as due to the bombardment of the Moon by projectiles coming from without."

He says he could find no statement of this theory more than twenty years old, or earlier than 1872, but that the idea is older and various obscure allusions indicate that it was earlier in print. The first definite statement is by PROCTOR who suggested it in 1873 ("The Moon," p. 346).

(3) With the coöperation of PROFESSOR R. S. WOODWARD, GILBERT calculates that the heat of collision of a body moving with the parabolic velocity at the Lunar surface would raise the temperature to 3500° Fahrenheit, or higher; so that much of the rock is molten, and he thinks this would explain the level surfaces of some of the craters. "MEYDENBAUER, as a corollary of certain conclusions in regard to meteoric matter, holds that the surface of the Moon is clothed with a mantle of cosmic dust, a deep layer of loose particles everywhere concealing the solid nucleus, and that the fall thereon of aggregates of similar dust produced the Lunar craters. By experimentation with various finely-divided substances

he has in this way produced small craters simulating several of the Lunar varieties. His results show raised rims analogous to the Lunar wreath, central hills, and arched inner plains, such as characterize a few of the Lunar craters. His published results do not include level inner plains, nor the association of inner plains with central hills; but, on the other hand, he does not extend this process to the largest craters and the maria. For them he suggests the collision of solid stars of sulphur or phosphorus, originally Moon's of the Earth's system, and he recognizes fusion as one of the results of their collision" (A. MEYDENBAUER: *Sirius*, February, 1882).

(4) He discusses EBERT's work on the ratio of the rim content to cavity content. "EBERT has compiled the available published data and computed the ratio of rim content to cavity content for ninety-two craters, ranging in diameter from eight to nearly one hundred miles (H. EBERT: *Ueber die Ringgebirge des Mondes, Sitzungsberichten d. Physik.-med. Societät.* Erlangen, p. 171, Munich, 1890). In twenty-eight instances he finds the rim content the greater; in the remaining sixty-four instances he finds it the smaller; and in about fifteen instances the rim volume is but a small fraction of the content of the cavity. He finds further that the rim is relatively large in the case of the larger craters. Though the imperfection of the data gives a large probable error to the determinations, there can be no question of the general fact that in many instances the rims of large craters are quite inadequate to fill the cavities they surround. This is an important fact, but it is not necessarily inimical to the impact theory." From some experiments GILBERT concludes that these differences may be explained by the degree of yielding of the material of the Moon's surface.

(5) GILBERT then considers the round form of the Lunar craters and the angles at which the satellites would have to fall to produce such an effect. To overcome the tendency to produce elliptic outlines by falling at angles considerably inclined from the vertical, he imagines that all impacts are due to a ring of moonlets revolving about the Earth analogous to the rings of *Saturn*. With the coöperation of PROFESSOR R. S. WOODWARD he shows that when the moonlets move in a single plane, the curve of distribution for falling moonlets has much the same form as the curve deduced from the observed ellipticities of craters*.

(6) GILBERT remarks that the area of the surface struck is as the square of the radius of the moonlet, but the energy applied to the area being as the mass of the falling body and therefore as the cube of the radius, more energy per unit of area is developed by the collision of large than of small bodies; and he thus

*It may be remarked that this result applies equally well to the swarm of Asteroids in which the Moon was revolving before its capture by the Earth. The supposed ring of moonlets about the Earth is scarcely admissible, so that this part of his reasoning seems to be vitiated.

accounts for the melting of the material indicated by the flat bottoms of some of the larger craters.

(7) He conceives the yielding of the mass in collision to be a flattening out, spreading, and overflow at the margins of the area of impact. The rim would settle, and by flowing towards the centre give rise to an upward movement, occasioning the central hill; and the elastic recoil of the Moon's mass aided this effect. The upper part of the moonlet may not have been fused, and by remaining central in the crater, it may have been uplifted by the recoil to constitute the surface of the central hill.

(8) Certain surface phenomena "indicate that a collision of exceptional importance occurred in the *Mare Imbrium,* and that one of its results was the violent dispersion in all directions of a deluge of material — solid, pasty and liquid. Toward the southwest the deluge reached nearly to the crater *Theophilus,* a distance of 900 or 1,000 miles, and southward it extended nearly to the latitude of *Thibet.* Northward and northeastward it probably extended to the limb. Westward it passed beyond *Posidonius,* and toward the east and southeast its traces are lost in the *Oceanus Procellarum.* Its more liquid portion gathered on the lowlands, giving rise to several maria and minor plains. The fact has been recognized by various students, notably by GREEN (E. N. GREEN: *Jour. Brit. Ast. Ass.*, April, 1891, p. 379) and MEYDENBAUER (A. MEYDENBAUER: *Sirius,* February, 1882) that many of the lunar plains are due to floods of molten material overspreading the low-lying tracts and burying the preëxistent irregularities of surface. At various points in such plains, and especially at their margins, crescentic hills project above them, recognized as portion of crater rims; and elsewhere the plains are divided by systems of cracks whose arrangement betrays the distribution of underlying ridges. The plains most closely associated with the sculpture system and the supposed viscous deposit are the *Sinus Roris, Mare Frigoris, Lacus Mortis, Lacus Somniorum, Sinus Medii, Sinus Estuum,* and *Mare Nubium.* The *Oceanus Procellarum* may have been created at the same time or may have been merely modified by this flood. The *Mare Serenitatis,* whose sharp outlines and circular form mark it as an old crater, doubtless received a new surface." . . . . "By considering the extent and probable thickness of the various deposits from the flood, it has been estimated that its volume may have equaled a sphere 80 or 100 miles in diameter, and there is perhaps no occasion for surprise that the results of the collision of a body of such magnitude were exceptional in character as well as in extent." . . . . "Thus, by the outrush from the *Mare Imbrium* were introduced the elements necessary to a broad classification of the Lunar surface. A part was buried by liquid matter whose congelation produced smooth

plains. Another part was overrun by a flood of solid and pasty matter which sculptured and disguised its former details. The remainder was untouched, and probably represents the general condition of the surface previous to the *Imbrian* event."

(9) In regard to the white streaks radiating from *Tycho*, *Copernicus* and other craters, GILBERT quotes a letter from MR. WILLIAM WÜRDEMANN of Washington, D. C., to DR. B. A. GOULD, as follows: "The most remarkable appearance on the Moon, for which nothing on Earth furnishes an example, is presented by those immense radiations from a few of the larger craters — perfectly straight lines, as though marked with chalk along a ruler — starting from the center of the crater and extending to great distances over every obstruction. My explanation is that a meteorite, striking the Moon with great force, spattered some whitish matter in various directions. Since gravitation is much feebler on the Moon than with us and atmospheric obstruction of consequence does not exist, the great distance to which the matter flew is easily accounted for."

(10) GILBERT supposes the Moon to have grown by the gathering up of the moonlets once constituting a ring about the Earth. "As the Moon's mass grew, the blows it received were progressively harder, and for a time their frequency also increased. The rate of heating probably reached and passed its maximum while the mass was materially less than now. During the whole period of growth the surface lost heat by radiation, but the process of growth cannot have been slow enough to permit the concurrent dissipation of all the impact heat. On the one hand, there should have been some storage of heat in the interior, and, on the other hand, the stored heat can never have sufficed for the liquefaction of the nucleus. Toward the close of the process, when blows were hard but rare, liquefaction was a local and temporary surface phenomenon, but the general temperature of the surface was low."

From this account it will be seen that while the theory of GILBERT as originally set forth by him included many excellent features, it was not without difficulties; the methods of exposition and the terminology appealed rather to the Geologist than to the Astronomer and Physicist. But the main reason why the theory was not taken up by Astronomers is that it was little known to them, while the LAPLACE-DARWIN theory that the Moon was once a part of the Earth could not readily be reconciled with the doctrine of such powerful collisions as were required to produce such enormous craters. With our present point of view that the Moon is a captured planet which once had its orbit in the region where the asteriods now revolve, the traditional LAPLACIAN objection is overcome, and at the same time other asteroids once revolving in this region become available for making the observed indentations.

§ 164. *Theory of Impact Adapted to a Satellite Impinging Against a Globe More or Less Covered with a Layer of Cosmical Dust.*

We now come to a statement of the general theory of the so-called Lunar craters, namely, *that they are simply Satellite Indentations in a surface of loose and largely uncemented cosmical dust and fragmentary rock, and not volcanic at all.* It is to be remembered all through this discussion that the Moon's gravity is only about one-sixth that of the Earth, and therefore any layer of dust or rock on the Moon would be less settled than on the Earth in this proportion; so that the material would be only very slightly compressed by gravity. Moreover, as there has never been any water or other fluid on the Moon, except perhaps melted lava in places, there has been no chemical agency which could cement the dust or fragmentary rock into a coherent mass, and it has therefore retained the form of a loose mass of stony character, somewhat analogous to the ashes blown out of a terrestrial volcano. Falling satellites, whether large or small, could therefore easily indent the surface by compression of this uncemented material under the force of impact.

Accordingly, if a satellite fifty miles in diameter collided with the Moon, it would sink down into the soft and uncompacted surface, and at the same time be flattened and spread out at the base, as shown in the upper figure on Plate XII. This flattening and basal spreading of the satellite would make a broad saucer-shaped crater, steepest on the inside, and surrounded with more or less debris driven out and scattered by the force of the impact, exactly as shown by the Lunar craters. In flattening and spreading at the base, the satellite would force the walls of the crater outward, and itself be reduced to fragments, resting on a fluid base, with the highest peak in the centre. This is exactly the aspect presented by the larger craters. The peak within the rim is essentially central, though often irregular and broken, and it is impossible to doubt that this is how it arises. These central peaks are conspicuous in all large craters, except when they have been made flat-bottomed by the fusion of the satellite, or largely filled up by the deposit of cosmical dust. And it is to be presumed that the same cause has formed the smaller-sized craters, though in this case there may be less spreading of the mass at the bottom; for as the walls are very near the peak in the centre it will be inconspicuous, because after the expulsion of the walls by the force of the impact some of the loose material in so small a crater naturally slides down to cover it up. In most cases these satellite masses break into fragments, and give several peaks. Thus *Copernicus* has six central mountains, while *Alphonsus* alone exhibits a true central sharp-pointed peak.

HUMBOLDT concludes that "one of the most remarkable objects, however, on the whole surface of the Moon is the annular mountain range of *Petavius*, in which the whole internal floor of the crater expands convexly in the form of a tumor or cupola, and is crowned besides with a central mountain" (*Cosmos*, V. p. 449). He adds that the form on the Moon is permanent, whereas in our terrestrial volcanoes such convexity is temporary, and soon relieved by the escape of vapor. In the downfall of many satellites it would almost inevitably happen that the convex form would occasionally be preserved, in spite of the flattening and embedding of the mass. This, then, it seems, is the true explanation, and not the force of elastic vapors imagined by HUMBOLDT, by analogy with terrestrial volcanoes.

In connection with the formation of these craters, CAPTAIN A. W. DODD, U.S.N., has made several valuable suggestions resulting from his large experience in various kinds of target practice. He tells me that when the resisting surface is not too hard, experiments with projectiles indicate that the crater will have about three times the diameter of the impinging shell. Accordingly for the Lunar surface, with typical craters about sixty miles across, the impinging satellites probably had a diameter of some twenty miles, about like the planet *Eros* or the smaller asteroids. The accompanying figure C, Plate XII, shows the effect of bullets fired against a leaden disc. The lead was forced out in the form of a rim, while the embedded bullet remains as a central peak, just as observed in the Lunar Craters. It is difficult to imagine a more satisfactory verification of the theory than is afforded by these simple experiments, which CAPTAIN DODD devised, for imitating on a small scale effects shown in the heaviest target practice, and obviously applicable to the indentations produced by satellites colliding with the surface of the Moon.

Summary of the Phenomena Explained by the Theory of Impact.

It is found that the impact theory explains the following facts:

(1) Both large and small craters, and their superposition over one another, some being older and others newer, as the case may be.

(2) The frequent occurrence of small craters on the rims of large ones, where they would scarcely arise from eruptive causes.

(3) The existence of craters in perfectly smooth plains, as well as in rough and broken regions; and the unequal density of the craters in different parts of the Lunar surface. Terrestrial volcanoes generally follow the mountain ranges along the seacoasts. On the Moon the craters are scattered indiscrimi-

nately, except that they are rare in the maria, for reasons which will hereafter appear.

(4) The greater steepness of the inner walls, and the great diameters of the larger craters, which could not well be explained by volcanic forces. If it be thought that more large craters ought to be elliptical than are observed, it may be recalled that, even if the first contact with the Moon produced such an outline, the impact of a large satellite would generate enough heat and underlying flow to force out the walls about symmetrically all around, and the final figure would be circular like the globular figure of the satellite. Thus craters which are, say, ten times as wide as they are deep, ought to be almost circular; while smaller craters would be more irregular and elliptical, as found by observation. This is because the forcing out of the material beneath small craters is less effective than in the case of large craters, and they retain more nearly their original shape of first contact.

(5) The very flat-bottomed craters, noticed in such regions as *Mare Nubium,* are due to the filling up of deeper and more irregular craters with cosmical dust, or with melted material which has assumed a level surface. This has at length become so deep as to leave only the walls visible about a level central area, while the central peaks have been nearly or entirely covered up.

(6) In many cases the Lunar photographs show that even the walls are practically covered up; for they can now be traced with difficulty, and merely as a faint outline. The walls are obliterated, especially in the so-called maria. So far as one can see, two and only two, explanations of these so-called "ghost" craters are possible: (1) The deposit of cosmical dust from the heavens, and from the conflagrations arising in the impact of satellites; (2) The partial melting down of the walls by the conflagrations, which produced the maria, so that only an outline of the original crater walls can be traced. The fact that the "ghost" craters occur chiefly in the level maria supports the conflagration and melting hypothesis, and this certainly is one of the leading causes. But since the earlier craters away from the maria also show the effects of age, as if tending to become obliterated by falling dust, this latter cause also is at work. Moreover, the two causes necessarily are related. Together they explain the ageing of the craters in the rough regions far from the maria, as well as the buried or "ghost" craters in the maria themselves.

(7) This shows that many craters have not only been obscured and partly blotted out by the falling dust, but also that a countless number of these objects have been permanently buried by the process of deposit and conflagration. The so-called seas are areas once leveled down by melting, in which few recent craters

have been formed. The seas of the Moon appear to be singularly level, and this can only point to terrible impacts at some time in the past, by which these whole areas were so fused that pretty much all inequalities of elevation disappeared. They have since been covered with a layer of cosmic dust, but have suffered relatively few large indentations. They generally appear dark, because the surface is nearly level, and the Sun's light when reflected is but little scattered and seldom so directed that the beam from any considerable part of the surface passes near the eye of the observer.

(8) If this view be admissible, it also indicates that the whole Moon was formed by accretion, and that the surface never did experience true eruptive phenomena, such as we observe on the Earth.

(9) The interior of the Lunar craters is generally below the level of the surrounding normal surface, and this cannot well be explained except by impact. Volcanic eruptions could not well produce depressions of the crater basins.

"The bottom of many of the craters are very deeply depressed below the general surface of the Moon, the internal depth being often twice or three times the external height" (HERSCHEL, *Outlines of Astronomy*, § 430).

This remark of SIR JOHN HERSCHEL shows that decided depression of the basins is common to all craters, both those with rims and those without. It is almost impossible for volcanic forces to produce such a result. One or two Hawaiian volcanoes are the only depressed craters on the Earth, and they are recognized to be exceptions to the general rule of elevation characteristic of our planet.

(10) It is evident that the craters have not been produced by the removal of material from the center and the piling of it up to make the surrounding walls; for in probably three-fourths of the cases, as PROFESSOR H. EBERT has shown, it is easily proved by calculation that the volume of the excavation exceeds the volume of the material contained in the wall. This remarkable volume relationship would be explained if the matter beneath the crater were compressed by the force of impact, and only a part of it and of the falling satellite forced out to form the surrounding walls.

(11) The shorter streaks radiating from such centers as *Copernicus* and *Aristarchus* are easily explained. It is sufficient to suppose that the collision was so forceful that matter was scattered far out in all directions, and perhaps heated to fusion in the process; yet, as the Moon has little or no oxygen, it did not burn and blacken as meteoric stones do in falling on the Earth, but simply took on a fused and glassy aspect, which, by reflection, gives the brightness of the shorter streaks radiating from *Tycho* and its associates. This explanation was given by

MR. WÜRDEMANN, of Washington, D. C., many years ago, in a letter to DR. B. A. GOULD, but it seems to be but little known to astronomers.

(12) The long rays from craters such as *Tycho* are similar optical effects of glassy material falling on walls of craters lying nearly in a straight line, and radiating from this center. This is shown by the photographs. Any crater which had matter ejected from it radially, in the process of formation, will have a system of rays, due to the effect of the sunlight on the higher elements of the surface traversed by the rays running from the crater as a center.

(13) As the Moon's force of gravity is feeble, the vapor and metallic and lithic rain due to impact might be carried hundreds of miles, and these streaks due to material falling on corrugations and ridges might extend out from the craters for a considerable distance, and sometimes appear to be prolonged by coincidence with other crater walls, ridges or rays.

(14) The considerable number of craters which are simple depressions without sensible walls are to be explained by the comparative looseness of the material of the Moon's surface layers — which allows the mass to yield downward without throwing up much of a wall about the depression produced.

(15) The clefts are paths cut by glancing satellites, which thus leave a straight or curved line, according to the nature of the surface and the resistance and rebound. Photographs confirm this origin of the clefts, and show that they are not cracks but actual cuts, sometimes more than a hundred miles in length.

(16) Rills are cracks or offsets along walls of craters which often are more or less hidden by later deposits. They pursue in some cases an irregular course, and often may be due to settlement of loose material, as in landslides on the Earth. The terrible impacts which have formed the great craters have been so violent that it would be remarkable indeed if faults or landslips did not develop under the force of these mighty oscillations of the Moon's frail globe.

(17) Changes in the aspects of a crater due to caving in, settlement, etc., are always possible; but to be entirely certain that the change is real, the illumination has to be exactly the same as the two epochs, which is seldom possible. If the suspected changes are real, photography will eventually establish this fact.

(18) The covering up of ancient cities on the Earth is due to deposits of waste, rubbish and dust traceable to meteorological causes connected with the atmosphere, such as sand borne by the wind from the desert, etc. On the Moon, however, there is no atmosphere sufficiently dense to carry dust, and it must therefore be scattered by impacts and by direct descent from celestial space. The fact that the older craters are visibly covered up, is a tangible proof of the important part played by cosmical dust in the course of ages.

(19) The different degrees of obliteration shown by the various Lunar craters is an impressive witness to the progressive falling of cosmical dust, in a celestial world devoid of rain or other meteorological disturbance of any kind.

(20) At zero degrees centigrade the maximum molecular velocities of the atmospheric gases are found by DR. JOHNSTONE STONEY to be as follows: oxygen, 1.8 miles per second; nitrogen, 2.0; water vapor, 2.5; helium, 5.2; hydrogen, 7.4. These values usually decrease with the fall of temperature, but the slight modification thus arising is not very considerable for small changes.

(21) Now, at the surface of the Moon, the parabolic velocity is 1.5 miles (2.37 kilometres, cf. *A.N.*, 3992, p. 136), and therefore none of these atmospheric gases can be retained. For, although we do not know the Moon's temperature very accurately, it would seem that during the Lunar night, it must approach the absolute zero, while during the day it cannot well exceed the boiling-point of water. Accordingly, the above values are not sensibly altered by the admissible variations of temperature.

(22) Observations on the refractions of stars occulted by the Moon prove that if any sensible atmosphere exists at the Lunar surface, it does not exceed $\frac{1}{5000}$ part of the density of the terrestrial atmosphere. We may therefore conclude that no sensible atmosphere has ever existed upon the Moon, either before or since the capture by the Earth; but that the vapors there arising have congealed into dust or constantly escaped into space.

(23) The cosmical dust that falls upon the Moon therefore encounters no atmospheric resistance, but plunges headlong against the Lunar surface. Any vapor due to the force of collision quickly cools, and, if it condenses into solid particles, is precipitated as dust, and nowhere amounts to a permanent cloud. If it remains true gas, the molecules gradually escape into space.

(24) If now we compare the Lunar photographs with the accompanying imprints made by raindrops, and by bullets fired into a leaden disc as a target, we shall notice the most remarkable similarity in the two effects. The raindrops, however, are all fluid, and leave only saucer-shaped imprints, and no central peaks; whereas the leaden bullets and stony satellites indenting the Lunar surface would necessarily leave central peaks, in accordance with observations. Thus the Moon's surface can be nothing but fragments of rock filled with finer dust; and it is evident that it has never been molten as a whole and has never shown true volcanic activity, as known upon the Earth.

The last conclusion is confirmed from another point of view by an exact calculation given in *A.N.*, 4053, p. 345, showing that the total gravitational heat of condensation of the matter of the Moon would raise an equal mass of water

PLATE VIII. GENERAL MAP OF THE FULL MOON, SHOWING THE LOCATION OF THE CRATERS AND OF THE MARIA.

(As Seen by the Naked Eye.)

Plate IX. The Moon, Seven Days Old, Photographed by Ritchie at the Yerkes Observatory.

(*Naked Eye View.*)

Plate X. The Moon, Eight Days Old, Photographed by Loewy and Puiseux at the Paris Observatory, Feb. 13, 1894 (from *Atlas Lunaire*, Plate A).

(*Naked Eye View.*)

Plate XI. The Moon, Ten Days Old, Photographed by Loewy and Puiseux at the Paris Observatory, Feb. 23, 1896 (from *Atlas Lunaire*, Plate B).

(Naked Eye View.)

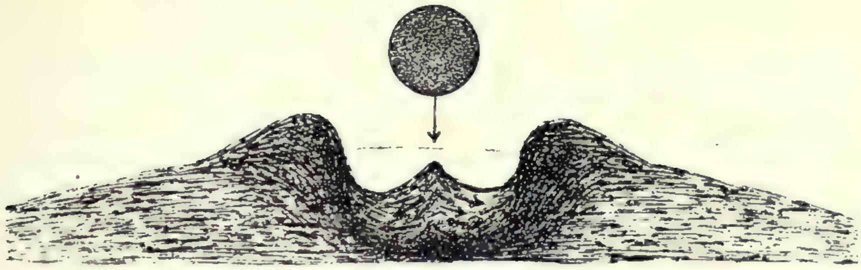


Fig. *a*. Satellite Colliding with the Lunar Surface.

Fig. *b*. Imprints of Raindrops, Made July 21, 1849, Kentville, Nova Scotia (from *Lyell's Geology*, p. 328).

Fig. *c*. Leaden Target Indented by Bullets, as an Illustration of Crater Formation, by Captain A. W. Dodd, U. S. N., Mare Island, 1909.

Plate XII. Illustrations of the Theory of Crater Formation by the Impact of Satellites Against the Lunar Surface.

PLATE XIII. LUNAR CRATER COPERNICUS AND SURROUNDINGS, PHOTOGRAPHED BY RITCHIE AT THE YERKES OBSERVATORY (*Publications*, VOL. II, PLATE VII).

(*Inverted or Telescopic View.*)

Plate XIV. Mare Serenitatis, Mare Tranquilitatis, and Surroundings, Photographed by Ritchie at Yerkes Observatory (*Publications*, Vol. II, Plate VIII).

(*Inverted or Telescopic View.*)

PLATE XV. MARE NUBIUM, BULLIALDUS, ETC., PHOTOGRAPHED BY RITCHIE AT THE YERKES OBSERVATORY (*Publications*, VOL. II, PLATE XVI).

(*Inverted or Telescopic View.*)

Plate XVI. The Moon, Fifteen Days Old, Photographed by Ritchie at the Yerkes Observatory.

(Natural or Naked Eye View.)

through only 408 degrees Centigrade. It is there pointed out (p. 348) that the development of such a small amount of heat, in the course of long ages, would not at any time give rise to a temperature that would produce fusion of rock. Even when radio-active substances are considered, the conclusion is the same—namely, that in the slow and almost insensible development of the Moon by accretion, enough heat to produce general fusion could not have arisen.

Accordingly we may dismiss the old volcanic theory once for all as false and misleading; and may look upon our satellite as a battered planet, which presents to us the most lasting and convincing evidence of the processes of capture and accretion by which the heavenly bodies are formed.

The strength of the present argument regarding the origin of the Lunar craters does not rest on one class of phenomena alone, but on several distinct classes of phenomena, which are all harmonized among themselves and brought into accord with the necessary processes of planetary growth. Since numerous worlds form in a nebula, it follows that impacts between some of them will necessarily occur; and the Moon's face shows the size of these masses by their imprints, which thus throw an unexpected light upon the state of the solar system in the remote past.

§ 165. *On the Temperature Produced by a Falling Satellite and on the Origin of the Atmospheres of the Planets.*

In regard to the origin of the atmospheres about the planets we may remark, that in the case of the Moon, a body moving towards the surface with the parabolic velocity of 2.37 kms. per second, would by collision produce a quantity of heat

$$Q = \frac{MV^2}{425 \times 2 \times 9.81} = \frac{MV^2}{8339} = \frac{[2370]^2}{8339} = 673.57 \text{ Calories,} \tag{369}$$

when the mass is one kilogram. This quantity of heat would raise the temperature of a kilogram of water 673 degrees centigrade. If the specific heat of the body be like that of most terrestrial stone, not exceeding 0.2, the effect on the meteoric stone striking the Lunar surface would be to raise its temperature through at least 3367° *C.*, provided all the heat were concentrated on the meteorite itself. But a considerable part of the energy would be transferred to the surface struck, and only the remainder would be available for raising the temperature. Some bodies, however, would collide with the Lunar surface with a velocity even greater than this, and perhaps as high as the parabolic velocity around the Sun, 42 kms. per second, giving for the value of Q 211530 Calories, and for stone a temperature of about a million degrees.

In view of these facts it is clear that much of the material falling on the Moon would be reduced to a state of dust and vapor. As there is no atmosphere there much of it would immediately fall down upon the Lunar surface as a metallic or lithic rain of very fine dust. The Lunar surface would therefore be covered with cinders and ashes somewhat analogous to that thrown out of terrestrial volcanoes. This cosmic dust would be soft and yielding like moderately settled volcanic ashes, and when a satellite struck it, there would be a scattering of debris and a throwing up of ridges such as we see about the craters. In some cases the satellite might have such velocity that after it buried itself in the Moon complete fusion would take place by the intense heat below. The crater would thus be filled with liquid lava and might solidify as an even surface perfectly level, such as we find in the case of the crater *Archimedes,* to which Professor Burkhalter kindly called my attention. In a vast number of such collisions there would naturally be every gradation between complete fusion and partial fusion, depending on the velocity of the impact; all classes of these phenomena seem to be presented to us on the Lunar surface. When the satellite traversed the surface at a small angle, the result would be a grazing collision, the satellite being cut away and vaporized by the resistance, and parts of the Lunar surface cut away and vaporized at the same time. This sudden collision would produce a vast cloud of vapor and lithic and metallic rain, like that which falls from Aereolites on the Earth; but in an hour or so it would again be precipitated upon the Lunar surface as fine dust. The surface of the Moon still shows that a number of such grazing collisions took place. In all cases the satellite kept on moving till it was burnt up or scattered in small fragments. In at least one case between *Triesnecker* and *Manilius* the satellite seems to have changed its course by rebounding from a Lunar mountain.

Now on a planet which can hold an atmosphere the vapors produced by impact, or by internal heat, gradually accumulate, and this at length gives the planet an atmosphere with gases and water vapor suitable to the maintenance of life. On a small body such as the Moon any gases produced are not retained, but lost into space, or absorbed in the rock. It is natural to suppose that the Moon has not lost her atmosphere, but simply never had one. The atmosphere of the Earth arose after our planet was considerably larger than the Moon is now.

In this connection one point called to my attention by Professor Burkhalter deserves attention. It has been noticed that in the so-called sea areas of the Moon, the craters have lower walls always toward these seas than on the opposite side; and the inference had been drawn that the movements of the water supposed to have once been in the Lunar seas had washed down the walls of the craters exposed to the tides. This would seem plausible enough, if real

seas of water could be admitted. But there is an even better explanation, as follows: When a satellite hits the Lunar surface, the walls thrown up on the different sides of the crater depend on the height of the surface at the point of contact, the lower wall being toward the lower ground. This law is verified in terrestrial experiments, and will always be true if the movement is so rapid that the projectile is not bent from its course by the unequal resistance on the several sides. Now the falling satellites near the Moon always move with considerable velocity, and such momentum, that the higher ground is crushed down and outward without any change in the direction of motion. The result is clearly a higher wall on the side of the higher ground, in accordance with observations. It is not strange therefore that the lower walls of the craters are towards the seas; for this naturally follows from the process of impact by which the craters are made. Moreover, if the inequalities once existing in the regions now covered by the maria have disappeared by melting, it would be natural for the walls towards the centre of the conflagration to be more perfectly obliterated than those nearer the margin of the area. This is another explanation of the phenomenon pointed out by PROFESSOR BURKHALTER. Accordingly we conclude that the craters have not been washed down by tidal oscillations in the Lunar seas, but that they always were lowest or were most melted on that side. There is therefore no evidence that seas ever existed on the Moon; and since water and air could not exist there now, as we infer from the theory of gases, and confirm by observation, it follows that water probably never did exist upon the Lunar surface, and there is no Lunar phenomenon really indicating this condition.

§ 166. *On the Imprints of Raindrops and on the Exact Process of Crater Formation as Inferred from Physical Experiments on Highly Elastic Solids.*

In connection with the formation of the Lunar craters it is important to recall that the experiments of TRESCA and ST. VENANT (*Sur l'Écoulement des Corps Solides, Memoirs des Savants Étrangers, Académie des Sciences de Paris, Tomes* 18 and 20) have shown that under the action of sufficiently great forces even very rigid and elastic bodies lose their rigidity and their elasticity and become plastic. Under such forces the most rigid solids yield and adapt themselves to the walls of a mold, and even flow from an open orifice as a heated mass of extremely high viscosity. Now in the impacts against the Moon's surface, as we have seen above, the forces are so enormous that not only could the satellites be crushed and made to flow, but also in many cases reduced to molten liquid or even vaporized. Moreover neither the satellite nor the Moon's globe is more rigid than average meteoric stone. Therefore it is certain that in collision both masses

would flow, the Moon's surface giving down while the satellite spreads out as a thin disc, thickest in the centre, the pressure forcing the yielding material out at the periphery and making a circular wall all around. This leads to the development of craters of exactly the type now observed on the Moon's surface, and there can, I think, be no possible doubt that this is the way in which the Lunar craters were formed.

The spreading of the falling raindrop as it imprints itself in the soft mud offers a case of crater-making where there is entire fluidity of the indenting globe and perfect plasticity of surface, but the forces producing the indentation and flow are very feeble. In the case of the impacts on the Moon both the globe of the satellite and the Lunar surface are moderately firm, but the forces producing the flow are tremendous. Therefore it is not surprising that the craters were produced and some of them melted in the process. One would naturally expect much vapor and dust to be scattered in all directions; and the streaks and spattering extending from the newer craters show unmistakably that this actually took place on a grand scale.

In the case of the filled crater *Wargentin*, which is a smooth circular tableland some fifty-four miles wide, standing several thousand feet above its base, and bearing a low wall about the greater part of its edge, we may suppose that when it was formed the Moon's surface did not give down as much as usual, and yet the force of the impact was such that very complete fusion ensued, and the level was only slightly lowered by a breach in the wall. The crater *Julius Cæsar* stands at a similar level, determined by the lowest breach in its south-western rim, but the surface is not so level as in the case of *Wargentin*, *Phocylides b* and some of their associates. The principal difference between *Wargentin* and *Archimedes*, for example, is in the level of the interior floors, the former being filled to overflowing, while the latter is only partially filled; but this is a difference of degree and not of kind, for the filling in both cases must be ascribed to exterior fusion and not at all to the rise of lava from beneath, as so many selenologists have supposed. As the Moon's globe as a whole has never been molten, it is not admissible to suppose that the interior poured forth fields of lava such as are occasionally found upon the Earth. All the molten rock noticed on the Moon appears to be superficial and local in character, as if determined by satellite impacts.

The criterion which seems absolutely conclusive against the volcanic theory is that based on the great size, peculiar shape and sunken character of the craters, and the fact that the volumes are greater than those of the rims about them. Moreover, it is to be noticed that no possible exertion of explosive forces directed from the top of the central peak could dig out the hollow circular trough between the

peak and the surrounding crater wall. This feature of the lunar craters is typical, and quite decisive against the volcanic theory. It seems practically impossible for a crater such as *Tycho* or *Theophilus* to have originated except by the impact of a satellite. It thus appears that the mountains on the Moon have no analogy whatever with those on the Earth. Terrestrial volcanoes have been uplifted by the action of explosive vapors, chiefly steam, and the active volcanoes are therefore situated near the sea, and our mountain ranges have been formed by the expulsion of lava from under the bed of the sea (cf. "The Cause of Earthquakes, Mountain Formation and Kindred Phenomena Connected with the Physics of the Earth," and three additional Memoirs by the writer in the *Proceedings of the American Philosophical Society*, Philadelphia, 1906-1908).

If it be remarked that the outer walls of a terrestrial volcano were formed by explosions which blew out the core, and the central peak has been built up by subsequent ejections from within; and the question be asked whether a Lunar crater might not have been formed by an enormous explosion producing a large hole in the center which was afterwards filled from beneath, we may answer this question emphatically in the negative, because if this had occurred the numerous flat-bottomed craters noticed on the Moon would not exist, no such craters being known on the Earth; and moreover, the central peaks in *Copernicus*, *Theophilus*, etc., would themselves have craters, which is not a fact. And finally, terrestrial volcanoes are all *elevated*, and their interior craters are perched much above the normal level of the surface; while on the Moon the craters are conspicuously depressed, and this situation cannot be reconciled with eruptive forces.

As a last objection against the impact theory it might be asked why, if the Lunar craters are due to the impact of satellites, the Earth does not exhibit batterings and indentations of the same kind. We answer that the geological changes on the Earth, due to the action of the oceans and the atmosphere, have entirely obliterated the original indentations of our globe; and since the continents were formed the Earth has suffered no important collisions with satellites, because the region in which it moves was long ago cleared of small bodies. This again indicates that the conspicuous indentations still visible on our Moon probably were made prior to its capture by the Earth; which affords us an impressive illustration of the enormous age of the solar system and of the great changes occurring on the Earth, while scarcely any have taken place on the Moon, because it has neither oceans nor atmosphere. As the Moon could not at present retain air or water, it is obvious that it has never had these elements; and for this additional reason it is impossible to suppose that the eruption of craters has ever been at work on our satellite.

Aside from the foregoing apparently conclusive arguments against the volcanic theory, the theory of satellite impacts presents the great advantage of harmonizing our conceptions of Lunar development with those known on other grounds to have operated in the growth of the planets; for all the largest bodies of our system have augmented their masses by the absorption of satellites. The impact theory thus accords with the known processes of cosmical evolution, which are simple and everywhere the same.

And as our Moon is a captured planet, on which surface changes are nearly insensible, owing to the absence of water and atmosphere, the study of its well-preserved surface phenomena is of the highest importance for throwing light upon the terrific process of planetary development by the capture of satellites.

§ 167. *Indications of Planetary Growth Furnished by the Indentations on the Moon.*

The conclusion that the Lunar globe as a whole has never been molten is confirmed from another point of view by an exact calculation given in *A.N.*, 4053, p. 345, reproduced in § 208 of this volume, showing that the total gravitational heat of condensation of the Moon would raise an equal mass of water through only 408 degrees centigrade. It is there pointed out that the development of such a small amount of heat, in the course of long ages, would not at any time give rise to a temperature that would produce fusion of rock. Even when radioactive substances are considered the conclusion is the same — namely, that in the slow and almost insensible development of the Moon by accretion, enough heat to produce fusion could not have arisen.

Accordingly we may dismiss the old volcanic theory once for all as false and misleading, and may look upon our satellite as a growing mass of cosmical dust, which presents to us the most lasting and convincing evidence of the processes of capture, collision, and accretion by which the heavenly bodies were formed.

The Moon now exhibits many large craters which are practically buried by the accumulation of molten rock and cosmical dust; and since these buried craters are no doubt as deep as those now observed in full view upon the surface, and therefore not far from five miles in depth, it follows that we have before us indisputable evidence of a layer of molten rock and cosmical dust covering the Moon to a depth of at least five miles. This is $\frac{1}{200}$ of the Moon's radius, and the natural inference is that the entire globe has been built up by this same process of accretion. In the past we might have entertained such an idea, but we could nowhere point to visible proof of the postulated process. If this be the true mode

of Lunar formation it will therefore prove extremely useful in the theories of Cosmogony.

As the South-Eastern quadrant of the Moon is that which is the most terribly battered by collisions, it is highly probable that this part was once in front, when our satellite was developing and revolving as an independent planet about the Sun. If this be so, it shows that after the Moon began to form, the position of the axis was sensibly shifted by collisions, but probably not greatly changed since this globe was captured by the Earth. Admitting this probable readjustment of axis, the present unsymmetrical position of the most strongly indented face of the Moon, shows that our satellite did not originate near the Earth, but came to us from the heavenly spaces. The indications are that the principal indentations on the Moon's face arose when our satellite was still an independent planet. To be sure, the contrast between her face and that of the Earth might be explained partly by the greater activity of atmospheric agents on our globe; but since no important impacts can be postulated on the Moon without implying corresponding catastrophes on the Earth, the essential absence of great terrestrial catastrophes during Geological History throws the Moon's construction back to a very remote epoch; and it therefore becomes more logical to hold that the history of the two bodies as a system does not run parallel further back than some four hundred million years. The outstanding inequality in the Moon's secular acceleration makes the rate of approach about one mile in 200,000 years, or 200,000 miles in 40,000,000,000 years. If the effect of resistance in the past averaged one hundred times what it is now, the total duration since the capture is reduced to about four hundred million years, and this result harmonizes very well both Astronomical and Geological phenomena. Whilst such estimates are very rough, they probably are sufficiently trustworthy to give us the correct order of the time involved, and that seems to be all we can hope for at present. Investigations such as these afford us an impressive illustration of the usefulness of the theory of capture and impact, without which many of the most wonderful phenomena in the heavens would remain utterly bewildering to astronomers.

§ 168. *The Surface Indications of the Planet Mercury and of the Other Satellites of the Solar System.*

The rough experience shown by the battered state of the face of the Moon gives us a valuable clue to the past history of the other bodies of the solar system. No doubt all the planetary bodies, both large and small, have gone through a similar experience, and we may conceive all of them, the Earth included, to have suffered

countless collisions with satellites of considerable size. Some of the bodies which collided with the Earth during its earlier history may have been as large as the Moon; while planets as large as the Earth may have collided with *Jupiter* and *Saturn*, *Uranus* and *Neptune*. It is clear that many such considerable masses have been swallowed up in laying the foundations of these immense planets, and a similar conclusion is even more emphatically true of the Sun. But the events here alluded to happened a long time ago, and no trace of the effect is now observable, unless it be in the great red spot on *Jupiter*, which may be a survival of such a catastrophe of comparatively recent date. All the large planets are covered by atmospheres which conceal from our view the internal state of the several bodies in question.

But if we consider a small planet such as *Mercury*, which cannot retain or build up an atmosphere of much density, owing to its feeble power of gravitation, we shall perceive that its surface ought still to be similar to that of the Moon, covered with great indentations and streaks due to the impact of satellites. Now in 1901, while observing under the best atmospheric conditions with the twenty-six-inch Equatorial at Washington, the author obtained the impression that the planet *Mercury* actually has a surface similar to that of our Moon. In view of our present knowledge of the causes which have produced the craters and larger markings on the Lunar surface, it is impossible to doubt that the impression gotten at Washington rests on a real foundation. We may therefore safely conclude that all the smaller planetary bodies, such as *Mercury* and the satellites, have battered surfaces essentially analogous to that of the terrestrial satellite, which alone admits of minute telescopic investigation.

Some markings have been observed on the satellites of *Jupiter* and *Saturn*, but their exact character has not been made out with entire certainty. The satellite *Iapetus* has a dark side, due no doubt to a dark area of large size similar to the so-called seas on the Moon. This satellite therefore is variable, and it has been possible to conclude from observations of its light that it shows always the same face toward *Saturn*. In the same way a number of the Asteroids are variable in brightness. They too have been battered up by collisions such as we see impressed on the face of our terrestrial satellite. It will be remembered that the planet *Eros* underwent considerable variations of brightness in 1901, and some observers then remarked that the light fluctuations would be most easily explained by the hypothesis that it is double. Though such a phenomenon would be extremely rare, it is clear that two asteroids might collide in such a way as to form a double planet of two globes cemented together. But whether *Eros* be double, or single, it is clear that the asteroid has suffered collisions in past ages, and now

has a figure of unequal brightness in its different parts. A similar remark applies to many of the Asteroids — they have been in collisions! And therefore their light fluctuations are not remarkable, but naturally to be expected. The intelligent study of our Moon therefore throws an unexpected light upon many celestial phenomena, which otherwise would remain very obscure.

On the Cause of the Variability of Certain Satellites of the Planets of the Solar System.

The observed variability in the brightness of certain satellites of the solar system has long been a source of perplexity to astronomers, and although several explanations of the phenomenon have been offered, probably none of them can be proved to rest on a true physical cause. The distance of the satellites is so great that in all probability our own Moon is the only one of these bodies which can ever be observed with much detail. It happens, however, that there is great similarity between our satellite and those of the other planets, and, moreover, that the cause which has operated in shaping the surface of our Moon is now apparently established beyond doubt. Our Moon always shows the same surface towards the Earth, and the same relationship seems to hold true for the principal satellites of *Jupiter* and *Saturn*. As pointed out in *A.N.*, 4343, these small bodies probably never had much axial rotation, but even that little has been destroyed by tidal friction of the central planet which now governs their motions. As an illustration of the variation of the light of the satellites, it will suffice to refer to the case of the *Saturnian* satellite, *Iapetus*. Soon after its discovery by *Cassini* in 1671, it became so faint that it was lost and not recovered until the following year. He then found that it regularly became invisible in the following half of its orbit, and his early conclusions were verified by Sir Wm. Herschel in 1792 (*Phil. Trans.*, 1792, p. 14) and are familiar to all modern observers. This regular fluctuation in the brightness has been explained by the circumstance that the satellite presents always the same face towards *Saturn*, just as the Moon does towards the Earth; and by the additional circumstance that different areas on the surface of the satellite are of very different degrees of brightness. But what is the cause of extreme dullness in certain areas, and great brightness in others? To answer this question it is sufficient to recall the dark areas on the Moon. These so-called maria are level plains in which the inequalities of surface have been obliterated by the heat of collision with a satellite of large size. Considerable sized areas have thus been melted, and now reflect but very little light, as we have seen in the above discussion on the origin of the Lunar Craters.

A similar cause has been at work on the surface of *Iapetus* and it thus happens that the face turned to us on the following side is unusually dark. The darkness is no doubt similar to that presented by the maria on the Moon and there is no reason why most of it should not be on one side. In fact, our Moon itself as seen from a distance would be slightly variable. Under the circumstances it is not remarkable that the larger satellites of *Jupiter*, as well as those of *Saturn* should exhibit unmistakable variability, depending on the conjunctions with their planets, as carefully investigated by Dr. Paul Guthnick in *A.N.*, 4023, and *A.N.*, 4098.

The brighter sides of these satellites are very rough and covered by craters such as we find abundantly on the Moon, while the darker sides have a preponderance of maria, and the result is the fluctuation in brightness found by observation.

The variability of the satellites of *Jupiter* and *Saturn*, photometrically investigated by Guthnick in *A.N.*, 4023, and *A.N.*, 4098, indicates that they, too, have maria covering their surfaces, due to collisions, as in the case of the Moon. For, as observed from a distance, the Moon also would be variable, according to the extent of the maria on the side towards the Sun. Lastly, the mathematical argument regarding the capture of the satellites and of the Moon is confirmed by Schroeter's observations, 1789-1793, showing that the planet *Venus* rotates in twenty-three hours, twenty-one minutes. For, if *Venus* has that period, the Earth never could have rotated faster than at present, and the Moon necessarily would be a captured planet. There is found also to be a theoretical reason why *Venus* ought to rotate faster than the Earth, so that the capture of the Moon is confirmed both by the observations of *Venus* and by mathematical theory, and the origin of the Lunar craters by impact is a necessary corollary to the capture theory of satellites.

§ 169. *The Terrestrial Mountain Ranges Entirely Different from Those on the Moon.*

The presence of considerable ranges of mountains on the Moon seems to call for a brief discussion of their mode of formation, and we should therefore pause to consider the mountains of the Earth, as the only other planet about which we possess any definite knowledge. It is a remarkable fact which cannot be too strongly emphasized, that the Terrestrial Mountains are entirely different from those on the Moon. We have seen that every indication furnished by the surface of the Moon points to impacts with other bodies as the origin of the scars now seen on her face. Not only the craters and smaller inequalities of surface, but also the longer ranges of mountains may be traced to this cause. For ranges such

as the *Alps* and *Apennines*, etc., are found to enclose areas such as the *Mare Imbrium, Mare Serenitatis*, and other so-called seas. And we have seen that the larger areas of the Moon, as well as the smaller ones, have been made by impact with other bodies like the Asteroids. Apparently the central areas have been melted, and walls thrown up around them, while much vapor has often been expelled in various directions. In the case of the Lunar Seas, the falling bodies have been of large size and the impacts of unusual violence; so that mountain ranges have been formed about the margins of the disturbed areas. A careful examination of the entire Lunar surface shows that this process has been general; the phenomena are so continuous that it obviously applies to all the ranges of mountains as well as to the craters observed on the surface of our satellite.

The process of satellite impact is entirely different from that which has been at work on the Earth, where the mountain ranges have arisen from the wrinkling of the crust produced by the leakage of the oceans and the expulsion of lava at the margins of the sea. This new Theory of Terrestrial Mountain Formation was developed by the writer in 1906-8, and published in four Memoirs included in the *Proceedings of the American Philosophical Society* held at Philadelphia.*

It is there shown that six great classes of phenomena, namely: (1) world-shaking earthquakes, (2) volcanoes, (3) mountain formation, (4) the formation of islands and plateaus, (5) seismic sea-waves, (6) the feeble attraction of mountains and plateaus, all depend on the movement of lava beneath the crust, due primarily to the leakage of the oceans. The Andes, for example, have been produced by the ocean injecting lava under the west coast of South America till it has finally erected a great wall along the border of the continent. The expulsion of the lava produces the terrible earthquakes, and the subsequent sinking of the sea bottom gives rise to the seismic sea-waves which have often proved disastrous to the western seaboard of South America.

The same process of mountain formation is beautifully shown in the Aleutian Islands, which are a mountain range still under water. The deep valley south

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*1. "The Cause of Earthquakes, Mountain Formation and Kindred Phenomena Connected with the Physics of the Earth." *Proc. Am. Philos. Soc.*, 1906.

2. "On the Temperature, Secular Cooling and Contraction of the Earth and on the Theory of Earthquakes Held by the Ancients." *Proc. Am. Philos. Soc.*, 1907.

3. "The New Theory of Earthquakes and Mountain Formation as Illustrated by Processes Now at Work in the Depths of the Sea." *Proc. Am. Philos. Soc.*, 1907; issued in March, 1908.

4. "Further Researches on the Physics of the Earth, and Especially on the Folding of Mountain Ranges and the Uplift of Plateaus and Continents Produced by Movements of Lava Beneath the Crust Arising from the Secular Leakage of the Ocean Bottoms." *Proc. Am. Philos. Soc.*, 1908; issued in September, 1908.

of the Aleutian Islands, formed by the sinking of the sea bottom when the lava is expelled from under it in great earthquakes, illustrates the process of mountain formation so plainly that no doubt can remain that the movement depends on the leakage of the waters of the ocean through the Earth's crust. The greatest plateaus of the globe, like the principal mountain ranges connected with them, face the greatest oceans, and they have all been uplifted by the same process.

The movement which results in mountain uplifts takes place in a thin layer of quasi-solid lava just beneath the crust. Otherwise the matter of the globe behaves throughout as solid, owing to the great pressure to which it is subjected. The rigidity of the Earth depends on the solidity produced by the pressure, both the temperature and pressure increasing towards the centre and keeping the matter everywhere solid and rigid, though at enormously high temperature.

From these considerations we see that mountain formation on the Earth and on the Moon depend on entirely different causes. On our planet, under present conditions, there would be no mountain formation without the oceans. As there have never been any oceans on the Moon, the formation of the Lunar mountains must necessarily have depended on a different process. And whilst the process of impact which still shows on the Moon's face has also operated in the primeval history of the Earth, it was earlier than the development of the mountain ranges now observed upon our planet. All traces of the earliest impacts of satellites against the Earth have been obliterated by the secular effects of the oceans and of the atmosphere, which have thus produced the whole series of sedimentary rocks that have preserved the records of the life history of the globe.

§ 170. *The Misleading Doctrine of the Secular Cooling and Contraction of the Earth.*

The misleading doctrine of the secular cooling and contraction of the Earth is so widespread in the literature of modern science that it calls for some notice in the present discussion. In the four Memoirs recently published in the *Proceedings of the American Philosophical Society*, to which allusion has already been made, it seems to be proved conclusively that the terrestrial mountains are formed by the sea, and that the shrinkage of the Earth is wholly insensible, and more than counteracted by a secular expansion of the globe, due to the leakage of the oceans. It is estimated by the best data now available that the expansion of the globe is from ten to one hundred times more rapid than the shrinkage due to secular cooling. No account is there taken of the secular development of heat

Fig. *a*. Map Showing the World Ridge (from Frye's *Complete Geography*, by permission of Ginn & Co., Publishers). It will be noticed that the high mountains and great plateaus everywhere face the outside, which is towards the water hemisphere. This map therefore bears impressive testimony to the truth of the new theory, and the world ridge stands as an everlasting witness to the secular action of the ocean in uplifting the land hemisphere of the globe.

Fig. *b*. Water Hemisphere, Which Has the World Ridge Around It.

Plate XVII. Illustrations of the Land and Water Hemispheres of the Terrestrial Globe, Confirming the Theory That the Mountains and Plateaus are Formed by the Oceans (from *Proc. Am. Philos. Society*, Vol. XLVII, No. 189, 1908).

due to Radio-Activity, because it is difficult to estimate this effect accurately; but if this could be included it seems certain that, aside from the expansion depending on the leakage of the oceans, it would be proved that the Earth is undergoing no secular shrinkage whatever.

To show the basis of this reasoning, without going into details, we may remark that, ignoring radio-activity, the loss of heat due to the secular cooling of the globe is such as to produce a contraction of the Earth's radius of only 1.5 inches in 2,000 years. Now in a single earthquake at Yakutat Bay, Alaska, in 1899, the sea-coast was raised for more than one hundred miles, by amounts varying from 3 to 47⅓ feet. Small subsidences also occurred in a few places, but the movement of elevation greatly predominated. As between a shrinkage of 1.5 inches in 2,000 years, and an uplift of from 3 to 47⅓ feet at a single disturbance, there is, of course, no comparison. How many such earthquakes would occur in 2,000 years cannot be accurately determined, but it is safe to say that one in a century would not be an excessive estimate. Obviously such an infinitesimal shrinkage as 1.5 inches in 2,000 years would produce no earthquakes whatever. Accordingly it is now generally recognized that earthquakes and mountain formation depend on the secular leakage of the oceans as their true physical cause.

Under the circumstances the antiquated doctrine of the secular cooling and shrinkage of the Earth cannot be too strongly condemned. It has been a prolific source of mischief, and wholly misleading in its effects upon scientific thought. Before we can make solid progress, this erroneous doctrine must be permanently given up; it is entirely devoid of foundation, and contradicted by the most obvious phenomena of Nature.

All the speculations in works on Geology and Physics of the Earth, implying a contraction of our globe, are therefore worthless. We must unreservedly abandon the idea that faults in the Earth's crust are traceable to such a cause; for it has been proved that all these movements are caused by the leakage of the oceans and by no other cause whatsoever.

For the same reason we are not to entertain the doctrine that faults on the Moon are due to the cooling of the Moon's globe. The entire Lunar globe has never been at high temperature, and its surface has never been wrinkled by cooling and contraction. Whatever may have produced the few faults suspected to exist on the Moon's face, it certainly was not secular cooling, as that erroneous doctrine has been applied to the Earth. In all probability the faults which may exist on the Moon are to be referred to the strains of the violent shocks accompanying the terrible impacts which our frail satellite has experienced in past ages. If the great elastic waves accompanying these terrible blows did not give

rise to some breaks of the Lunar surface, it would be very remarkable indeed. It is to such causes, both tangible and real, that we should refer all the phenomena of the Lunar surface. The theory of secular cooling cannot be admitted for two reasons: (1) The Moon, as a whole, was never at very high temperature, and therefore never cooled greatly; (2) secular cooling has produced no sensible effects on the terrestrial globe, and therefore is not likely to have been more effective on a globe which has cooled much less than our own.

CHAPTER XV.

New Theory of the Origin of the Planetary System.

§ 171. *Three and Only Three Admissible Hypotheses Regarding the Mode of Origin of the Planets and Satellites.*

(1) That the planets and satellites were originally detached from the central bodies which now govern their motions, by acceleration of rotation, or formed from the condensation of matter thus thrown off, as was imagined by Laplace in 1796.

(2) That the planets and satellites are all captured bodies, the planets having been captured by the Sun and the satellites captured by their several planets, as announced by the author in *Astronomische Nachrichten*, 4308.

(3) That the planets and satellites were formed right where they now revolve, by the agglomeration of scattered particles of cosmical dust.

These are all the hypotheses which need be considered regarding the formation of the Solar System. In the first hypothesis the bodies are thrown off, and thus go from the center outwards or at least remain behind in the assumed shrinkage of the central mass; in the second hypothesis they come from a distance inwards; while in the third hypothesis they neither approach to nor recede from the central masses, but remain at the same distance. Any intermediate hypothesis will partake of the properties of these three. Whatever be the true mode of formation of the planets and satellites, it will necessarily be included under one or more of these hypotheses, and all others are excluded.

1. As to the first hypothesis, that the bodies were detached from their central masses by acceleration of rotation, as imagined by Laplace, we need do nothing more than recall Babinet's criterion given below, in § 173, to assure ourselves, by the results of exact calculation, that the supposed detachment did not occur. It is thus shown that the centrifugal force was always much too small to overcome the centripetal force of gravity. Even in the case of *Saturn's* equator, where the conditions are most favorable for a separation, it never ex-

ceeded one-seventh part of the centrifugal force required to detach a satellite (cf. *A.N.*, 4341-42, and § 101, p. 212).

BABINET'S criterion is therefore absolutely decisive against the LAPLACIAN doctrine of detachment, and we are compelled to admit that it did not occur, even for the rings or any of the satellites of *Saturn*, and still less could it have occurred elsewhere in our system.

2. In regard to the second hypothesis, that the bodies have been captured, and added on from without, and thus come in from a distance, the proof already advanced is sufficient, and need not be repeated here.

3. That the bodies could be formed out of scattered dust right where they now revolve will not be admitted by any one who has considered the feebleness of the force of gravity and its inefficiency in producing aggregation, where disturbing and disrupting forces are at work.

It is recognized that gravitation is such a feeble force that it may produce globes out of scattered material only if disturbing causes are absent, otherwise aggregation cannot take place. As far back as 1861 KIRKWOOD had reached the conclusion that comets are disintegrated, and in 1884 he became impressed with the difficulty of imagining scattered masses to condense into globes. The tidal action of the sun had been shown to have a tendency to spread a loose non-resisting mass into a ring of matter diffused around the orbit; especially when the eccentricity is high, and the perihelion distance small. "Analysis seems to indicate that planets and comets have not been formed from rings but rings from planets and comets" (*Proc. Am. Philos. Soc.*, Vol. XXII, p. 109). So much was inferred from mathematical theory, and the theory found verification in the observed diffusion of meteoric trains along the paths of comets, which are dispersed in many cases with surprising rapidity.

This objection obviously does not hold against a mass of some size revolving in an orbit nearly circular. It is only in the case of such considerable masses, or smaller masses revolving at greater distances from the Sun, that we have stable conditions permitting of growth by accretion. It is well known that, within ROCHE'S limit, the bodies could not be detached as single masses, because they would be torn to pieces by tidal action. And if detached as fragments, within this limit, aggregation could not occur so long as the pieces remained near a large central body, because the disrupting tendency would prevent the small bodies from uniting. And at any moderate distance from a dominant central mass the particles would still have unequal velocities of angular movement, so that they could not unite under the feeble attraction of their own gravitation; and the result would be a swarm of dust analogous to what we see in

the rings of *Saturn* and in the matter revolving about the Sun and producing the Zodiacal Light.

Accordingly, while the planets and satellites already formed could gather up and do actually gather up some waste material where they now revolve, they could not possibly have started as mere particles of dust so close to the large masses which now govern their motions. In a region so near large masses the dispersive and disruptive tendency would prevent the development of sensible nuclei from scattered particles of fine dust. It is therefore impossible for the planets and satellites to have formed where they now revolve, and the last of the three hypotheses is wholly excluded from consideration.

As the first hypothesis is excluded by BABINET'S criterion, while the third is eliminated by the considerations based on the tendency to dispersion just adduced, there remains only the second as a possible mode of formation of the planets and satellites. By the logical process of exclusion we are thus restricted to the Capture Theory, and it therefore follows that the planets have been captured by the Sun, and the satellites captured by their several planets. This is the only admissible hypothesis respecting the mode of formation of the solar system, and the mathematical theory of the capture of these bodies already developed shows that it is the true law of Nature.

§ 172. *Statement of the Objections to* LAPLACE'S *Theory.*

Although BABINET'S criterion establishes the inadmissibility of the LAPLACIAN theory, and it therefore can no longer be considered to represent a true cosmical process, yet owing to the important part played by this now abandoned theory in the past history of Cosmogony, it seems advisable to give a condensed resumé of the principal objections to this celebrated theory.

1. A system of distinct rings could not form and separate as imagined by LAPLACE; but if the rotation were rapid enough to produce detachment, the matter lost would be in the form of a continuous zone of uniformly scattered particles, without noticeable division into rings. And obviously such a swarm of particles could not condense into planets and satellites, but would remain a mass of cosmical dust.

2. If a system of rings existed, they could not condense into a system of planets or satellites, because the feeble attraction of gravity would not enable the diffused particles of dust to get together into one mass. The criticism of NEWCOMB on this point (*Popular Astronomy*, edition of 1878, pp. 504-6; p. 523), is still valid, and could hardly be improved upon.

3. In general, the nebulae of the heavens are shown to take the spiral form, because the coils of the nebulae are not in equilibrium, but gradually settling and winding up under the attraction of gravitation. In such a spiral nebula there is little or no hydrostatic pressure; and therefore there is no connection between the coils of a spiral nebula and Laplacian Rings, because by hypothesis the latter are thrown off under conditions of fluid equilibrium, while these hydrostatic conditions are seldom or never present in a spiral nebula.

4. In the Laplacian Theory a satellite cannot revolve faster than its planet rotates on its axis; this Laplacian condition is contradicted, however, by the observed motion of *Phobos*, the inner satellite of *Mars*, and by the rapid revolution of the rings of *Saturn*.

5. And in the table of data deduced from the application of BABINET'S criterion, we see that nearly all of the phenomena of the solar system unmistakably contradict the theory of LAPLACE; so that the classical nebular hypothesis is entirely untenable and must be permanently abandoned.

6. MOULTON'S *Criticisms of the Laplacian Theory.* In the *Astrophysical Journal*, Vol. XI, March, 1900, PROFESSOR F. R. MOULTON has several destructive criticisms of LAPLACE'S Theory which are of considerable weight. We shall notice especially that based on the lack of constancy in the moment of momentum, which mechanically is closely related to BABINET'S criterion, and leads to similar results. Let M represent the moment of momentum of a sphere of the radius R rotating with the angular velocity ω, and σ the density; then we have for the solar nebula the expression:

$$M = \omega \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \sigma r^4 \sin^3 \theta \, d\theta \, d\phi \, dr. \tag{370}$$

Suppose the density depends upon the distance from the centre; then $\sigma = f(r)$. Substituting in (370) and integrating, we have

$$M = \frac{8}{3} \pi \omega \int_0^{R} f(r) r^4 dr. \tag{371}$$

Using the laws of density employed by DARWIN in his paper of the "Mechanical Condition of a Swarm of Meteorites," p. 25, MOULTON finds by quadrature the following values of the moment of momentum:

$M = 32.176$, when the nebula extended to the orbit of *Neptune*;
$M = 13.250$, when the nebula extended to the orbit of *Jupiter*;
$M = 5.690$, when the nebula extended to the orbit of the *Earth*;
$M = 3.400$, when the nebula extended to the orbit of *Mercury*;
$M = 0.151$, the value in the system at present.

"Instead of being a constant, the moment of momentum is found to vary in a remarkable manner. On account of the approximations made the first number is somewhat too small, while the last is too large, as the Sun was assumed to be homogeneous in computing the moment of momentum which enters into it. Notwithstanding these errors in opposite directions, the moment of momentum in the first case is 213 times that in the last. It follows from these figures that if the mass of the solar system filled a spheroid extending to *Neptune's* orbit, and rotated with a velocity sufficient to make its moment of momentum equal to that of the present system, and if it then contracted with the law of density always that adopted above, the centrifugal force would not equal the centripetal until it had shrunk far within *Mercury's* orbit. Such an enormous difference cannot be ascribed to uncertainties in the law of density or to approximations in the mechanical quadratures; but it points to a mode of development quite different from, and much more complicated than, that postulated in the nebular theory under discussion."

7. The rarity of the solar nebula postulated by LAPLACE, namely, two hundred and sixty million times less than that of atmospheric air at sea level, was such that it could not transmit hydrostatic pressure from the centre outward; this objection was strongly urged over forty years ago by KIRKWOOD and PEIRCE; and it alone would exclude the possibility of the process imagined in the classic theory, which requires the exertion of hydrostatic pressure.

8. Retrograde satellites like those of *Jupiter* and *Saturn* cannot be reconciled to the Laplacian hypothesis, without assuming planetary inversion, which is not admissible. An equally great difficulty arises in any attempt to explain the retrograde systems of *Uranus* and *Neptune*.

Many other objections more or less related to these could be urged against the hypothesis of LAPLACE, but it is not necessary to thrice slay the slain; and we therefore dismiss the subject as requiring no further consideration. Even if the form of the Laplacian hypothesis be modified as suggested by FAYE, ROCHE and other writers, the above objections will still be quite insurmountable, and there is therefore no course open to us but to permanently and unconditionally abandon it.

§ 173. BABINET'S *Criterion Based on the Mechanical Principle of the Conservation of Areas.*

To establish the truth of the proposition that the planets and satellites have not been detached by acceleration of rotation, as imagined by LAPLACE, but on the contrary have been captured and added on from without, and have since had their orbits reduced in size and rounded up under the secular action of the nebular resisting medium formerly pervading the solar system, it is sufficient to repeat the argument advanced in *A.N.*, 4308, and to apply to the motion of these bodies a criterion based on the mechanical principle of the conservation of areas, which was proposed by BABINET in 1861 (*Comptes Rendus*, Tome 52, p. 481, March 18, 1861). In the paper of BABINET here referred to it is shown that if ω be the Sun's angular velocity of rotation with radius r, and ω' and r' the same quantities when the matter of this globe is expanded into a sphere of radius r'; then by the law of the conservation of areas

$$\omega' r'^2 = \omega r^2 . \tag{372}$$

As to the derivation of this formula it is sufficient to recall that the square of the radius is the same as that which appears in the moment of inertia of a rotating mass, and the product of this by the angular velocity ω gives the moment of momentum, which is always a constant in any free rotating system subjected to no forces except the mutual attraction of its parts. Thus

$$C = \sum m r^2 \omega = \omega \sum m r^2 = \omega' \sum m r'^2 . \tag{373}$$

In the case where the matter of the Sun is expanded to fill the orbit of the Earth, we take $r = 109.5$, and $r' = 23445$, and get ω' $(23445)^2 = \omega$ $(109.5)^2$.

And the time of the Sun's rotation, when the matter of that globe is expanded to the Earth's orbit, becomes

$$25^d.3 \frac{\omega}{\omega'} = 25^d.3 \left(\frac{23445}{109.5}\right)^2 = 3192 \text{ years.} \tag{374}$$

And when the matter of the Sun is imagined expanded to fill the orbit of *Neptune*, at a mean distance of thirty, we have for the time of rotation of the hypothetical solar nebula

$$25^d.3 \frac{\omega}{\omega'} = 25^d.3 \left(\frac{30 \times 23445}{109.5}\right)^2 = 2888533 \text{ years.} \tag{375}$$

Adopting the system of constants for the solar system employed in the paper in *A.N.*, 3992, with the latest results for the new satellites of *Jupiter* and *Saturn*,

we find for the entire solar system the following table, which requires no further explanation.

TABLE SHOWING THE APPLICATION OF BABINET'S CRITERION TO THE PLANETS AND SATELLITES, WHEN THE SUN AND PLANETS ARE EXPANDED TO FILL THE ORBITS OF THE BODIES REVOLVING ABOUT THEM.

Solar System.

Planet.	R_0 The Sun's Observed Time of Rotation.	P_0 Observed Period of Planet.	R_c Time of the Sun's Rotation Calculated by BABINET'S Criterion.
Mercury	25.3 days = 0.069267 years	0.24085 years	479 years
Venus	....	0.61237 "	1673 "
The Earth	....	1.00000 "	3192 "
Mars	....	1.88085 "	7424 "
Ceres	....	4.60345 "	24487 "
Jupiter	....	11.86 "	86560 "
Saturn	....	29.46 "	290962 "
Uranus	....	84.02 "	1176765 "
Neptune	....	164.78 "	2888533 "

Subsystems.

Planet.	Satellite.	R_0 Adopted Rotation of Planet.	P_0 Observed Period of Satellite.	R_c Time of Planet's Rotation Calculated by BABINET'S Criterion
The Earth	*The Moon*	1 day	27.32166 days	3632.45 days
Mars	*Phobos*	$24^h.62297$	7.6542 hours	190.62 hours
	Deimos	....	30.2983 "	1193.52 "
Jupiter	V	9.928	11.9563 hours	64.456 hours
	I	....	1.7698605 days	14.60 days
	II	....	3.5540942 "	36.900 "
	III	....	7.1663872 "	93.933 "
	IV	....	16.7535524 "	290.63 "
	VI	....	250.618 "	10768.8 "
	VII	....	265.0 "	11602.4 "
	VIII	....	930.73 "	61997.2 "
Saturn	Inner edge of ring	10.641	0.236 days	0.6228 days
	Outer edge of ring	....	0.6456 "	2.383 "
	Mimas	....	0.94242 "	4.2902 "
	Enceladus	....	1.37022 "	7.0615 "
	Tethys	....	1.887796 "	10.822 "
	Dione	....	2.736913 "	17.751 "
	Rhea	....	4.517500 "	34.620 "
	Titan	....	15.945417 "	186.05 "
	Hyperion	....	21.277396 "	273.06 "
	Iapetus	....	79.329375 "	1580.1 "
	Phœbe	....	546.5 "	20712 "
Uranus	*Ariel*	10.1112 (cf. *A.N.* 3992)	2.520383 days	33.714 days
	Umbriel	....	4.144181 "	65.435 "
	Titania	....	8.705897 "	176.05 "
	Oberon	....	13.463269 "	314.83 "
Neptune	Satellite	12.84817 (cf. *A.N.* 3992)	5.87690 days	141.8 days

It will be seen from a study of the remarkable data given in this table that the hypothetical solar nebula, when it extended to the orbits of the several planets, as imagined by LAPLACE, could not have rotated with sufficient velocity to detach any of these masses. This inference was already drawn by BABINET in 1861. Indeed he first applied the criterion to the cases of the Earth and *Neptune*. I have developed the table to show moreover that none of the satellites could have been detached from their planets, and thus the argument against the detachment of the planets and satellites is complete and unanswerable.

By the logic of exact data based on a mechanical law of unquestioned validity and without any assumptions as to the law of internal distribution of density except that it remains unchanged, we are thus compelled to admit that the premise adopted by LAPLACE was false and unjustifiable, when he supposed that the planets were detached from the Sun, and the satellites detached from the planets by acceleration of rotation. Thus this venerable explanation of the roundness of these orbits falls to the ground.

Now the planets and satellites could be formed in but one of two possible ways: (1) They might conceivably have been detached from the central masses which now govern their motions, by acceleration of rotation, as supposed by LAPLACE. (2) They might have been original nuclei captured in the midst of the solar nebula, and afterwards gradually built up by the agglomeration of more cosmical dust, while at the same time the orbital motion in this resisting medium would have reduced the major axes and eccentricities of their orbits and thus produced the near approach to perfect circularity now observed in our solar system.

The third hypothesis considered in § 171 has already been disposed of, and need not be considered here.

We have, however, just proved, by the application of BABINET's criterion, based on the law of areas, that these bodies could not have been detached from the central masses about which they now revolve. Accordingly it follows that they were all captured, and have since had their orbits reduced in size and rounded up under the secular action of the resisting medium formerly pervading the planetary system.

§ 174. *The True Physical Cause of the Roundness of the Orbits of the Planets and Satellites is to be Found in the Resisting Medium Formerly Pervading Our Planetary System.*

As was first pointed out in *A.N.*, 4308, January 1, 1909, there is absolutely no escape from this unexpected conclusion. For we may prove it by the following reasoning. The effect of a resisting medium in reducing the major axis and eccen-

tricity of the orbit of the resisted body is fully recognized, and has been known for more than a hundred years. The formulae for the changes of these two important elements may be reduced to the form (cf. equations (172) and (178))

$$\delta a = -\frac{2a^2}{p^2(1+m)}[A' v + \text{periodic terms}], \tag{376}$$

$$\delta e = -\frac{2}{p}[A e v + \text{periodic terms}], \tag{377}$$

where p is the parameter of the orbit, a the semi-axis major, e the eccentricity, v the true anomaly, and m the mass of the resisted planet, and A and A' constants. As both of these expressions are negative, it follows that under the secular action of a resisting medium the major axis and eccentricity always decrease. In deriving these formulae, however, the density of the resisting medium is supposed to increase towards the center, conformably to what is observed in the nebulae and shown to result from the theory of gases.

Laplace himself has discussed this question with characteristic penetration in the *Mécanique Céleste* (Liv. X, Chapter VII, § 19). He shows that when the density of the medium, represented by $\phi\left(\frac{1}{r}\right)$, increases towards the Sun, the semi-axis major and eccentricity always decrease; and finally remarks: "Therefore at the same time that the planet approaches towards the Sun, by the effect of the resistance of the medium, the orbit will become more circular." It is surprising that it did not occur to the author of the *Mécanique Céleste* that the roundness of the orbits of the planets and satellites could be explained by a resisting medium quite as easily and simply as by the theory of a rotation which would gently detach these masses and set them revolving in orbits which are nearly circular, especially since the nebular hypothesis itself necessarily implies the existence of such a medium where the planets and satellites now revolve. Laplace merely remarks that if the nebula filled the whole of this space the bodies would encounter such resistance as to cause them to fall into the Sun; but in making this statement he overlooked the fact that most of the nebulous matter did go into the Sun and planets, and it is from this circumstance that the central masses became so preponderant, while the attendant bodies are in all cases so very small.

During a recent conference with my friend Professor George Davidson, I mentioned Laplace's proof that a resisting medium had formerly acted against *Jupiter's* Satellites I, II, III, to bring about a near approach to commensurability in their mean motions, and thus enable their mutual attraction to establish a rigorous relationship under the influence of this slowly acting cause. This vener-

able astronomer justly remarked: "LAPLACE had the true cause in sight, but he did not carry it far enough to discover the actual process by which the solar system was formed."

Evidently LAPLACE had not tested his nebular hypothesis by the criterion based on the conservation of areas, afterwards proposed by BABINET, and it simply did not occur to him that the circularity of the orbits pointed unmistakably to the secular action of a resisting medium. As the very existence of a nebula implies resistance to bodies revolving within it, this oversight is the more remarkable; and unfortunately not only was LAPLACE's reasoning vitiated, but an equally disastrous effect exerted on all other investigations in Cosmical Evolution for more than a century, because all mathematicians followed the same line of thought, on the false premise that the planets and satellites were detached from the central bodies which now govern their motions.

If in the light of this new theory of the shaping of the orbits under the secular effects of resistance we examine our solar system carefully we shall find many phenomena confirming the former existence of such a medium in our system.

It must suffice here* to call attention to but a very few of the numerous survivals of the primordial resisting medium still shown by our system:

(1) The rapid motion of *Phobos*, the inner satellite of *Mars*, which has been brought down near the planet by resistance, till it now revolves in less than a third of the time of the planet's rotation. It is true that PROFESSOR SIR G. H. DARWIN explains this motion of *Phobos* by a tidal retardation of the axial rotation of *Mars*, but in view of the large part undeniably played by the resisting medium in the formation of our system as a whole this explanation will not hold, though a very small part of the observed effect may be traceable to Tidal Friction.

(2) The famous inequality in the motions of the three inner *Galilean* satellites of *Jupiter*, which points unmistakably to a resisting medium, as was sagaciously pointed out by LAPLACE in 1796. His remarks on this subject are as valid and convincing as any which could be made to-day.

(3) The observed rapid motion of the inner ring of *Saturn*, which greatly exceeds the axial rotation of the planet. The rings evidently were never detached from the planet, but simply survive out of a much larger mass of cosmical dust which has been absorbed in building up the mass of *Saturn*. All the data in the table relative to BABINET's criterion bear on this same question.

(4) The general fact that the satellite orbits are so round, and in general rounder and rounder the nearer we approach the planets, confirms the capture of these bodies in a medium which was denser towards these centers. The round-

* This summary is left substantially as it stood in *A.N.*, 4308.

ness of the satellite orbits shows that resistance was very effective from *Mars* to *Neptune*, and therefore no doubt throughout our whole solar system.

(5) The retrograde motion of *Saturn's* Satellite *Phœbe* and *Jupiter's* Eighth Satellite is likewise to be explained by the capture of these bodies. Whatever may have been their original eccentricities at the time of capture, even retrograde directed bodies could have survived, because the medium against which they revolved was of very slight density at that great distance from the planets. A very small density of the resisting medium at this distance is also indicated by the survival of considerable eccentricities in the orbits of these two satellites. The eccentricity of the orbit of *Phœbe* is given as 0.22, that of *Jupiter* VIII as 0.44, which are certainly anomalous enough to excite our suspicion. It is not by chance that retrograde motion in these two cases is associated with the highest eccentricities observed among all the satellites thus far discovered.

(6) The orbits of the Asteroids have been gathered into their present positions mainly by the action of *Jupiter* and of the resisting medium. Originally they were more widely distributed over the whole system than at present; but even now they overlap the orbits of *Jupiter* and *Mars* and there may be others of still wider range.

(7) The extreme roundness of the orbit of *Neptune* is a clear indication that this planet moved for a long time against a vast amount of nebular resistance. Therefore it is very improbable that our planetary system terminates with *Neptune*. In all probability there are several more planets beyond the present boundary of the system, some of which may yet be discovered.

(8) The equatorial accelerations noticed on the globes of the Sun and of *Jupiter* and *Saturn*, are to be explained by the falling in of matter revolving in vortices about these bodies. As the orbital motion of this matter near these bodies exceeds that depending on the axial rotation, the falling particles necessarily produce an equatorial acceleration. This process may still be going on; at any rate it has been in progress so recently that the effects still continue.

(9) The solar system was formed from a spiral nebula, revolving and slowly coiling up under mechanical conditions which were essentially free from hydrostatic pressure. And spiral nebulae themselves arise from the meeting or mere settling of streams of cosmical dust. The whole system of particles has a sensible moment of momentum about some axis, and thus it begins to whirl about a central point, and gives rise to a vortex. In the actual universe the spiral nebulae are to be counted by the million, and it is evident that they all arise from the automatic winding up of streams of cosmical dust, under the attraction of their mutual gravitation. The two opposite branches of the spiral nebulae, so often shown

on photographs, represent the original streams of cosmical dust which are coiling up and forming gigantic spiral systems.

(10) When the nebula rotates and the coils wind up in such a way as to leave open spaces between the coils, or at least freedom from sensible hydrostatic pressure, the usual result is the development of a system made up of small bodies, such as the planets compared to the greatly preponderant Sun, or the satellites compared to the much greater planetary masses which control their motions. In the solar system where the conditions are accurately known this is proved to have occurred; and it was repeated so many times, always with uniform results, giving a large central mass and small attendant bodies, that the general law for this condition is clearly established.

(11) If the streams so converge that the nebulous mass becomes very concentrated at the center, so as to become a figure of equilibrium under the pressure and attraction of its parts, the nebula may divide into a double star, as I have elsewhere inferred from the researches of POINCARÉ and DARWIN on the figures of equilibrium of rotating masses of fluid.

(12) Now both of these forms of development are abundant in the actual universe, and probably almost all of the apparently single stars are surrounded by systems of planets. Evidently there is one continuous process by which both types of systems are produced, and it appears to depend on *nebular* as distinguished from *fluid fission*, considered in § 112, p. 235. Therefore, whilst the usual result of the condensation of a spiral nebula is the development of a large central mass attended by much smaller bodies, the circumstances may be such that most of these smaller globes unite with the largest attendant planet and the system thus becomes a double star. Hence the extreme types of cosmical systems.

The effect of this work will be to give the Theory of the Resisting Medium the highest importance in all researches relating to the History of the Universe. It is very remarkable that the principal secular effects of this cause are exactly opposite to those due to Tidal Friction as investigated by DARWIN. For while Tidal Friction usually increases the major axis and eccentricity of an orbit, the resisting medium as regularly decreases both of these elements. In the actual physical universe both causes are at work together, sometimes one influence predominating and then the other. Resisting medium is relatively most effective in a system made up of a large central Sun, and small attendant bodies, such as the planets of our solar system; and as the systems of satellites dominated by large planets. Tidal Friction is most effective in systems made up of two large masses, such as the double stars.

It has seemed advisable to call attention to the cause of the roundness of the

orbits of the planets and satellites, because it appears likely that the criteria now introduced may go far towards clearing up the mystery which has always surrounded the origin of our solar system.

§ 175. *Further Considerations on the Theory of the Rotation of the Principal Planets, and on the Growth of the Minor Globes Which Have Finally Become Satellites.**

In Chapter XI, on the "Origin of the Lunar-Terrestrial System by Capture," (cf. *A.N.*, 4343), attention has been called to the fact that under the conditions existing in nature it is impossible for bodies of small mass to acquire rapid rotation. We shall now examine this question in more detail, so as to make clear the conditions which may lead to rapid rotation, and *vice versa*. If we adopt the Laplacian law of density (cf. *A.N.*, 3992, eq. 21), we have

$$\sigma = \sigma_0 \frac{\sin (qx)}{qx} = \frac{L \sin \left(q \frac{r}{a}\right)}{\frac{r}{a}}; \tag{378}$$

where for the Earth $\sigma_0 = 11.215$, water $= 1$; $q = 144° \ 53' \ 55''.2 = 2.528959$ radians; $x = \frac{r}{a}$, a being the Earth's radius, and r the radius of any shell; $L = 4.43463$; and we shall find for the mass

$$M = 4\pi \int_0^a \sigma r^2 dr = 4\pi a^3 L \int_0^a \frac{r}{a} \sin \left(q \frac{r}{a}\right) d\frac{r}{a} = \frac{4\pi a^3 L}{q^2} [\sin q - q \cos q]. \tag{379}$$

If I represent the Earth's moment of inertia for the Laplacian law, when the figure is considered spherical, we shall have

$$I = \int r^2 \sin^2\theta \, dm = \int_0^{2\pi} \int_0^{\pi} \int_0^a \sigma r^4 \sin^3\theta \, dr \, d\phi \, d\theta \, . \tag{380}$$

If we use the law indicated in (378) for σ, we shall get

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^a \frac{L \sin\left(q \frac{r}{a}\right)}{\frac{r}{a}} r^4 \sin^3 \theta \, d\theta \, d\phi \, dr = \frac{8\pi a^5 L}{3q^4} \{3(q^2-2)\sin q - q(q^2-6)\cos q\}. \tag{381}$$

Introducing the result indicated in equation (379), this becomes

* cf. *A.N.*, 4358.

$$I = \frac{2}{3q^2}\left\{\frac{3(q^2-2)\sin q - q(q^2-6)\cos q}{\sin q - q\cos q}\right\} Ma^2 . \tag{382}$$

In homogeneous spheres the value of I is found to be $0.4Ma^2$, but for LAPLACE's law as applied to the Earth, the value is found to be less; namely,

$$I = 0.331278\, Ma^2 . \tag{383}$$

We see therefore that the fraction 0.331278 is determined by the expression

$$k^2 = \frac{2}{3q^2}\left\{\frac{3(q^2-2)\sin q - q(q^2-6)\cos q}{\sin q - q\cos q}\right\} . \tag{384}$$

Now the moment of momentum of axial rotation H is the product of the moment of inertia by the angular velocity:

$$H = I\omega = k^2\omega Ma^2 . \tag{385}$$

And this equation enables us to recognize the different factors which enter into the expression for the moment of momentum of axial rotation. Let the mass and the mean radius be fixed; then it is evident that H will be large only when I and ω are large. Or if k^2 also be fixed by LAPLACE's law, and taken to be 0.331278 in the case of the Earth, then the value of H will depend simply on ω. Thus H will depend wholly on the impulse by which rotation is established and the angular velocity developed.

If k^2 be not fixed, but variable in any manner, then with constant mass and mean radius H will depend on k^2 and ω conjointly. To make k^2 a maximum the density has to be a maximum at the surface of the sphere; but this condition is dynamically unstable, and such arrangements neither arise in nature, nor would they long endure if started by artificial means. In the observed nebulae the density increases fairly rapidly towards the center, and the same law obviously holds among the stars, and planets (cf. *A.N.*, 4053).

If the arrangement of the internal density followed the law for a monatomic gas, as outlined in *A.N.*, 4053, and there applied to the Sun, major planets and fixed stars, we should have (cf. Chapter XVII, eq. (ζ))

$$\left.\begin{aligned} \sigma &= \sigma_0 \left\{\frac{1}{x^2}\frac{d\mu}{dx}\right\} \\ &= \sigma_0 \{1 - \alpha_1 x^2 + \alpha_2 x^4 - \alpha_3 x^6 + \alpha_4 x^8 \;.\;.\;.\;.\; \} \end{aligned}\right\} . \tag{386}$$

The coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \;.\;.\;.\;.\;$ in this series are given in equation (ζ), Chapter XVII. Following equation (ρ) it is there shown that the radius of inertia of a monatomic sphere is 0.45 ; so that for a monatomic sphere

$$I = (0.45)^2 Ma^2 \tag{387}$$

Accordingly for a monatomic globe we should have

$$H = (0.202483)\,\omega\ .\ Ma^2\,; \tag{388}$$

So that k^2 is about one-fifth, and with this modification the above reasoning would still hold true.

Finally, in a nebula essentially devoid of hydrostatic pressure, the satellites prior to their absorption into the principal planets are revolving as free planetary bodies; and their moment of momentum of orbital motion is by collision added to that of the planet's rotation about its axis. Any satellite contributes an element of orbital moment of momentum given by the expression

$$M\left(\frac{m_i r_i}{M+m_i}\right)^2\sqrt{1-e_i^2}\,\Omega_i + m_i\left(\frac{M r_i}{M+m_i}\right)^2\sqrt{1-e_i^2}\,\Omega_i = \frac{M m_i}{M+m_i}\sqrt{1-e_i^2}\,r_i^2\,\Omega_i. \tag{389}$$

And all the satellites revolving within the planet's control will contribute to the moment of momentum of axial rotation

$$\sum_{i=0}^{i=i}\left\{\frac{M m_i}{M+m_i}\right\}\sqrt{1-e_i^2}\,r_i^2\,\Omega_i = M\sum_{i=0}^{i=i}\left\{\frac{m_i}{M+m_i}\right\}\sqrt{1-e_i^2}\,r_i^2\,\Omega_i\ . \tag{390}$$

This expression is large when $\sum_{i=0}^{i=i} m_i$ is large compared to M; so that if the planet has a large swarm of satellites, with a total mass which is considerable, each moving at its appropriate distance r_i, and with angular velocity Ω_i then the bringing of them all down upon the planet by the influence of the resisting medium will very materially augment the moment of momentum of axial rotation. The planet's central attraction at any distance r is increased as follows:

$$\left.\begin{aligned} g(1+\gamma) &= \frac{M+\sum_{i=0}^{i=i} m_i}{r^2} = \frac{M}{r^2}\left(1+\frac{\sum_{i=0}^{i=i} m_i}{M}\right), \\ \text{or}\qquad \gamma &= \frac{1}{g}\left(\frac{\sum_{i=0}^{i=i} m_i}{r^2}\right). \end{aligned}\right\} \tag{391}$$

But as the increase in the moment of momentum by (390) is more rapid than that of gravity, since each mass m_i is multiplied by $r_i^2\,\Omega_i$; and therefore the larger $r_i^2\,\Omega_i$, the larger the product $\sum_{i=0}^{i=i} m_i\,r_i^2\,\Omega_i$ is, the eccentricity e_i being so small as to be disregarded in this discussion; we perceive that the moment of momentum and the oblateness will frequently increase with the growth of the central mass. Therefore the larger planets, on the whole, have the most rapid rotations, and have thus been rendered quite oblate;* while all the smaller planets,

*The orbital velocity of a revolving particle about a planet is always much larger than the velocity of a surface particle due to axial rotation; the precipitation of a revolving particle against the surface therefore accelerates the axial rotation.

such as the Earth and *Mars*, have slower rotation and smaller oblateness. To produce large moment of momentum of axial rotation there must be a large central mass, so as to give a strong central force, and a large amount of matter $\sum_{i=0}^{i=i} m_i$ added to the planet from the vortex circulating about it. This gives large moment of momentum about the axis of rotation.

The moment of momentum of axial rotation is $H = \omega \,.\, k^2 \,.\, Ma^2 = \omega I$; so that the increase of mass by the addition of satellites affects M , a , and k^2, as well as ω. Relatively the largest change is in ω ; for as the mass grows the central attraction grows in proportion, but the added moment of momentum is

$$\sum_{i=0}^{i=i} m_i r_i^2 \Omega_i ,$$

and thus augmented by the factors depending on the increased radius and angular velocity, which are themselves enlarged by the increase of the central mass.

These are the general conditions of the problem, without regard to how the vortex is started about the planet; but it may be noticed also that in small bodies the hour-glass shaped space connecting with the Sun's sphere of control is so small and narrow that but few particles enter it, and what few do enter will experience a more nearly equal division between retrograde and direct revolutions about the planet. It thus appears that on the whole the larger masses have a tendency to augment the moment of momentum of axial rotation and oblateness which is greater than in small masses.

The planet *Saturn* has a decidedly larger closed space about it than *Jupiter*, because although of smaller mass it is at greater distance; and hence as measured by the resulting oblateness, has effectively the most rapid rotation. But this also depends on the density, and the conditions for giving large oblateness in the case of *Saturn* are most favorable. *Uranus* and *Neptune* also have large closed spaces about them, but their masses are smaller, the density greater, and the resulting oblateness is therefore no doubt smaller than in the case of *Saturn*. The observed oblateness of *Uranus* exceeds that of *Jupiter* (cf. *A.N.*, 3992), but there is reason to think that the values found are too large. It is doubtless true that some matter comes under the control of the planets without passing through the neck of the hour-glass space defined by the surfaces of zero velocity extending around the Sun; but the amount thus gathered from miscellaneous sources probably is small, and may not be a very important element in the theory of satellites and of planetary rotation.

From the considerations already adduced in Chapter X, on the "Dynamical Theory of Satellites," it is evident that detachment of masses by rapid rotation, if it occurs at all, must be exceedingly difficult to bring about. If the descending stream of matter was supplied to the rotating spheroid in a certain way, which, however, will seldom arise in nature, because it would all have to be directed against the periphery of the rotating mass, so as to give maximum angular velocity of rotation; a process of partial fission might in the course of time develop. Yet even if this augmentation of velocity should come about, it is more than probable that the matter subsequently detached, by acceleration of rotation, would be in the form of a swarm of particles, and would have great difficulty in collecting together into one mass. All the well-known objections to LAPLACE's theory of ring formation and condensation could be urged here with full effect. If the separation was in the form of a lump or nucleus, it might survive, provided the action of the resisting medium did not again bring about its precipitation upon the central mass, which, however, would be almost certain to follow. It is evident, therefore, that, while the separation of masses by accelerated rotation is not impossible, this process requires such very special conditions that it seldom takes place in Nature.

If we consider, for example, the case of the Earth and Moon, where the primordial central mass would have had to acquire an enormously rapid rotation, in less than $2^h\ 50^m$, it will become evident that there are in Nature no forces which could produce such very rapid rotation. That is, there is *no regular process* at work, which could produce such an effect. A grazing collision of two already existing globes, if properly aimed with suitable velocities, might give rise to one common mass spinning so rapidly that scattered portions of it would be detached, and after separation circulate around the residual central mass. But collisions of nearly equal globes are so rare, owing to their infrequency and very small size compared to the large vacant spaces in which they move, and so nearly impossible to effect under the conditions ordinarily existing in cosmical systems dominated by central forces, that this hypothesis has little interest. Moreover, even if the primordial mass were disrupted in this way, the scattered fragments could never get together to form a single globe like the Moon.

§ 176. *On the Original Extent of the Planetary System.*

It appears that the only place in which such globes as the Moon can be started is in the midst of a diffused nebula, where the nuclei are not disrupted or prevented from growing by the strong attractive forces of neighboring masses. The

absence of great attractive centres allows the smaller nuclei to grow, both because their spheres of influence are more extended than when near large masses, and because the large masses have not yet swallowed up all of the surrounding nebulosity; so that the smaller masses have a good chance to grow by accretion.

It is very evident that the origin of the planets and satellites dates back to the earliest nebular stage, and that the embryo of a body such as our Moon was at one time on the very outskirts of the system, where *Neptune* now revolves, or even beyond. Matter equivalent to twenty-seven million such globes was swallowed up in laying the foundation of the Sun; and the fact that Moons or planets of rather small size have been captured by all the principal planets, from the Earth to *Neptune*, indicates how widely diffused such globes were in the condensing nebula which originally formed the solar system. These globes have grown somewhat in later times, by the gathering up of cosmical dust, but their main growth was attained in the nebular stage of the system, which has now quite disappeared.

These considerations afford us some conception of the immeasurable ages which have elapsed since the foundations of the solar system were laid in a whirlpool nebula. The total duration of time involved is certainly to be reckoned in billions of years. The extent of the system has grown less with the lapse of ages, and the orbits have grown rounder as well as smaller. And since there are good reasons to believe that even now unseen planets will be found to circulate at least three times as far away as *Neptune*, we see how vast must have been the extent of the solar nebula when that primordial cosmical vortex was just starting. It may easily have extended to one thousand times the distance of the Earth from the Sun.

Considerations of this kind explain the great length of time involved in the development of cosmical systems. For in such a tenuous nebula the process of transformation is slow, because the resistance of the diffuse nebulosity is slight, and a dominant central Sun has not yet developed. This line of thought also enables us to understand the vast extent of the spiral nebulae, and the insensible velocity of their rotatory movements. Unless the central mass is enormous, these gigantic cosmical vortices must necessarily revolve with extreme slowness. Therefore it is not probable that motion can be detected in less than centuries, and in many instances the period required to disclose a whirling movement is more likely to be reckoned in thousands of years.

If we recall the extreme tenuity of the nebulae observed in the depths of space, and their transparency to the light of the faintest stars, we shall perceive that our solar system must have been in a similar state, which can also be proved

by calculation from known data. But as the major planets at length gathered up the nebulosity in the regions where they revolved, it is clear that they have exerted their influence over wide belts. This could be done partly by the orbital motion of the nebulosity which would periodically bring it near the planets, and partly by the secular decrease in the size of the planetary orbits. As *Neptune's* orbit is nearly circular, one cannot doubt that the solar nebula originally extended much beyond the present bounds of the solar system. It is difficult to form exact estimates of the original extent of our system, but the bounds of the primordial nebula can hardly have been less than one thousand astronomical units.

Accordingly if the embryo *Neptune* was originally on the outskirts of the primordial system, this would make his present orbit only about one thirty-third of the original sweep of the solar nebula. However this may be, it seems certain that *Neptune's* path is much smaller than it was originally, and the same remark applies to the orbits of *Jupiter, Saturn* and *Uranus*. Even after these planets had attained half their present size their orbits can hardly have been diminished by amounts less than the intervals which now separate them. We may infer this partly from the fact that each of these planets has no doubt been chiefly instrumental in clearing up the spaces next beyond it, and partly from the circumstance that the eccentricities are in all cases so very small. The eccentricities of the embryo planetary orbits can hardly have been smaller than 0.5 and they may have been 0.8 or even higher; and a change which would extinguish such an eccentricity would also greatly reduce the major axis, since these two elements are closely related and a modification of one affects also the other. We may then in all probability consider that the primordial orbits of the major planets when half their present masses had been attained were at least twice as large as they are now, and a secular decrease to one-third of their original size seems not improbable.

Since publishing the paper on the "Cause of the Remarkable Circularity of the Orbits of the Planets and Satellites" (*A.N.*, 4308), January 1, 1909, it has seemed advisable to emphasize more strongly the conclusion that there is certainly one, most likely two, and probably three unknown planets beyond *Neptune*. In December, 1904, I examined the evidence bearing on the place of the planet next beyond *Neptune* and concluded that this body, which was then designated as *Oceanus*, was most likely near longitude 200°, and at a distance of 42.25, with a period of 272.2 years. Such a body harmonizes all known data, but the location in longitude is subject to considerable uncertainty. The longitude cannot be much smaller than this, but it might be appreciably larger. The other planets I have placed at distances of 56 and 72 respectively, but they will be much more

difficult to discover than that situated between 42 and 44. In view of the light now thrown upon the mode of formation and constitution of the solar system, I would recommend a persistent photographic search of the region of the ecliptic between longitudes 200° and 250°. Different persons will form different estimates of the validity of the grounds on which this extension of our solar system outward has been based, but I am satisfied that time will show the prediction to be well founded. *To suppose the planetary system to terminate with an orbit so round as that of Neptune is as absurd as to suppose that Jupiter's system terminates with the orbit of the Fourth Satellite. The force of this analogy is not appreciably weakened by the fact that Jupiter's satellites constitute a sub-system of our solar system.*

Since the above was written two important papers on the subject of a trans-Neptunian planet have appeared, one by PROFESSOR W. H. PICKERING (*Annals of the Observatory of Harvard College*, Vol. LXI, Part II), and the other by M. GAILLOT of the Paris Observatory (*Comptes Rendus*, March 22, 1909). The problem does not admit of definite solution by the method of inverse perturbations, which was employed by ADAMS and LEVERRIER in the search for *Neptune*, but PICKERING finds indications of a planet in $7^h\,48^m$ right-ascension or longitude 105°.8. GAILLOT obtains evidence of two planets, at distances of 44 and 66 respectively, the remoter one being in longitude 108°, near the position assigned by PICKERING. This coincidence in position is not decisive as to the location of a real planet, yet the subject promises to become one of increasing interest in the future. We now know with certainty that the system does not terminate with *Neptune*, but must be of enormously greater extent.

§ 177. *Remarks on the Origin of the Asteroids and on the Mass of the Planet Mercury.*

In *A.N.*, 4308 (p. 192, paragraph 6), attention has been called to the mode of formation of the swarm of asteroids between *Mars* and *Jupiter*, which were gathered into their present positions mainly by the action of *Jupiter* and of the resisting medium formerly pervading our solar system. The action of *Jupiter* in capturing periodic comets and throwing their orbits within his own is very well known. This process of transformation is impressively illustrated by a diagram of the orbits of the comets of *Jupiter's* group, given by PROFESSOR W. W. PAYNE, in *Popular Astronomy*, for October, 1893. This diagram merits attention from every one interested in this subject, and has already been discussed in the Theory of the Capture of Comets, Chapter IX.

The late M. CALLANDREAU, of the Paris Observatory, reached the conclusion, from extensive mathematical researches on the theory of the capture of comets,

that there is a connection between the mode of formation of *Jupiter's* group of comets and of the zone of asteroids. The nature of this connection is now much clearer than it has been heretofore. Aside from the perturbative action of *Jupiter* in transforming orbits which cross his path, the secular effects of the resisting medium upon two spheres of the same density but unequal radius, are in the inverse ratio of their radii. Small bodies therefore revolving against resistance rapidly approach the Sun, while the orbits of the large bodies are but little reduced in size by the secular effects of the resisting medium. These two causes have operated to gather the Asteroids within the orbit of *Jupiter*.

That many of the Asteroids formerly extended much beyond *Jupiter's* orbit is indicated by the survival of the so-called *Achilles* group, which has at least four members, with orbits of this type. This fact and the analogy with the periodic comets is satisfactory proof of the former state of our system; but still more impressive evidence is afforded by the fact now established that the satellites of *Saturn* are all captured planets which once revolved in independent orbits about the Sun. The "Dynamical Theory of the Capture of Satellites" (*A.N.*, 4341-42) thus throws a clear light on the state of our system in the remote past. It indicates beyond doubt that small planets were once numerous in the zone between *Jupiter* and *Saturn*, but that in time they were worked out of this region, some being captured by *Jupiter*, others by *Saturn*, to build up their respective systems of satellites; while still others were swallowed up in the globes of these planets and in the Sun. The residue of the primordial group survives as satellites of the several planets and as Asteroids.

In regard to the mass of *Mercury* it is to be observed that the satellite *Titan* is really a planet with a mass 1 : 4700 that of *Saturn*, or 1 : 16450000 of the mass of the Sun. By means of a method originally suggested by Dr. G. W. Hill, *Mercury's* mass has been found by the writer to be 1 : 14868548 (cf. *A.N.*, 3897), which is but little larger than that of *Titan*. Now the diameters of *Mercury* and of *Titan* are 4350 kms. and 5048 kms. respectively (cf. *A.N.*, 3992); and as both are shown to be planets, one captured by the Sun, the other by *Saturn*, we see that we are quite justified in taking the mass of *Mercury* smaller than the values previously used by astronomers. Since no satellite has a density exceeding 3.76, which is the mean specific gravity assigned to *Jupiter's* Satellite II, while all the rest are smaller, with an average of about 2.49 (when given weight proportional to the volume the value is 1.78), (cf. *A.N.*, 3764, p. 336), we see that a mean density of 3.09 (cf. *A.N.*, 3897, p. 140) for *Mercury* must be considered highly probable. At any rate, we have not the least ground to think *Mercury's* mass and density should be increased; on the contrary, if any change is made in these elements,

they are likely to be decreased. Accordingly, the proof that the satellites are captured planets, appears likely to give us, as an indirect outcome of the new theory of their origin, an additional means of correcting the mass of *Mercury*, while at the same time affording a clear view of the origin of the great belt of Asteroids between *Mars* and *Jupiter*, which has long been so mysterious to astronomers.

During the early stages of the recent work on the formation of the solar system, in July, 1908, I had the advantage of several conferences with PROFESSOR R. T. CRAWFORD, of the University of California. Some of the suggestions resulting from his large experience in dealing with the theory of the Asteroids proved valuable, and aided in clearing up this problem, when the solution was by no means so clear as it is to-day.

§ 178. LEUSCHNER'S *Researches on the Origin of the Comets.*

As we have seen in Chapter VI, § 68, LAPLACE adduced reasons for holding that the comets came to our system from outer space. The recent researches of LEUSCHNER, of the University of California, have tended to modify this result in important particulars. He no longer finds that so many of these bodies have orbits which are really parabolic, as was once supposed; but on the contrary, when the observations are criticially discussed by the most refined modern methods, he finds over half of the orbits to be elliptic, though the eccentricity often is high. LEUSCHNER gives the following important conclusions on the laws of cometary orbits, in a recent address to the Astronomical Society of the Pacific (*Publications, A.S.P.*, Vol. XIX, No. 113, Apr. 10, 1907):

"An accurate knowledge of the eccentricities of comet orbits is of importance in determining the origin of comets. It is, therefore, advisable to study the eccentricities from as many points of view as possible. Two methods of classifying the eccentricities have occurred to me which do not seem to have entered into the analysis hitherto. Both are related to the accuracy of the observational material from which the orbits are derived. One is to classify the eccentricities on the basis of the general accuracy of the observations, the other on the basis of the observed heliocentric arc.

"Marked progress has been made during the last century in the methods of observation and in the construction of telescopes, so that observations have become more and more reliable, and the number of days during which comets of the same brightness may be followed has constantly increased.

"Ever since the first computation of a comet orbit was made, it has been customary to derive a parabola as a first approximation to the orbit, and to attempt

a more general solution only if the deviations of the observed positions from the places computed from the most probable parabola were in excess of the probable errors of observation. This custom has become so thoroughly fixed in astronomy that even now it would be considered absolutely unwarranted to suspect a comet of moving in an ellipse if by a little stretching of the probable limits of observational error a parabola could be found to represent the observed positions.

"A prejudice has always existed, and exists now, in favor of the parabola. This prejudice is not entirely due to statistical investigations of the orbits of past comets. A further excuse for the same may be found in the fact that the first geometrical and analytical methods for solving a comet orbit were parabolic. The solution of an elliptic orbit was originally possible only in cases like HALLEY'S Comet, where more than one appearance has been observed, so that one of the unknowns, the period, became known.

"GAUSS'S general solution had its first application on the asteroid *Ceres* at the dawn of the 19th century, and it was not until some time later that general methods were also applied to comets.

"It is well recognized fact that when the observed arc is short and the probable error of observation on a comet is large, the solution of the orbit will be uncertain, or, in other words, in such cases a large number of different orbits will be found to satisfy the observations. The 'Short Method' which has been used extensively in the Berkeley Astronomical Department during the past three years is well suited for estimating the limiting values of the elements. The range of possible periods and eccentricities is far greater than has perhaps been supposed hitherto. A cursory examination of many definitive observations shows that in many cases a long-period ellipse will often answer as well as a parabola. The ellipse is then generally dismissed with the statement that there is no reason to suspect a deviation from the parabola. It would be just as consistent to conclude that there is no reason to suspect that the comet moves exactly in a parabola. In accordance with existing belief regarding the eccentricities of comet orbits, DR. KREUTZ in his biennial reports to the *Astronomische Gesellschaft* adopts the parabola whenever it is found sufficient.

"Before proceeding to an examination of the published lists of elements, it is therefore well to emphasize that possibly in no case where an ellipse or hyperbola alone is given can the observations be represented by a parabola, but when a parabola is given the observations may frequently be consistent with an ellipse and sometimes with a hyperbola.

"If, in spite of this fact, it can be demonstrated that by far the majority of well determined orbits is elliptic, then the time has come when astronomers should

abandon their prejudice for the parabola, by investigating and stating the complete range of possible solutions in each case.

"OLBERS'S, GALLE'S, and WINLOCK'S lists were not available when a preliminary examination of the eccentricities was undertaken. The excellent list, however, contained in E. WEISS'S edition of '*Littrow's Wunder des Himmels,*' is well suited for the purpose of the preliminary investigation, especially because it gives the duration of visibility in days. This list runs to 1885. The results of the preliminary examination were, however, roughly revised just before publication, on the basis of GALLE'S and WINLOCK'S lists and KREUTZ'S biennial reports to the *Astronomische Gesellschaft* to 1904, which latter are contained in the *Vierteljahrsschrift.* Comets discovered between 1885 and 1895 were added to WEISS'S list only when the duration of visibility was included in the data at hand. Periodic comets are, of course, counted only for their first apparition.

"For the purpose of classifying the orbits on the basis of the general accuracy of the observational material or more nearly of the observed positions, the percentage of parabolic orbits was ascertained for each of three groups, in the order of time as given in Table I.

TABLE I.

Dates	$e = 1$
–1755	99 per cent.
1756–1845	74 per cent.
1846–1895	54 per cent.

"It is safe to assume that there has been a progressive and pronounced advance in the accuracy of observation in these three periods of time. Hyperbolic orbits were not included in the totals on which the percentages of Tables I and II are based. From the more accurate observations of the fifty years from 1846 to 1895, we may therefore conclude that it is no more probable that a comet is parabolic than that it is not.

"In Table II the eccentricities have been grouped on the basis of the duration of visibility in days. The percentage of parabolas is given for each group. The comets discovered before 1756 have been excluded in the totals from which these percentages were derived, as their orbits can throw little light on the subject under consideration.

TABLE II.

Duration of Visibility	$e = 1$
1– 99 days	68 per cent.
100–239 days	55 per cent.
240–511 days	13 per cent.

"These figures are certainly striking. They show that the longer a comet is under observation the more probable it becomes that its orbit cannot be satisfied by a parabola.

"This result is in entire accordance with the opinion held by some astronomers that few, if any, orbits are strictly parabolas. In the last group only eight comets were available, which are all given as elliptic by WEISS and for one of these KREUTZ'S later reports give a parabola, which has been adopted, the same as every orbit has been considered parabolic in these tables for which the observation could be satisfied by a parabola. It is therefore extremely doubtful whether a parabola is definitely established for any comet having remained visible two hundred and forty days or more. It would have been better if Table II could have been based on the length of the observed heliocentric arcs, but these are not immediately available, and in a first approximation for a large number of comets, the average of the number of days of visibility may be taken to correspond to the average heliocentric arc.

"Percentages have also been derived for various ranges of eccentricity. These, however, will not be published until the final investigation has been concluded.

"The average eccentricity of periodic orbits is very high. In applying the short method it has been found that whenever a short arc yielded a considerable range of periodic solution, a longer arc would yield solutions for the eccentricity nearer the upper than the lower previous limits. The explanation of the high eccentricities lies in the nature of visibility from the Earth unless their orbits are highly eccentric. The others must remain invisible until the power of our telescopes is still further increased.

"From the average brightness of comets at unit geocentric distance the maximum perihelion distance at which a comet may be seen in opposition from the Earth with the more powerful instruments may be derived. The values of the eccentricity corresponding to this maximum for a given value of the semi-major axis or period will then be the minimum eccentricity which the orbit of a comet of average brightness and of given period must have in order to be visible from the Earth, under the most favorable circumstances.

"This question will be studied in connection with a proposed further study of comet orbits. The theory that, in general, comets are permanent members of our solar system, seems to have been greatly strengthened by the foregoing preliminary statistics."

This important report of LEUSCHNER is here given in his own words, because it does not admit of condensation without omitting some considerations which should be included. Moreover the elliptic character of cometary orbits has been largely overlooked heretofore, and the subject deserves the prominence thus

assigned it. Accordingly we may take it to be a fact that nearly all of the orbits of comets are elliptic and only in very exceptional cases are they parabolic or hyperbolic; and whilst orbits of these latter classes may occasionally arise, they are very rare indeed. Whence it follows that we cannot properly regard the body of the comets as foreign to our system. It is still possible that a few foreign bodies may enter our system, as LAPLACE has maintained, but it is doubtful if we are able to observe them except on the rarest occasions; and the vast majority of the comets come from a shell on the outer limits of the Sun's sphere of attraction. They are no doubt the outlying wisps of nebulosity with which the sphere of the Sun's attraction was orginally* more or less filled. All the interior parts of this material have long since been drawn together to form the planetary system, and only the waste nebulosity from the periphery of the sphere now remains to furnish us comets. The orbits have been transformed in certain cases, and short-period comets have thus arisen; but most of the comets retain their aphelia at the remote region where they originally came. This also accords with the theory that with the lapse of ages the orbits of the bodies of our solar system have undergone a great shrinkage. Accordingly the further study of the orbits of comets may throw much new light upon the former state of the solar system, and this line of research cannot be too strongly commended to the attention of astronomers.

§ 179. *Answer to a Question of* DR. G. W. HILL.

In a letter to the author, dated April 5, 1909, DR. G. W. HILL asks the following question: "And what reason is there for supposing that when the planets start out on their revolutions they must necessarily have large eccentricities?

"If all eccentricities are equally probable we should expect some of the eccentricities to be large, others small. If all are small (however *Mercury's* eccentricity ought not to be called small) this seems to indicate that at the outset the motions were nearly perpendicular to the radii vectores and of such amount as to make the centrifugal and centripetal forces nearly equal. Certain conditions in the antecedent state of the system might compel this."

The general character of the theory developed in this work may be considered a sufficient answer to this reasoning of the great mathematician who has done so much for American astronomy; yet there may be some advantage in examining the problem a little more critically. In the first place we remark that as the planets are captured bodies and have been added to the system of the Sun from without, it necessarily follows that they came from a distance; and when the

* At a still earlier period much of the cosmical dust now condensed into comets may have come to our nebula from the fixed stars, since nebulae also are built up gradually by the process of capture.

embryo planets first approached the nucleus of our system, they must have moved in orbits of large eccentricity, just as the comets do now. This result follows immediately from the capture theory, unless we suppose that as the embryo planets neared our system for the first time the motion in each case was so adjusted in direction and in magnitude as to give a definite velocity suitable to an ellipse of small eccentricity. The probability that these motions were originally so restricted as to give this extraordinary result is so very small as to entirely disappear, and we conclude that it could not have occurred.

Moreover, according to WHEWELL's theorem, the velocity at any point of an ellipse is equal to that which would be acquired by a body in falling from rest or with zero velocity from a circle with radius equal to the major axis of the ellipse; this is also called VAN DER KOLK's theorem (*A.N.*, 1426), but WHEWELL was the first to discover it. As the planetary orbits are all nearly circular, this is equivalent to saying that the matter out of which they were formed came from distances but little greater than twice their present mean distances—which is inadmissible, as not harmonizing with the great tenuity and extent of systems observed among the nebulae.

Again, in starting from the circle of zero velocity, the matter naturally was at rest before falling towards the Sun. But the present orbital motions of the planets are nearly perpendicular to the radii vectores, while matter falling straight to the Sun would be moving parallel to these same lines; yet if there is a whirling vortex about the center, that which survives as planets — obviously an insignificant part of all the matter falling into the Sun — might thus acquire a direction nearly perpendicular to the radii vectores. Now such a motion of great circularity has been acquired, and any supposition that the matter so set in motion by whirling came from so short a distance as that fixed by the diameters of the planetary orbits, is even more inadmissible than if the orbits were less circular. It follows, therefore, that the planets must have come to the Sun from a great distance, and their original orbits must have been of large eccentricity.

Accordingly we see that the motion of the streams of primordial cosmical dust which formed our system must have been such as would correspond to orbits of comparatively close approach at perihelion and which would carry it back to an original great distance if undisturbed. Consequently the original velocities could not have been such as to give round orbits, and all the original orbits must necessarily have been highly eccentric. Revolution against resistance, and gravitative disturbances due to the nebulosity and solid planetary bodies near the Sun would gradually reduce the major axes of these elongated ellipses, and their eccentricities, and thus in time give us orbits of great circularity and small size, like the planetary orbits are observed to be.

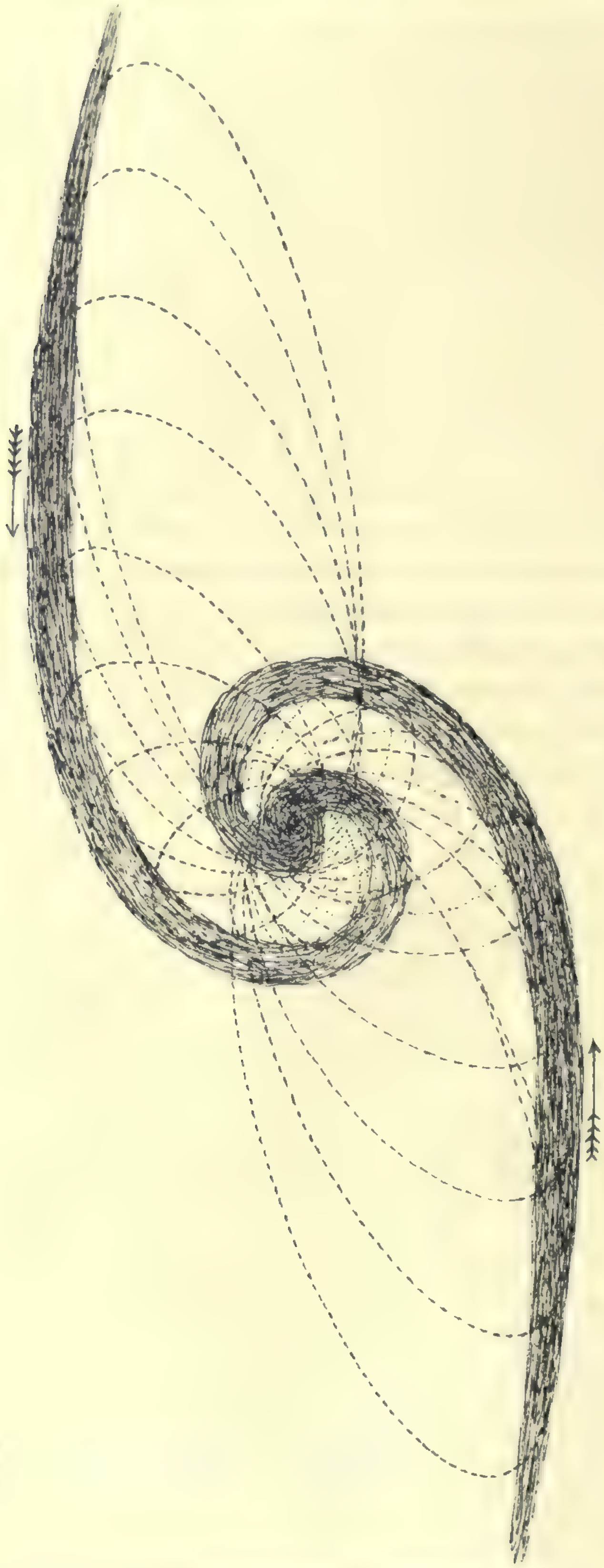

FIG. 35. DIAGRAM ILLUSTRATING THE TENDENCY OF SPIRAL MOVEMENT.

If central attraction alone acted the particles would describe ellipses as here represented, but under the action of the rest of the mass the paths are not re-entrant; yet there is a gradual rounding up of the orbits that would be described, the nearer we approach the center, because resistance is everywhere at work, and the central nucleus is more and more dominant the nearer it is approached.

Finally we see this theory verified by what is observed in the satellite systems. Originally the satellite orbits were all much larger and much more eccentric than they are now observed to be; but in time they have been reduced in size and rounded up under the action of a resisting medium. And just as the satellite orbits have been reduced in size and rounded up, so also have the planetary orbits undergone a similar transformation. The accompanying figure illustrates the mode of transformation of these orbits. The law of attraction in the case of a diffused spiral nebula is such that the path of any particle is not an elliptic orbit, but a series of constantly shifting and non-re-entrant curves which grow smaller and rounder with each revolution. At length most of the nebulosity is absorbed into the Sun and planets, and the orbits of the planets which survive near the center all have small eccentricities. This is the true secret of the mode of formation of the planetary system.

§ 180. *Expressions for the Forces Acting Upon a Particle in a Nebula of Any Form or Extent, and Between Two Nebulae of Any Figure Whatever.*

In the midst of a spiral nebula it is to be observed that the particle does not obey merely the force of gravitation exerted from the fixed center or nucleus of the mass, but also the forces exerted by every other particle in the nebula. The potential of the forces acting on the particle whose coördinates are x', y', z' in a nebula of density σ' and of unlimited extent is

$$V = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma'\, dx'\, dy'\, dz'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}}. \tag{392}$$

And the forces along the coördinates axes are

$$\left.\begin{aligned}
X = \frac{\partial V}{\partial x} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma'(x'-x)\, dx'\, dy'\, dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}},\\
Y = \frac{\partial V}{\partial y} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma'(y'-y)\, dx'\, dy'\, dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}},\\
Z = \frac{\partial V}{\partial z} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma'(z'-z)\, dx'\, dy'\, dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}.
\end{aligned}\right\} \tag{393}$$

When the boundaries are finite they may be specified by corresponding changes in the limits of these integrals, but in the case of actual nebulae the extent is

always enormous, and infinite limits are therefore appropriate unless the contrary is indicated.

If we have two nebulae interpenetrating each other in any manner the mutual potential energy of the two systems of particles is given by the sextuple integral:

$$E = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma\sigma'\, dx\, dy\, dz\, dx'\, dy'\, dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}, \tag{394}$$

where σ is the density of the first mass at any point (x, y, z). And the forces along the coördinate axes would be found by simple derivation under the sextuple integral, just as in the above example of the triple integral.

If R denote the resultant force acting on any particle at (x, y, z), the exhaustion of gravitational energy produced by bringing a vast number N of equal masses from rest at an infinite distance to an equally spaced distribution through a sphere of radius r is easily shown to be (cf. THOMSON and TAIT'S *Nat. Philos.*, § 549; and *Baltimore Lectures* of LORD KELVIN, 1904, p. 270):

$$E = \frac{1}{8\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R^2 dx\, dy\, dz = \frac{3}{10} Fr\,, \tag{395}$$

where F denotes the resultant force of the attraction of the N equal masses on a material point, of mass equal to the sum of their masses, placed at the spherical surface of radius r.

By means of methods such as these, it would be possible to determine the resultant force exerted by any nebula upon a particle within it (or without it), if we knew the magnitude of the component bodies and their distribution in space; but in default of knowledge we can only recognize the general fact that the forces at work are very complex, and cannot possibly give a re-entrant path to any revolving particle. On the contrary, just as the nebula as a whole under the mutual attraction of its parts becomes more and more symmetrical by revolution and condensation, so also will the paths of the individual particles become smaller and more circular as the mass condenses towards the center and forms a Sun surrounded by a system of planets and satellites.

In our theory of the spiral nebulae, we have seen that the streams which coil up under their mutual attraction, and as the result of relative motion, slowly condense towards the center. For a long time these streams in the outer parts of such a nebula have no hydrostatic connection with the nucleus. The coils move independently of each other, and as they settle towards the center by repeated

convolutions, each time becoming more and more circular, they also near each other. The particles in these whirling streams or vortices spread themselves around a closed circuit more or less elliptical, but not exactly re-entrant; for the paths are still spiral and not perfectly closed at the end of a revolution. The larger masses in these spirals gather more and more material to themselves, while at the same time they revolve incessantly against a resisting medium.

The resistance has the following double effect: (1) It decreases the radii vectores and thus aids in building up the central mass, till it begins to exert an attraction sufficient to cause the particles to revolve in approximate ellipses. (2) When a mass begins to move in an ellipse, the resistance decreases the major axis of the orbit and also the eccentricity. The orbital motion of the body through the nebula brings to it new material to increase the size of the embryo planet. As the planet by revolving incessantly against the resisting medium gradually nears the Sun, it sweeps up in time all the cosmical dust in a wide zone, and thus the system is gradually cleared of nebulosity.

In this way our solar system originated. When the nebula began to condense, it was made up of two or more streams of matter exerting no hydrostatic pressure except where they were in collision. As the winding up of the streams progressed, the coils gradually became rounder and rounder and the central mass grew larger and larger and finally became powerful enough to compel the embryo planets to move in approximate ellipses. Afterwards the continuation of the process made the ellipses nearly circular and much smaller than they had been originally. The central parts of the nebula and the streams which converged there formed the Sun, while those which circled high above were gradually collected into planets moving in orbits smaller and smaller and becoming rounder and rounder. Thus at length was produced the planetary system, with such beautiful orbs as *Jupiter* and *Saturn, Uranus* and *Neptune,* now revolving in nearly circular paths, yet never set in motion by hydrostatic pressure from the center, as imagined by LAPLACE.

As the planets condensed the streams which formed a vortex about them and gave material whirling in spiral paths would likewise include captured satellites, having properties very similar to those of the planets. The satellites are for the planets exactly what the planets are for the Sun; so that we cannot doubt that the mode of genesis was in all respects similar. Thus both the planets and satellites are captured bodies which have since grown by accretion, and arose originally in diffuse nebulae, devoid of hydrostatic pressure; and as that process was repeated so frequently in our solar system, it is evident that the heavens must contain vast multitudes of planetary systems of similar character, in which

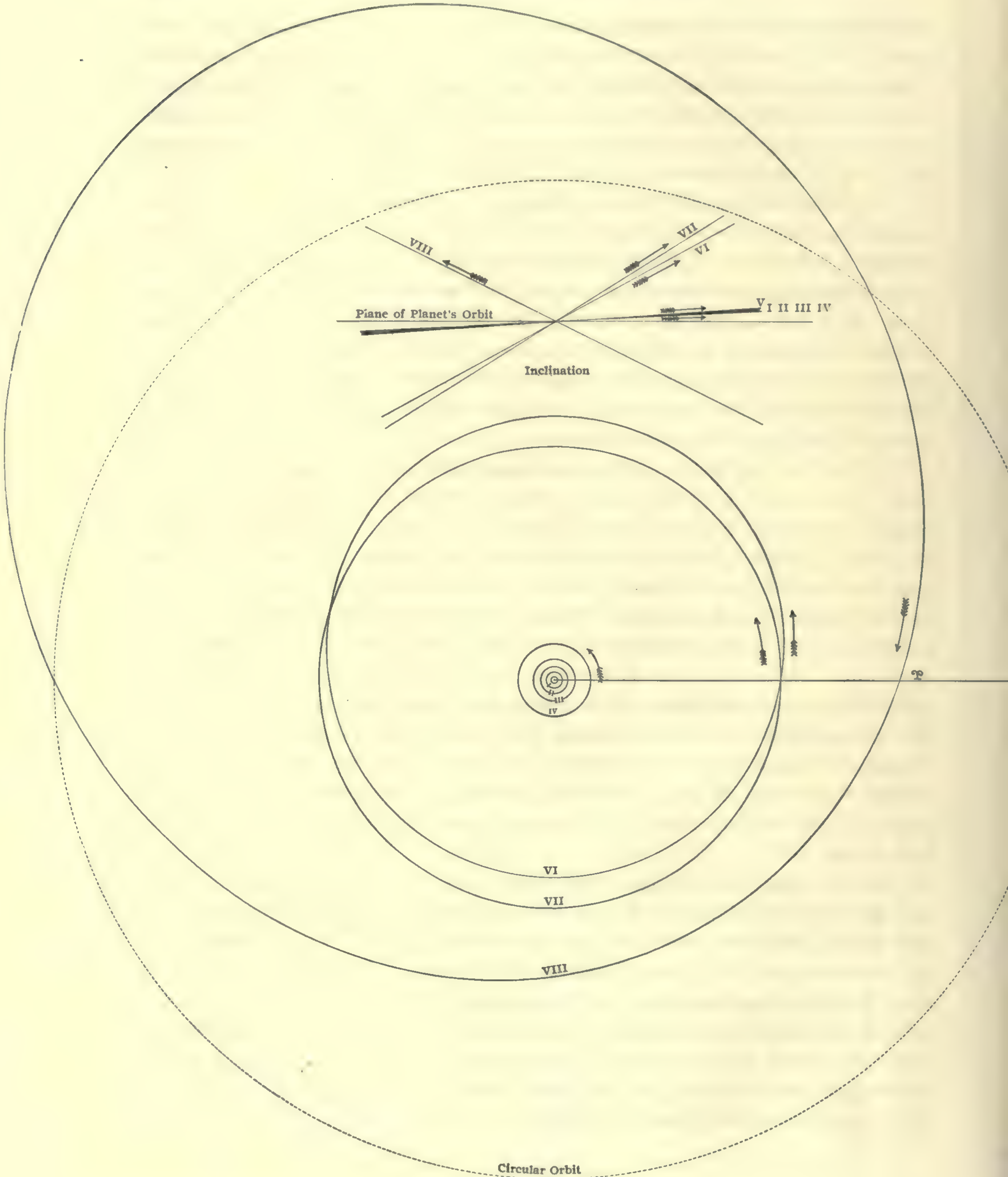

FIG. 36. DIAGRAM OF THE SYSTEM OF JUPITER DRAWN TO SCALE AND SHOWING THE RETROGRADE MOTION OF THE EIGHTH SATELLITE.

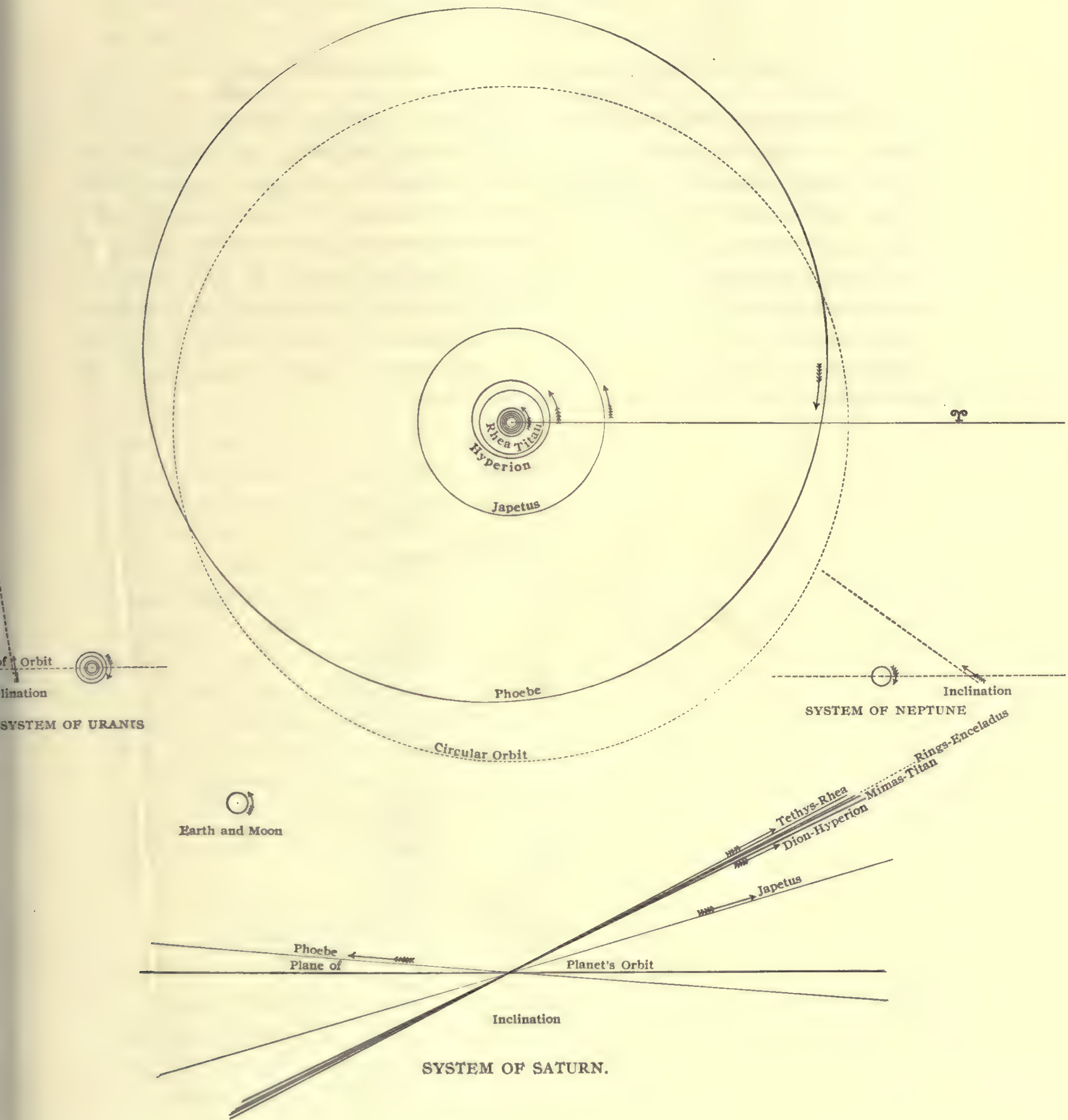

Fig. 37. Diagrams of the Systems of Saturn, Uranus, Neptune, and the Earth and Moon, Drawn to Same Scale. The Retrograde Orbit of Phœbe is Shown in the Diagram of the Satellites of Saturn.

all the bodies have been captured and have had their orbits transformed by the resisting medium in which they have revolved.

§ 181. *On the Eccentricities of the Orbits of the Satellites.*

If we examine the eccentricities of the orbits of the satellites of *Jupiter* and *Saturn*, we shall be surprised to find that the two retrograde satellites have by far the largest eccentricities; in fact these satellites have the largest eccentricities of any of the satellites of the solar system. In the case of *Jupiter's* Eighth Satellite, the eccentricity found by CRAWFORD attains the unprecedented figure of 0.44; while the eccentricity of *Saturn's* Ninth Satellite, found by W. H. PICKERING, is just half as large. In each system the eccentricity of the outer orbit is remarkably large and difficult if not impossible to reconcile with the classic Laplacian nebular hypothesis, even as modified by STRATTON. The following table shows the eccentricities of the principal satellite orbits.

Planet	Satellite	Eccentricity
Earth	*Moon*	0.05491
Mars	*Phobos*	0.0217
	Deimos	0.0031
Jupiter	V	0.00308
	I	0.0000
	II	0.0000
	III	0.001335
	IV	0.007278
	VI	0.1550
	VII	0.0246
	VIII	0.44
Saturn	*Mimas*	0.0190
	Enceladus	0.0046
	Tethys	0.0000
	Dione	0.0020
	Rhea	0.0009
	Titan	0.02886
	Hyperion	0.1043
	Iapetus	0.02836
	Phœbe	0.22
Uranus	*Ariel*	0.000
	Umbriel	0.000
	Titania	0.000
	Oberon	0.000
Neptune	Satellite	0.00292

It will be seen from this table that the eccentricity is small for all of *Jupiter's* satellites, except the Sixth (0.15) and Eighth (0.44). In the system of *Saturn* a precisely similar condition exists, and the eccentricity is small except in the case of *Hyperion* (0.10) and *Phœbe* (0.22). Thus the two retrograde satellites have by far the largest eccentricities of all the bodies of this class in our system, and a strong presumption arises that the high eccentricity, the retrograde motion and the position on the outer borders of these systems are all intimately connected. As the high eccentricities are permanent and not due to perturbations merely, their occurrence on the outer parts of these systems seems to indicate that they are survivals of more extreme eccentricities in the primitive orbits.

If these retrograde satellites had been captured and added to the outer parts of the systems by encountering of resistance in passing near these planets, it seems certain that immediately after their capture the eccentricities would most likely have been comparatively high, like those of the periodic comets, say 0.65 or 0.70. In time, however, the action of the resisting medium would have greatly reduced both the mean distance and the eccentricity; yet if the medium were quite rare, at that distance from the planets, as seems probable, a considerable part of the original eccentricities would survive, and give us the very phenomena which are now observed. Accordingly the large eccentricities as well as the retrograde motion are essentially inconsistent with the theory of LAPLACE; but both of these phenomena find a natural and simple explanation in the capture theory here developed. A new theory which harmonizes such anomalous phenomena will inevitably have a strong claim to acceptance.

It may be noted also that besides the large eccentricities of the retrograde satellites the difficulty of explaining the inclinations of the Sixth and Seventh Satellites of Jupiter by the Laplacian theory is generally recognized. The inclination of *Jupiter's* Sixth and Seventh Satellites are so high as to be quite remarkable. According to DR. ROSS (*Lick. Obs. Bulletin,* No. 112 and 82) the inclination of these bodies to *Jupiter's* orbit, together with the mean distances and periods, are as follows:

	Inclination	Mean Distance	Period
Satellite VI	28° 44′.8	3037″.0	250.618 days
Satellite VII	32°	3152″.4	265.0 days
Satellite VIII	145° 48′	7287″.0	930.73 days

The corresponding elements for the Eighth Satellite are added from the recent work of PROFESSOR CRAWFORD (*Lick Obs. Bulletin,* No. 137). The capture theory gives a natural and simple explanation of all these phenomena, and undoubtedly represents the true process of Nature.

§ 182. *On the Planar Arrangement of the Planetary System.*

Before entering upon the problem of the obliquities of the planets, which must receive careful attention, it seems advisable to notice one criticism of the previous work, in order to make clear the general principles underlying the capture theory. In *Nature*, of July 29, 1909, a reviewer of the paper in *A.N.*, 4308, closes his account by asking: "Why, for instance, on the hypothesis of capture, are the vast majority of the orbits near the plane of the ecliptic and their motion direct?" This is because our system was formed from a spiral nebula, itself produced by the unsymmetrical meeting of two streams of nebulosity or by the mere gravitational settling of a single nebula of curved and unsymmetrical figure, thus giving a rotating cosmical vortex, or spiral nebula, but without hydrostatic pressure as imagined by LAPLACE. In *Lick Observatory Publications*, Vol. VIII, Plate 34, will be found an illustration of *H.V.* 2 *Virginis*, a spiral nebula of unsymmetrical figure just beginning to coil up and form a system. What will happen in the later stages of this nebula is sufficiently shown in the Lick Photographs of the other nebulae given in this volume. As the mass whirls and condenses under resistance, it will necessarily retain and draw down most of the nebulosity into the principal plane of motion. This is exactly what has given the observed arrangement of the solar system.

All these bodies revolve in the same direction and nearly in a fundamental plane of maximum areas, which was discovered by LAPLACE in 1784, and by him proved to be invariable. Owing to numerous changes in the physical universe this Invariable Plane is the only geometrical element of any system which remains rigorously fixed, whatever be mutual action of the component bodies; and the fundamental plane thus defined cannot be disturbed except by the action of the fixed stars, which are too far away to exert a sensible influence.

It is impossible for a cosmical vortex to form and develop in this manner without producing motion confined essentially to one plane. Such oblate vortices are found everywhere among the nebulae. All well developed spiral nebulae are whirling in a comparatively thin plane, just as is observed in the arrangement of the bodies of our planetary system. The photographs of the spiral nebulae show this planar arrangement beautifully. It is easily understood mechanically, as a gyrostatic effect, arising from the way in which the system is started; and being an observed fact in the arrangement of the planetary system, it shows that our system was no exception to the general order of nature, but originated from a whirlpool nebula.

Any nebulous mass whirling as a vortex and winding up will necessarily

have its coils mostly in one plane, and under the action of resistance any nebulosity not in that plane will tend to be drawn into the dominant plane of motion to which most of the matter is confined. Moreover the effect will become more marked as the whole nebula condenses and greatly contracts its dimensions. When a nebula in developing takes an unsymmetrical figure, and then settles under the mutual gravitation of its parts, it will form a spiral nebula and continue to coil up and attain greater and greater symmetry, as the cosmical vortex condenses and decreases in size under gravitation and resistance.

If the nebula arises from the meeting of two or more streams of cosmical dust, the whole system thus constituted will necessarily have some principal plane towards which it gravitates as it settles. And as such a nebula is a vastly expanded mass of excessive tenuity, it will revolve a long time in condensing, and therefore finally acquire arrangement in a plane. This oblate arrangement usually is not due to centrifugal force, as in the figures of equilibrium, because hydrostatic pressure seldom exists in a true nebula, but is due principally to the gyrostatic and centripetal tendency, as the nebula whirls and slowly condenses, producing a system of planets revolving around the Sun in the center.

§ 183. *On the Physical Cause Which Has Produced the Small Obliquity of the Planet Jupiter* (cf. *A.N.* 4367).

We have seen that the planets are as old as the Sun itself, and have gradually neared the Sun from a great distance. Therefore large bodies like *Jupiter*, *Saturn*, *Uranus* and *Neptune* acquired independent rotations, when they were much further from the Sun than at present, and originally these rotations may have been in any planes. As they neared the Sun and became more like the inner planets, the rotations would tend to become direct, because of the vortices of captured satellites revolving about the planets in planes passing through the Sun. These vortices increase the masses of the planets by the precipitation of satellites, and thus tend to bring their equators into the general plane of movement of the planetary system. The retrograde motion of the satellites and the high tilting of the planetary axes is therefore natural enough on the outer parts of the system; but it ought not to persist in the planets nearer the Sun; and here, fortunately, theory is in complete accord with observation.

In the critical investigation of this problem of planetary obliquity, we begin with *Jupiter*, the greatest of the planets, because here the influence at work has left on the Jovian system so distinct an impress of its mode of operation that there can be no doubt of the correctness of the assigned physical cause. We have

found that all the planets have increased greatly in mass, with the lapse of ages, by the capture and absorption of satellites. And it naturally follows that *Jupiter* above every other planet has augmented his mass at the expense of the smaller bodies of our system.

This is shown by the part this great planet plays in the capture and transformation of the orbits of comets, and by the fact that he has gradually worked the Asteroids out of the regions beyond his orbit, and thrown them into the comparatively stable region within where they now revolve. That *Jupiter* has thus transformed the paths of thousands and millions of small bodies once crossing over his orbit, there is not the slightest doubt. Remarkable survivals still existing in our system prove this as clearly as if we had actually witnessed these effects within the historical period covered by exact observations.

On the basis of the phenomena presented by the comets and the Asteroids, therefore, we may confidently assert that *Jupiter* has not only transformed the paths of countless small bodies once crossing his orbit, but also that he has greatly built up his own mass by the capture and absorption of small bodies, which for brevity of diction we may speak of as satellites. If we consider the past we see that there was a time when vast quantities of these bodies revolved near *Jupiter*, in planes passing through the Sun, and having an average motion coincident with the plane of his orbit. The average satellite of this kind when captured and precipitated upon the planet would obviously augment the planet's mass and oblateness, and also tend to bring the Jovian equator into coincidence with the plane of the planet's orbit. Accordingly whatever may have been the original position of the planet's axis, this process of growth has tended to make it more and more nearly perpendicular to the plane of the Jovian orbit, which is also the mean position of the swarm of satellites revolving about the Sun and being gradually captured and absorbed by this giant planet.

Now it is a fact of observation that the axis of *Jupiter* is inclined from the perpendicular to the plane of his orbit by only about 3°; and this small obliquity tells the story of the capture of vast quantities of satellites so plainly that one cannot mistake its meaning. Moreover, the observed equatorial acceleration of *Jupiter*, similar to the equatorial accelerations observed in the Sun and *Saturn*, shows that the process of capture and precipitation of satellites is still going on. The small obliquity of *Jupiter* is therefore a beautiful illustration of what happens to any planet when the process of capturing satellites has gone far enough; namely, the axis tends to become perpendicular to the plane of the orbit, and the obliquity vanishes.

§ 184. *On the Relation Between the Obliquities of Jupiter and Saturn.*

If we consider the phenomena presented by *Saturn*, we perceive that his obliquity is less developed than that of *Jupiter;* for the mass of the former is less than one-third that of the latter, and the distance nearly double. And just as *Jupiter* has worked the satellites out of the regions beyond his orbit, so also he has both grown on the material thus captured, and at the same time robbed his remoter neighbor of material which otherwise would have augmented the mass of *Saturn* and diminished the obliquity to a smaller value than it has to-day.

It is easily shown by calculation that when *Jupiter's* mass was as small as that of *Saturn* and his distance as great, his obliquity may have been fully as large as that of *Saturn* is now.

The moment of momentum of a planet about its axis of rotation is $H = k^2 \omega M a^2$, where k^2 is the square of the principal radius of gyration, and therefore a constant depending on the law of density, ω the angular velocity of rotation, a the radius, and M the mass. For a second planet following the same law of density we have $H' = k^2 \omega' M' a'^2$. If new matter be added to the first planet to produce the second, we may imagine the shell thus deposited to be rotating about a different axis, and the two rotations may be compounded geometrically according to the principle of the parallelogram of forces. Now on this hypothesis, with $M' = 3M$, or the matter of the shell equal to $2M$, we have $\omega' > \omega$, $a' > a$, while k^2 remains unchanged, and $H' > 3H$. For the equatorial accelerations observed in the Sun, *Jupiter* and *Saturn* show that as the mass increases by the precipitation of satellites, the angular velocity ω also increases. Two spheres with volumes as 1 to 3, have radii as $1 : \sqrt[3]{3} = 1 : 1.4422496$; and the difference of the moments of momentum about the axes gives that of the shell:

$$H_0 = H' - H = k^2 \{\omega' M' a'^2 - \omega M a^2\} = k^2 \{3 \omega' (1.44)^2 - \omega.\} = k^2 \omega \left\{ 3 \left(\frac{\omega'}{\omega}\right) 2.0736 - 1 \right\} \cdot \qquad (396)$$

Accordingly even if ω' did not exceed ω, $H_0 > 5H$, or the moment of momentum of the shell exceed that of the nucleus by more than 5 to 1.

The increase in the angular velocity ω', over ω the original angular velocity of the nucleus, depends on the proportion of satellites revolving with the vortex, and accelerating the rotation by falling upon the planet, to those retarding the rotation or falling in all manner of contrary directions. This ratio is not known, but from the proportion of direct to retrograde satellites in the solar system is believed to be about $\frac{4}{5}$. Trebling the mass trebles the force of gravity at a fixed distance, and even if the radius be increased from 1 to 1.44, there will still

be a large increase in the angular velocity. The problem of the acceleration of the angular velocity probably does not admit of entirely rigorous treatment, but the following process of calculation will give approximately the effect produced. In order to determine the mechanical effect of the downfall of cosmical dust upon the planet's rotation we have to evaluate the triple integral:

$$\omega' = \omega \cdot \frac{4}{5} \cdot \frac{1}{2\pi} \cdot \frac{1}{\pi} \int_{a_1}^{a_2} \int_0^{\pi} \int_0^{2\pi} \frac{M}{a^2} \frac{a^2 da \sin\theta d\theta d\phi}{a \sin\theta}, \tag{397}$$

where $dm = a^2 da \sin\theta d\theta d\phi$, and the divisor $a \sin\theta$ gives the torque about the axis.

But as the distribution of material may be taken to be uniform as respects the angle ϕ, we may immediately integrate relatively to this variable; it is also obvious that we may restrict θ to one of the two symmetrical hemispheres. Thus we get:

$$\omega' = \omega \cdot \frac{4}{5} \cdot \frac{2}{\pi} \int_{a_1}^{a_2} \int_0^{\frac{\pi}{2}} \frac{M}{a^2} \frac{a^2 da \sin\theta d\theta}{a \sin\theta} = \omega \frac{16}{15} \frac{2\pi}{\pi} \int_{a_1}^{a_2} \int_0^{\frac{\pi}{2}} \sigma a^2 da d\theta \,. \tag{398}$$

When the density σ is taken to be uniform and the matter incompressible, this gives

$$\omega' = \omega \frac{16}{15} \pi \int_{a_1}^{a_2} a^2 da = \omega \frac{16}{15} \pi \left[\frac{a^3}{3}\right]_1^{1.44} = 2.234\, \omega \,. \tag{399}$$

Using this value in (396) we obtain finally

$$H_0 = H' - H = k^2 \omega \{13.8 - 1\} = 12.8 k^2 \omega \,. \tag{400}$$

In this calculation there are several factors to which considerable uncertainty attaches, but it seems certain that the moment of momentum of axial rotation due to the layer including two-thirds of the whole mass could not well be much less than ten times that of the original moment of momentum of axial rotation; so that the original obliquity of 27° would be reduced according to the relations

$$\frac{\sin\beta}{\sin\alpha} = \frac{1}{10}, \quad \alpha + \beta = 27^\circ \,; \tag{401}$$

and thus β become less than three degrees. We conclude, therefore, that if *Saturn's* mass should be trebled by the capture of satellites moving in the plane of his orbit, his obliquity would be practically obliterated, and the new axis of

rotation would be almost perpendicular to the plane of his orbit about the Sun. Under the circumstances it is clear that *Jupiter's* obliquity may at one time have been even larger than that of *Saturn* is now, and yet it would have been almost destroyed in the course of time by the capture of satellites.

§ 185. *On the Obliquities of Uranus and Neptune and of the Terrestrial Planets.*

In regard to the obliquity of *Uranus*, we have to judge principally by the planes in which the satellites move, though the planet is found by observation to have an oblateness of between 1 : 25 and 1 : 12 (cf. *A.N.*, 3992). As the satellites are observed to move sensibly in one plane, and their orbits show no recognized secular displacement, it is to be assumed that they move in the plane of the equator. In *A.N.*, 4341-42, we have pointed out that these Uranian satellites are excessively and almost unaccountably near the center of the large closed space about *Uranus*. It is evident that the major axes have been enormously reduced since these bodies were originally captured. Under the circumstances it is not surprising that in this great reduction of the mean distances the eccentricities have been entirely obliterated.

In connection with this great decrease of the mean distance it is easy to see how the satellites have gradually worked down into the plane of the Uranian equator. The effect here is similar to that observed in the case of the inner satellites of *Saturn* and of the particles composing the ring. Under the action of the oblateness and of the vortex of nebulosity whirling about the planet the revolving particles tend to be drawn into the plane of the equator as the path of least resistance. Satellites revolving about an oblate planet and suffering great reduction in the mean distance, by the action of a resisting medium, thus come finally to revolve nearly in the plane of the equator, as we see also in the case of the inner satellites of the system of *Jupiter*. For departure from this equatorial plane is resisted more and more, under the effect of oblateness and of the vortex of nebulosity, as the planet is approached; and the wear and tear of the vortex finally leaves the surviving particles adjusted to move exactly in the plane of the equator. This is the origin of the beautiful system of rings about *Saturn*, and the satellites near the planets everywhere show the same tendency to exact adjustment in the plane of the equator, whether it be in the system of *Mars* or *Uranus*. And just as the satellites which now survive about the planets are but a small part of those which once existed, so also the material in the rings of *Saturn* has been destroyed and renewed many times. This cosmical dust is within ROCHE's limit and cannot form a satellite, but the symmetrical arrangement with respect to

the equator of *Saturn* illustrates the action of the oblateness and of the revolving vortex in working the particles into the plane of the equator as the path of descent offering least resistance.

We may therefore take the equator of *Uranus* to coincide with the plane of the orbits of the satellites. Why then has the obliquity of *Uranus* remained so large? Simply because the primordial rotation was largely arbitrary and started when the planet was at a great distance from the Sun; and in later times the supply of satellites coming in to build up the planet's mass has been cut off by the action of *Jupiter* and *Saturn* in clearing up the solar system. *Jupiter* has robbed *Saturn* of building material, and *Saturn* has joined with *Jupiter* in robbing *Uranus*. This effect is also unmistakably exhibited in the system of *Neptune*, which has only a single satellite, with zero eccentricity, and very small distance compared to the large closed space about the planet, showing that a great interval of time has elapsed since the capture of the satellite took place.

It is thus a remarkable fact, clearly demonstrated by surviving phenomena in the solar system, that the inner of the major planets have robbed the outer by gradually cutting off the supply of satellites. This is the significance of the fact, that although *Uranus* and *Neptune* have larger closed spaces than *Jupiter* and *Saturn* (cf. *A.N.*, 4341-42), their surviving satellites are concentrated quite near the planets with the outer regions apparently vacant. For these reasons these outer regions may be in fact as vacant as they seem, and search for remoter satellites may prove to be in vain. The failure to find additional satellites about *Uranus* and *Neptune* seems to show that no satellites of sensible magnitude have been available for capture by these planets in recent times.

Moreover, since *Jupiter* has been able to rob *Saturn*, while *Saturn* joined him in robbing *Uranus*; and since *Neptune* in turn was robbed by all three of the major planets within his own orbit, we perceive in this state of fact an indication that the orbits of the satellites thus cut off from the outer planets were of such considerable eccentricity that they crossed the orbits of the planets next within. This is another answer to DR. G. W. HILL's question (§ 169) to the present writer, as to a reason for thinking that the primitive planetary orbits were necessarily of large eccentricity.

It follows therefore that if *Jupiter* and *Saturn* had been removed from our system at an early epoch *Uranus* would have had a much more extensive system of satellites than he now has, and the mass of *Uranus* would have grown so much that the obliquity would have been reduced to an insignificant value. But with *Jupiter* and *Saturn* cutting off the Uranian supply of material revolving about the Sun, the present high obliquity and the narrow system of satellites concen-

trated close to the planet has necessarily resulted. And the causes which have operated to prevent the obliteration of the obliquity in the case of *Uranus* have acted even more powerfully in the case of *Neptune*.

The arrangement of the obliquities and systems of satellites among the major planets is therefore entirely clear, and it is seen to have been the direct outcome of the available supply of satellites circulating around the Sun. The phenomena still surviving in the solar system thus throw a flood of light upon the state of our system in the remote past, and indicate with certainty that the primordial orbits of the satellites were nearly always sufficiently eccentric to overlap the orbit of the planet next within; otherwise the supply of material for the outer planets would not have been cut off by the inner ones, so as to arrest the development of *Uranus* and *Neptune*. The development of *Jupiter* is typical, and that of *Saturn* almost normal; while that of *Uranus* and *Neptune* is clearly and unmistakably arrested, in spite of the comparatively large masses of these outer planets. And this state of fact can only mean that the supply of satellites was long ago cut off by *Jupiter* and *Saturn;* and furthermore that the orbits pursued by the satellites were eccentric enough to have caused them to cross over the orbits of the two inner major planets. From these considerations we get an unexpected light on the development of our solar system. If there are other planets beyond *Neptune*, they will have suffered still more than the known planets by the cutting off of their supply of building material, and at this late epoch might be quite devoid of attending satellites of any kind.

If we now turn our attention to the inner or terrestrial planets within the orbit of *Jupiter*, we shall find indications that the same causes have been at work which have operated in the outer parts of the planetary system. The Asteroids are the chief survival of the swarm of small bodies formerly traversing the regions of the major planets; but *Jupiter* has thrown most of them within his own orbit, and almost cut off the supply of material both for himself and for the other planets beyond. By this same analogy we cannot doubt that the terrestrial planets, *Mercury*, *Venus*, the Earth, the Moon (formerly an independent planet), and *Mars*, at a remoter epoch also crossed over the orbit of *Jupiter*, and were finally thrown entirely within. Thousands of such planets were engulfed in the Sun, while only those with fairly round orbits escaped absorption. The surviving large eccentricity of the orbit of *Mercury* is therefore in no sense an exception to the present theory, but the large inclination may have aided this small planet in eluding the influence of *Jupiter*.

The Moon must have been captured by the Earth after these bodies were well within the orbit of *Jupiter* and probably even within the present orbit of

Mars. The fact that the Moon is not very near the Earth, but at a distance scarcely less than half of that at the time of capture, shows that while the events occurred a long time ago, it was comparatively late in the history of the solar system.

As for the obliquities of the inner planets, we observe that all these globes are small, and therefore they could not have greatly increased their masses by the capture of satellites entering the closed spaces within the HILL surfaces about these bodies. The obliquity of *Mars* is about 24°.8, that of the Earth 23°.5, while that of *Venus* is not certainly known, but is believed to be small. This is in accord with theory, for a body as small as *Mars* and more remote than the Earth, other conditions being equal, ought to have a larger obliquity; and in the same way the obliquity of *Venus* ought to be smaller than that of the Earth.

Accordingly we conclude that throughout the solar system the universal tendency has been to obliterate planetary obliquities, but the deficiency in the supply of material has left the resulting effects on the different planets very unequal, though tending everywhere in the same direction of zero obliquity, as exhibited in the typical case of the planet *Jupiter*.

§ 186. *Summary of the Phenomena Explained by the New Theory of the Origin of the Planetary System.*

1. The arrangement of the motions of the planets in a narrow zone, near the fundamental plane of the system, making the mutual inclinations of their orbits very small.

2. The circularity of the planetary orbits, and their mutual spacing, so as to give great stability to the resulting planetary system.

3. The circularity of the orbits is due to the secular action of the nebular resisting medium formerly pervading the system, and not at all to detachment in round orbits by gradually accelerated rotation, as imagined by LAPLACE.

4. Motions which were not stable have not long endured, but have passed by gradual transformation into a more stable condition, usually by the bodies concerned dropping nearer the dominant central mass, the planets descending towards the Sun and the satellites descending towards their several planets.

5. The Asteroids were originally distributed over the entire outer part of our system, but the attraction of *Jupiter* rendered their motions unstable, and instead of crossing his path as formerly they have now been transformed so as to lie wholly within his orbit.

6. The satellites of *Saturn, Uranus* and *Neptune,* were originally planets moving in independent orbits around the Sun. Their capture and survival near the several planets which now govern their motions throws an interesting and impressive light upon the condition of our system in the remote past; and enables us to make out with certainty that many such bodies once circulated about the Sun and traversed the spaces between all the planets.

7. At present the only known surviving group of such bodies is the Asteroids, between *Mars* and *Jupiter,* but later researches may establish the existence of small bodies of this class in other interplanetary spaces.

8. Observation shows that no bodies comparable in brightness to *Saturn's* larger satellites now exist between *Jupiter* and *Saturn*; and therefore it seems that *Jupiter* has done his work of clearing up the system very effectively.

9. The satellites are all captured planets, and since they became attached to their several primary planets have had their orbits reduced in size and rounded up under the secular action of a resisting medium.

10. Satellites may be retrograde as well as direct, but the chance of retrograde satellites surviving is very slight, except on the outer parts of the satellite systems, where the vortex of nebulosity has a very small density.

11. This explains the retrograde satellites of *Jupiter* and *Saturn* and the considerable eccentricities of their orbits, which are survivals of still larger eccentricities dating from the epoch of capture.

12. The capture of the periodic comets is similar to that of the Asteroids, except that some comets may once have come to our system in parabolic paths, while most of the Asteroids were no doubt parts of the original solar nebula, and have now been simply worked into the region within the orbit of *Jupiter.*

13. *Jupiter's* capture of a vast number of satellites or small planets once moving about the Sun, in the course of long ages, has greatly increased his mass. As the planes in which these bodies moved passed on the average about parallel to *Jupiter's* orbit around the Sun, the final effect of these successive additions to *Jupiter's* system was to give the resulting axis of the planet's rotation a situation almost perpendicular to his orbit. Whatever may have been the original obliquity, this situation with the planet's axis nearly perpendicular to the plane of the orbit has necessarily resulted from the capturing of vast quantities of satellites moving near the plane of *Jupiter's* orbit about the Sun.

14. When *Jupiter* was as remote from the Sun as *Saturn* is now, and had a correspondingly small mass, his obliquity may have been even greater than that of *Saturn,* which is about 27°. If the solar system could supply enough material to more than treble *Saturn's* mass (as it might do in a sufficiently long

time if *Jupiter* were out of the way), the obliquity of *Saturn* might become as small as that of *Jupiter*, by the time the former reached the present orbit of the latter.

15. The still higher obliquity of *Uranus* and *Neptune* therefore is not remarkable, but rather to be expected, since the axes originally may have had any positions whatever. The retrograde systems of satellites on the outer parts of the solar system therefore present no difficulty.

16. The known satellites of *Uranus* are all extremely near the planet, and the total extinction of their eccentricities indicates that originally they were at much greater distance. In working down towards the planet they have probably worked into the plane of the equator just as in the analogous case of the inner satellites of *Saturn* and of the particles now constituting *Saturn's* rings. This is a natural effect of oblateness where resistance has greatly reduced the size of the orbits. The oblateness of *Uranus* is found by observation to be between 1 : 25 and 1 : 12 (cf. *A.N.*, 3992, pp. 119-120).

17. *Neptune* too has a sensible oblateness, as shown by the shifting of the plane of the orbit of his satellite; yet in this case the satellite is not in the plane of the equator, but inclined to it by an angle of something like 20° (cf. *A.N.*, 3992, p. 121).

18. The satellites of *Uranus* and *Neptune* were captured early in the history of our system, as we may infer from the present smallness and roundness of their orbits. This also follows from the subsequent action of *Jupiter* and *Saturn* in clearing our system of numerous bodies of this class once revolving across their orbits.

19. And just as the Asteroids in the course of ages have been thrown into the comparatively stable region within *Jupiter's* orbit, so also no doubt were the terrestrial planets at a much earlier epoch. This transformation may have occurred when *Jupiter* was about as far away from the Sun as *Saturn* is now. At any rate it was at a very early epoch, and when *Mercury*, *Venus*, the Earth, the Moon (formerly an independent planet) and *Mars* were thrown within *Jupiter's* orbit, their transformed orbits had various eccentricities, some fairly large and others comparatively small. As all the eccentricities have been decreased by the secular effects of the resisting medium, in which the planets have since revolved, it is not remarkable that the eccentricities of *Mercury* and *Mars* are still fairly large.

20. The considerable eccentricity of *Mercury's* orbit is therefore no exception in the new theory of the formation of the solar system. For on the one hand orbits much more eccentric than the original orbit of *Mercury* might have caused that planet to work into the Sun, and many planets of the size of *Mercury* undoubt-

edly were thus swallowed up in laying the foundations of that immense central mass; while on the other if the eccentricity had not been sensible it might have been more difficult to have escaped capture by *Jupiter*. The high inclination of the orbit of *Mercury*, however, also aided it in escaping capture by this giant planet.

21. Just as *Jupiter* in the course of ages engulfed millions of satellites, so also the Sun engulfed millions of planets, some of them of the size of the Earth, or even as large as *Jupiter* himself, which has less than a thousandth of the mass of the Sun.

22. If the planets moved near the central plane of the system it must have been difficult for them to escape both *Jupiter* and the Sun, and undoubtedly the surviving terrestrial planets are only a few of the fortunate bodies among vast multitudes which perished. In clearing up the interior region of his system the Sun swallowed up all bodies which did not revolve in fairly round orbits, and the disturbing action of the other planets often aided in this destructive work which has left the inner parts of our system so vacant.

23. As the periodic comets are captured, it is not remarkable that they too, with few exceptions, have motions in the same direction as the planets, which have gradually transformed their orbits. This is a natural outcome of the planar arrangement of the solar system, which itself resulted from the primordial whirling of the spiral nebula from which the system was formed.

24. The present theory explains such phenomena as the *Gegenschein*, the Zodiacal Light, the Rings of *Saturn*, etc.; and shows us that every planet has a cosmical vortex revolving about it; but at the great distances of the planets the satellites are the only bodies which are bright enough to be seen in our telescopes.

25. All the planetary bodies are growing slightly by the gathering up of cosmical dust, and this is the cause of the outstanding secular acceleration in the mean motions of the Sun and Moon.

26. As for the axial tilts of the terrestrial planets they are in accordance with theory, *Mars* having a higher obliquity (24°.8) than the Earth (23°.5). All these planets are so small that they could not capture enough satellites to entirely obliterate their original obliquities, though in all cases the obliquities have been materially reduced.

27. As for the obliquity of *Venus* it may be confidently asserted that it does not exceed that of the Earth, and probably is considerably smaller. The estimates of SCHROETER, DEVICO and others among the older observers, which give an obliquity between 75° and 53°, must be regarded as doubtful; but

Schroeter's period of $23^h\ 21^m\ 7^s.977$ is more probable, since DeVico obtained $23^h\ 21^m\ 22^s$ from 10,000 observations in 1839-41.

28. The Earth never rotated on its axis much if any more rapidly than at present; and it is impossible for it ever to have detached the Moon, as was once believed. The Moon was originally a planet, but subsequently captured by the Earth, just as in the case of the satellites of *Jupiter* captured by that giant planet.

29. All the geological speculation which has been based on the supposed former rapid rotation of the Earth is therefore absolutely without foundation. The Moon is of celestial origin, and has never been near the Earth; and while tidal friction is a real physical cause, producing some secular changes, it has exerted much less influence than we have heretofore believed.

30. It is therefore probable that *Venus* has a rotation period of about a day, like the Earth and *Mars*. But *Mercury's* rotation was originally insensible like the Moon's, and it probably now shows one face towards the Sun, as observed by Schiaparelli in 1882.

31. In view of the roundness of *Neptune's* orbit, it is impossible to believe that the solar system terminates at *Neptune*. There must be at least two or three more planets, and the outer boundary of the system is likely to extend to 100 astronomical units. The original extent of the spiral nebula from which our system was formed probably was ten times larger, or 1,000 times the radius of the terrestrial orbit.

32. This shrinkage of the solar system with the lapse of ages is a necessary result of gravitation, and verified by the observed fact that the satellites in nearly all cases are near the centres of the closed spaces about their several planets.

33. The Moon is the only notable exception to this rule, the descent toward the Earth having been so gradual as to leave the present distance about half of the probable distance at the epoch of capture. This would seem to indicate that the Moon was captured late in the history of the solar system, when the resistance in the region where the Earth revolves was not great enough to bring this satellite rapidly towards its primary planet.

34. The capture of the Moon certainly occurred since the Earth came within the orbit of *Jupiter*, and probably since the distance was less than the present distance of *Mars*. Our Moon therefore once revolved as an independent planet in the space lying between the present orbits of *Mars* and the Earth.

35. The orbit of the Moon prior to its capture by the Earth probably had a sensible eccentricity, of the same order as that of the present orbit of *Mars*, but not larger than that of *Mercury*, nor smaller than that of the Earth.

36. It follows from this new theory of the origin of the solar system that the spiral nebulae are not made up of gas and fine cosmical dust merely, but also of solid globes of planetary size. The nebulae are therefore full of embryo planets, and it is only in such a vastly diffused mass free from disturbing centers of attraction that nebulosity can collect together to form planets. The planets and their systems are afterwards perfected by the capture of satellites, but the nuclei were already well developed in the primordial nebula from which the whole system arose.

CHAPTER XVI.

On the Figures and Dimensions of the Planets and of the Ring System of Saturn, and on the Internal Constitution of the Planets Resulting from Laplace's Law of Density.

§ 187. *Observations of the Figures of the Planets and of Their Absolute Dimensions, Which Depend also on the Most Probable Value of the Solar Parallax, Required in Researches on Cosmogony.*

To ascertain the attractive forces at work near the planets and satellites, and also the modifications of these forces, due to rotation, as required in researches on cosmogony, it is necessary to have accurate determinations of the masses and dimensions, and also of the figures and rotation periods of these bodies. We shall therefore examine this subject with care, in order to have before us the most precise data now available.

The determination of the masses of the planets is a problem in gravitational astronomy, and the results adopted in this work have already been considered in Chapter III, § 42. The problem of the absolute dimensions of the planets involves the solar parallax, and also the troublesome effects of irradiation. The latter influence indeed has usually been neglected, but a method for eliminating this disturbing cause was devised by the author at Washington in 1901-1902, and by him applied to every planet and satellite of the solar system which has a measurable disc. This method for eliminating the effects of irradiation depends on observations taken by daylight as well as by night, and is more fully explained below.

The problem of the solar parallax is a very old one and need not be discussed at length in the present work. In *A.N.*, 3897, an account is given of an investigation made in 1895, by which the author was led to the value $8''.796 \pm 0''.006$, agreeing exactly with the value $8''.7965$ independently found by Newcomb two years later (cf. *Astronomical Constants*, p. 166). In *A.N.*, 3866, Dr. Boris Weinberg, of Odessa, has discussed all of the methods of finding the solar parallax,

by a process of adjustment based on the theory of probability, and reached the value $8''.8004 \pm 0''.00243$. WEINBERG did not use DOOLITTLE'S aberration constant of about $20''.530$, which in combination with the MICHELSON-NEWCOMB velocity of Light (namely $V = 299860 \pm 30$kms.) gives $\pi = 8''.777$. In *A.N.*, 3897, it is pointed out that the present writer assigned the highest weight to the aberration method, and that depending on the opposition of *Mars* and the small planets, which was substantially the method followed by NEWCOMB. It may also be noticed that the aberration constant found by PROFESSOR A. HALL, JR., from a careful discussion of the Ann Arbor Meridian Observations of *Polaris* in 1898 was $20''.55$, corresponding to $\pi = 8''.769$. Accordingly it appears that the values of the solar parallax resulting from recent work on the constant of aberration are slightly smaller than WEINBERG'S mean value, and the effect of including them would be to reduce his solar parallax a little. In the *Monthly Notices* for June, 1904, MR. A. R. HINKS, of Cambridge, England, finds from the *Eros* observations $8''.7966 \pm 0''.0047$. He has since dealt with the problem in a series of papers, the concluding value in the *Monthly Notices* for May, 1909, being $\pi = 8''.807 \pm 0''.0027$. In Lick Observatory *Bulletin* No. 150, PERRINE gives the value $8''.8067 \pm 0''.0025$, from all the photographs made with the Crossley Reflector in 1900. The *Eros* observations taken by the author at Washington, with the micrometer, gave the value $\pi = 8''.806$.

Under the circumstances it is clear that we may still adhere to the value of the solar parallax deduced in 1895. If the true value is any larger than $8''.796$, the difference will be too small to be of much consequence; and, in view of the smaller values given by the aberration method, any departure from the values heretofore used by NEWCOMB and the author would not be justifiable.

§ 188. *On the Magnitude of the Irradiation of the Planets and of the Refraction Due to Their Atmospheres.*

With regard to planetary irradiation, and the observations by which the author sought to eliminate this disturbing cause, reference must be made to *A.N.*, 3984, and a series of papers on the diameters of the planets as seen at night, affected by irradiation, and as measured by daylight, free from this enlarging cause, in *A.N.*, numbers 3665, 3670, 3676, 3737, 3750, 3757, 3764, 3768, where it has been shown that at night the discs of the planets and satellites are surrounded by sensible spurious zones due to the irradiation. We have determined by careful observations about what these zones are in angular magnitude, when the bodies are viewed in large refracting telescopes. The following table gives these irradiation

zones, and their equivalents in kilometres at the mean distances of the several bodies:

Planet and Satellite	Irradiation in the Diameter	Absolute Equivalent at Mean Distance	Irradiation on One Limb = One-half
	″	km	km
Mercury	0.30	217	108
Venus	0.36 – 0.72*	261 – 538	130 – 269
Mars	0.38 – 0.76*	307 – 615	153 – 308
Jupiter	0.75	2847	1424
Saturn	0.56	3901	1950
Uranus	0.39	5425	2712
Neptune	0.20	4376	2188
Jupiter's Satellite I	0.25	915	458
" " II	0.23	864	432
" " III	0.34	1277	638
" " IV	0.27	1025	512
Titan, Sat. of *Saturn*	0.14	955	478

In the case of *Mars* the irradiation varies from about 0″.70 to 1″.02, with an average value of about 0″.84 (cf. *Report of the Superintendent of the U.S. Naval Observatory for* 1902, p. 18). When we see a planet in a telescope at night, the zone of the irradiation always extends much beyond where the atmosphere of the planet really terminates. And hence the real position of the atmosphere is within the apparent disc of the planet, by an amount almost equal to the constant of irradiation for the limb.

We cannot see a star after it appears to touch the planet's disc; and hence we never can see a star enter the atmosphere, because the irradiation zone overlaps it in general many times its greatest admissible depth. Thus from the very nature of the case observation of refractive effects in the planetary atmospheres appears to be impossible.

If a star should be occulted bright enough to be seen by day, when the planet's disc is free of irradiation, under the daylight illumination of the sky, and both images should be steady enough to give a perfect contact, there would be theoretically a chance of seeing the phenomenon heretofore searched for in vain; but the probabilities of realizing these conditions in practice are so small that no hope is held out that such an effort would be successful.

During the past century numerous occultations of stars by the planets have been witnessed by different observers, not a few of whom hoped in this way to find sensible effects of refractive atmospheres about these bodies. It seemed natural to expect that the light of a star in passing very near the disc of a planet

* The irradiation in the telescope is the larger value; reduction to mean distance about halves the angle subtended.

would suffer a double horizontal refraction; and judging by the amount of the horizontal refraction in our own atmosphere (about 2000″) the effect ought to be sensible in some cases, if the observer could see the star after its light really enters the planet's atmosphere. The probable heights of the atmospheres have been computed, and found to represent very small arcs, of the order of 0″.3 or less; yet if the observer could gaze directly upon this atmospheric rim about the planet, it might be possible to see a star slightly affected in the time of its disappearance or reappearance. In disappearing it ought to hang on the limb of the planet, and be somewhat delayed in its time of extinction; while in reappearing it ought to come out a little early, and remain nearly still till the planet moved over the space represented by double the amount of the horizontal refraction of the atmosphere. But so far as we recall not one of the many attempts to observe this phenomenon of planetary refraction has ever been unequivocally successful. Nevertheless, since refraction is a true cause, and must be at work in the atmospheres of the planets, a few observers have noted slight irregularities (probably of an optical or terrestrial nature), which were erroneously attributed to the atmosphere of the occulting planet.

The above considerations will probably make it clear that the phenomenon here sought can never be observed.

It appears, therefore, that owing to the irradiation, refractive effects are completely hidden and can never be observed in stars occulted by the planets, and all such effort is likely to be vain and useless.

§ 189. *On the Diameter and Rotation of Venus and on the Efficiency of Tidal Friction in the Solar System.*

The following discussion of the author's investigation of the diameter of *Venus* is taken from the *Astronomische Nachrichten*, No. 3676.

During the past two hundred years, the diameter of *Venus* has been determined by more than fifty separate investigators, and in the course of the Nineteenth Century has been carefully studied by the Government parties in the transits of 1874 and 1882, as well as by numerous individual observers equipped with Heliometers, filar and double-image Micrometers and other apparatus of special design; yet in spite of all the labor which has been bestowed upon the subject it appears that there is still no standard value in general use among astronomers.

The measurement of the diameter of *Venus* presents among others the following peculiar difficulties:

(1) The enormous change in the geocentric distance of the planet renders the apparent diameter extremely variable. And unfortunately the phase is so arranged as to give merely a thin crescent when the body is nearest the Earth, and when the disc is more rounded out and diminished in angular diameter, the distance is so much increased that errors entering into the measures affect the reduced diameter greatly.

(2) The line-like horns of the very thin crescent which the enlarged disc presents when near inferior conjunction, are so delicate that small atmospheric irregularities set them in violent motion, and it is difficult to locate their quiescent positions with the Micrometer wire. Even greater difficulty is experienced in forming accurate contacts of the images produced by the Heliometer and double-image Micrometer.

(3) When the crescent is enlarged the horns are broadened and more steady, but the apparent diameter is less, and an error in the setting enters with enlarged effect into the final result.

(4) When the planet approaches superior conjunction, and the disc is nearly round, these difficulties diminish, but the diameter is then so small that the advantage sought is more than lost, through the enlargement of errors of observation in the final result. Practically these difficulties are augmented by the circumstance that the planet is then near the Sun, and must necessarily be observed at a time of day when the air is much disturbed by the heat.

(5) During the transits across the Sun's disc, the image is both large and round, but measurement of its diameter is vitiated by unknown causes, like irradiation, a halo due to the planet's atmosphere, the black drop, etc., which, however, have been carefully investigated.

(6) *Venus* is always extremely bright, and its light very white, which renders the irradiation large and the secondary spectrum troublesome.

The experience of the past year (1900) led to the following conclusions:

(*a*) When the planet has a large angular diameter and the horns are very thin, they are tremulous and difficult to see sharply. Errors of observation under such conditions are increased, and produce in the reduced diameter an effect just about equivalent (within a certain range) to that due to the errors which arise when the horns are thicker and the planet apparently smaller and more remote from the Earth.

(*b*) The irradiation at the horns of the crescent is sensibly the same while the planet moves from East to West elongation; but in order that the horns may be sufficiently thin to be similarly affected by the state of our atmosphere, observations should not be taken when the planet is more than about ten weeks from inferior conjunction.

(c) When the observations are confined to this period of five months about inferior conjunction, the horns of the crescent are so thin that with accurate bisections the irradiation may be entirely disregarded. It has the well known effect of blunting the point of the crescent, enlarging it inwardly, and outwardly, by about equal angular amounts, and if the centre of the Micrometer wire be accurately placed on the point of the horn, it will be tangent to the true limb of the planet, without any correction for irradiation whatever.

In view of these considerations no correction for irradiation has been applied to the diameter given by the Washington observations; and all the individual determinations of diameter have been given equal weight, except as affected by the state of the seeing. The arithmetical mean of the 32 measures on 22 days is 16″.787; the weighted mean increases this quantity but slightly, and we have $D = 16''.800 \pm 0''.022$. Using 8″.796 for the Solar parallax, this concluded angular diameter, at distance unity, makes the absolute diameter of *Venus* 12181.7 ± 16 kilometers.

A very complete table of the results found by previous investigators is given in *A.N.*, 3676, but it need not be repeated here.

The value 16″.820 for the diameter of *Venus*, deduced by Dr. Auwers in 1894 from the transits of 1874 and 1882, is especially worthy of attention. Besides resting upon a most rigorous and exhaustive discussion of all available material, it has the advantage of resulting from observations taken when *Venus* had a diameter of 63″.5. Under these circumstances errors in the observed diameters affect the value at the mean distance by only about one-fourth of their original amount. Though the elimination of such influences as the irradiation is difficult and attended with some uncertainty, it hardly seems possible that his value of the diameter can depart from the truth to any great extent. The good agreement between the diameter found at Washington and that obtained by Auwers may be considered to fix the true diameter of *Venus* at about 16″.80 ± 0″.02, making the uncertainty only about 1 part in 840.

On the Rotation Period of the Planet Venus, and on an Observational Criterion for Testing the Efficiency of Tidal Friction in the Solar System.

In Chapter X, on the "Origin of the Lunar-Terrestrial System by Capture," § 123, pp. 256–7, the author has briefly considered the rotation period of *Venus* and pointed out grounds for holding that it cannot differ much from that of the Earth. The theory of planetary rotation depending on the impact of satellites, developed in that chapter, and the similar investigations since made on the obliquities of

the planets § 183, pp. 393–7, and the origin of the Lunar Craters, Chapter XIV, has now enabled me to throw considerable additional light upon the problem of the rotation of *Venus*.

On examining the records of the older observers, the author has been much impressed with the consistency of their conclusions, and the care with which their observations were taken. From the time when J. D. CASSINI first found the rotation to be about 23 hours, in 1667, till 1890, there was very satisfactory agreement among painstaking and careful observers that *Venus* rotated in 23 hours 21 minutes. SCHROETER's work in 1789-93 seems very conclusive indeed; and SIR WM. HERSCHEL, who gave considerable attention to the planet for a time, remarks (*Phil. Trans.*, 1793, p. 214) that the rotation period cannot be so long as 24 days, which BIANCHINI had deduced from inadequate data. The observations of DEVICO at Rome, based on a long series of over 10,000 observations taken with the Cauchoix refractor, seem to be especially satisfactory; and he only changed SCHROETER's period from 23 hours 21 minutes 19 seconds (*Phil. Trans.*, 1795, p. 153) to 23 hours 21 minutes 21.934 seconds (*Roma Oss.*, 1840-1850, cf. HOUZEAU's *Vade Mecum de l'Astronomie*, p. 467).

The accompanying table gives the conclusions of the principal observers on this interesting question:

Epoch	Time of Rotation	How Observed	Authority	Source
1665	"A little less than a day, about 23hrs."	A bright spot	J. D. CASSINI at Bologna.	HOUZEAU'S *Vade Mecum*, p. 466.
1726	24 days, 8 hours.	Surface markings, making obliquity 75°.	BIANCHINI at Rome.	" "
1732	23h. 15m.	Observations of J. D. CASSINI, 1667.	J. CASSINI at Paris.	" "
1732	23h. 20m.	Discussion of BIANCHINI's observations.	J. CASSINI at Paris.	" p. 467.
1789	23h. 21m. 19s.	Observations of the horns, apparently showing obliquity of 72°.	SCHROETER.	" "
1793	"Subject to considerable uncertainty but it can hardly be so slow as 24 d."	With reflecting telescopes which showed faint spots.	SIR WM. HERSCHEL.	*Phil. Trans.*, 1793 p. 214.
1801	23h. 22m.	By the return of same indentation of phase.	FRITSCH.	HOUZEAU'S *Vade Mecum*, p. 467.
1811	23h. 21m. 7.977s.	Rediscussion of observations.	SCHROETER.	" "
1840–50	23h. 21m. 21.934s.	By the study of over 10,000 observations with large CAUCHOIX refractor at Roman College Observatory; obliquity 49° 57′.5.	DE VICO	" "
1890–95	225 days.	Brilliant white spots near terminator.	SCHIAPARELLI	*A.N.* 3304.
1891	23h. 21m. 22s.	Observations at Brussels.	NIESTEN.	CLERKE'S *Hist. of Astron.*, p. 252.
1900	One day.	Spectrographic observations at Poulkowa.	BELOPOLSKI.	*A.N.* 3641.
1903	Period probably not short.	Spectrographic observations at Flagstaff.	LOWELL and SLIPHER	*A.N.* 3891.

Plate XIX. The Planet Venus, as Observed by Professor E. E. Barnard with the 12-inch Equatorial Telescope at Lick Observatory, 1889, May 29^d, 11^h 12^m, A.M.

The older observational evidence, therefore, which was free from any theoretical bias, nearly all supports a period of $23^h\ 21^m\ 21^s$, which is about 35 minutes less than our sidereal day ($23^h\ 56^m\ 4^s.09$). Now *Mars* rotates in $24^h\ 37^m\ 22^s.67$, or in a period about 41 minutes longer than a terrestrial day. On carefully reflecting over this excess of the Martian day, and the nearly equal deficiency of the Cytherian day, compared to the terrestrial as standard, I have been much surprised at such equable differences in the rotation periods of two planets about equally distant without and within the orbit of the Earth respectively. And the impression has grown on my mind that there must be some physical cause for this orderly arrangement connecting the axial rotations of the planets with their distances from the Sun. What, then, is this cause?

To answer this question it is sufficient to remark that (in *A.N.*, 4343) the axial rotations of the planets have been traced to the capture of satellites, which by impact against these bodies has given them a direct rotation, similar to the prevailing revolutions of the satellites in their orbits. This conclusion is confirmed by the investigations of the obliquities of the planets and of the origin of the Lunar craters.

To understand why *Venus* rotates more rapidly than the Earth, while *Mars* rotates more slowly, it is sufficient to recall that the orbital velocities of all satellites within the Earth's orbit are greater than those without. When moving in any kind of orbits and colliding with the body of *Venus*, they gave that planet a greater impulse in its rotation about its axis, than would arise under similar conditions at the distance of *Mars* or the Earth. This follows at once from the augmentation of the velocity in that region and from the nature of the closed Hill surfaces at different distances from the Sun, as discussed in *A.N.*, 4341-42; and has been treated by PROFESSOR LOWELL from a somewhat different point of view in *A.N.*, 4351. The vortex about the planet always has a systematic tendency to direct rotation, though a few of the individual particles move retrograde, just as shown in the observed motions of the satellites.

If, therefore, a planet such as *Venus* revolving at a considerable distance within the orbit of the Earth should have the mass increased by the capture of satellites moving with higher average velocities, there would be increased force in the impacts, and these cumulative impulses, in the course of ages, would give it a more rapid axial rotation. This, then, is the true physical cause of the connection existing between the rotation periods and the distances of the planets *Venus*, the Earth and *Mars*. The mass of *Venus* is only slightly smaller than that of the Earth, while that of *Mars* is much smaller; all of which is consistent with the causes here at work and the secular effects which would arise from these causes.

And as the rotations have thus been determined by the nature of the vortex revolving about the planets, it is clear that the obliquities have been correspondingly modified by the same influence. This is an additional reason for thinking that the obliquity of *Venus* probably does not exceed ten or twelve degrees, as inferred from observation by SCHIAPARELLI and other recent investigators.

The reinvestigation of the rotation period of *Venus* by means of the spectograph is greatly to be desired; but the light now thrown upon the problem from the theory of the impact of satellites, it seems to me, is such that the chances are almost infinity to one that SCHROETER, unbiased by any theory, found the true rotation period over a hundred years ago. If this inference is justifiable, it will show that tidal friction has exerted but little influence on the past history of the solar system, and that the impact of satellites has everywhere proved to be the more dominant cause. It follows from this line of investigation that the fall of meteoric matter is still accelerating the rotation of the Earth and counteracts the secular effects of tidal friction.

In order to show this it is sufficient to recall that if r be the radius of the Earth, r' that of *Venus*, ϱ and ϱ' the distances of these planets from the Sun, and M the mass of the Sun, the principal terms of the potentials of the solar tide-generating forces become

$$V = \frac{3}{2}\frac{Mr^2}{\varrho^3}\left(\cos^2 z - \frac{1}{3}\right) \;;\; V' = \frac{3}{2}\frac{Mr'^2}{\varrho'^3}\left(\cos^2 z' - \frac{1}{3}\right); \qquad (a)$$

where z and z' are the zenith distances of the Sun from the disturbed particles on the two planets. As *Venus* has an abundance of clouds, and water vapor, the extent of the oceans on the two planets may be assumed to be comparable; and in *A.N.*, 4104, the writer has shown that the average rigidity of the two globes is about as steel and iron, respectively, so that the bodily tides would be of similar character.

Now the efficiency of tidal friction in retarding the axial rotation is proportional to the square of the tide-generating forces, or

$$\frac{V^2}{V'^2} = \left(\frac{\varrho'}{\varrho}\right)^6\left(\frac{r}{r'}\right)^4\frac{(\cos^2 z - \frac{1}{3})^2}{(\cos^2 z' - \frac{1}{3})^2}. \qquad (b)$$

Neglecting the factor depending on coefficient of LEGENDRE, as being about the same when extended to all parts of the surfaces of both planets, and using 0.723 for the ratio of planetary distances, and putting $r = 8''.796$, $r' = 8''.40$ (cf. *A.N.*, 3676), we find the efficiency of solar tidal friction on the Earth and *Venus* to be

in the ratio of 1 : 5.823 (cf. *A.N.*, 4343, p. 382). The solar tidal friction is therefore nearly six times as large on *Venus* as on the Earth.

But to get the total friction tending to retard the rotation of the Earth, we have to take account also of the Lunar tidal friction. Now the Lunar tide-generating force is easily shown to be about 2.17 times that of the Sun. Consequently the total tidal friction on the Earth is to that on *Venus* as

$$\frac{(2.17)^2 + 1}{5.823} = \frac{5.7}{5.823}. \qquad (c)$$

Thus the combined tidal friction of the Sun and Moon is nearly but not quite equal to that of the Sun alone on *Venus*. Therefore, if the distances of the bodies have been sensibly the same as now, throughout past ages, *Venus* ought to have been somewhat more retarded than the Earth. But on the other hand, if the Moon had formerly been nearer the Earth, as DARWIN supposed, then the tidal friction on the Earth would have exceeded that on *Venus*.

To get some additional light on this question, we may turn to the planet *Mars*, the satellites of which are so small that solar tidal friction is the only cause which could modify the planet's rotation. Now it is easily found by equation (*b*) that, when the two planets are taken to have the same constitution, solar tidal friction alone is 160 times more powerful on the Earth than on *Mars*. The Earth, however, has a 4.7 times greater tidal friction depending on the Moon, and thus the total solar and lunar tidal friction working against the Earth's rotation is about 912 times greater than the solar tidal friction retarding the rotation of *Mars*. But notwithstanding nearly a thousand-fold greater retardation of its axial rotation for indefinite ages, the Earth still rotates 41 minutes faster than *Mars*. This result seems irreconcilable with the theory that the rotations of these two planets have been modified by tidal friction.

As the observations indicate that *Venus* rotates in 23 hours and 21 minutes, it will be seen that this rotation in a shorter period than the Earth is equally unfavorable to the theory that tidal friction has exerted any sensible influence in past ages.

Accordingly, as matters now stand, a confirmation of the rapid rotation of *Venus* found by CASSINI and SCHROETER, would show that the rotations of these three planets had been determined by causes other than tidal friction. And if the Earth never rotated appreciably faster than at present, it would be impossible for the Moon ever to have been a part of the terrestrial globe detached by rapid rotation; and the theory that the Moon is a captured planet would admit of observational confirmation.

With regard to the markings on *Venus*, it is sufficient to say that the indentation of the southern horn of that planet seen by SCHROETER and drawn by him, in the *Philosophical Transactions* for 1792, p. 360, and 1795, p. 176, I have myself seen, with the 26-inch refractor at Washington, in 1900. I remember that, on one occasion, I made a drawing of the indentation of the southern horn almost exactly like that given by SCHROETER. This indentation was clear and unmistakable, but as it was not seen on more than one or two occasions, the observation is reconcilable only with a rapid rotation, such as that found by SCHROETER. The observations of 1900 were devoted to the measurement of the diameter of the planet, and the observer therefore did not have opportunity to make a prolonged study of the planet's rotation. But as the indentation on the inside of the southern cusp was plain and so easily recognized on one occasion that I made a drawing of it, I am satisfied that nothing but good atmospheric conditions and persistence are required to confirm the short rotation period with certainty.

It is really not possible that the rotation period can be identical with the sidereal revolution. Since I observed *Venus* on 22 days between April 23 and August 18, 1900, and only drew the indentation of the southern cusp on one day, whereas on June 5, I noted "Two cusps identical," these observations alone are irreconcilable with a period of 225 days. For if *Venus* showed the same face towards the Sun, there would be no libration of a planet revolving in an orbit nearly circular that could account for the observed indentation of the southern horn. Under the hypothesis of a 225-day rotation-period, the indentation of the southern horn, if it showed at all, would show all the time, and this certainly was not the case. Observers who may take up the problem of the rotation of *Venus* will find this somewhat rare but unmistakable indentation of the southern horn the most satisfactory method of settling the question of rotation and obliquity, as well as the importance of tidal friction in the development of the solar system.

If *Venus* should thus be shown to have a rotation in a shorter period than the Earth, there would be two planets on which tidal friction should be nearly a thousand times more powerful than on *Mars*, and yet after the lapse of long ages, both still rotating the most rapidly, and *Venus* more rapidly than the Earth, notwithstanding its greater tidal friction. This, then, would be a valuable observational criterion which would authorize us to reject tidal friction as an appreciable factor in planetary development, which at the same time would confirm the capture of the Moon by the Earth, by showing that as tidal friction had not retarded the Earth's rotation, no such rapid rotation as had been postulated for the detachment of the Moon could ever have existed.

§ 190. *Concluded Absolute Diameters of the Planets and Satellites and of Saturn's System of Rings.*

The constants of irradiation of the planets have been found by comparing the diameters taken at night with those found under the best conditions by daylight, the latter being free of irradiation and therefore absolute so far as this may be attained by eliminating known disturbing causes. This is explained in *A.N.*, 3757, where it is shown that the values thus derived are to be preferred to those obtained with the heliometer, which is also supposed to eliminate the effects of irradiation. The results of measurements on the planetary diameters made with the heliometer appear to exhibit slight systematic errors which are not yet fully understood; under the circumstances, it seems probable that the micrometer by daylight, in the hands of a skillful observer, will give better results than could be obtained with the heliometer. The discussion of the diameters of *Jupiter*, in *A.N.*, 3757, closes with the following remarks:

In view of the differences here brought to light, it is obvious that two sets of planetary diameters should come into use among astronomers: One representing the apparent size of the planet as seen by observers in their telescopes at night, and to be used in physical observations and ephemerides, and in work on the satellites, where these objects are referred to the limbs; the other representing the true dimensions of the spheroid independent of its illumination by the Sun. This absolute diameter is to be used in the theory of the figure of the planet, and in the study of its physical constitution, such as the law of density, the moments of inertia, momentum, etc.

I am not aware of any previous determination which gives these two sets of diameters separately, although for many years it has been recognized that the irradiation renders the illuminated planetary disc sensibly too large. In view of the interest attaching to the absolute dimensions of the planets, it is a little remarkable that so little attention has been paid to the subject by practical astronomers. The supposed difficulty of evaluating the irradiation with accuracy is doubtless the cause of this neglect.

The observations of the equatorial diameter of *Jupiter* were taken by daylight, either before or immediately after the setting of the Sun. The observations were made in the interval of good seeing which occurs at sunset, when the atmospheric currents have ceased ascending from the heat of the day, and have not yet begun to descend from the cooling effects which follow the advance of night. This period of quiescence, depending on the thermal equilibrium of the

atmosphere, varies in duration on different days, and at different seasons of the year, but may be estimated approximately at one hour.

It was necessary to take the measures within this quiet period, and also when the light of day was just strong enough to take off the glare and eliminate the irradiation, without cutting down the disc of the planet, which is always hazy on the phase limb. After a little practice it was found that this could be done with entire satisfaction; and measures of the highest precision were secured. The color screen used was filled with the usual solution of Picric Acid and Chloride of Copper, dissolved in water. The planet appeared of a soft greenish yellow in a fainter field of the same color. The daylight sky was quite bright, so that dark wires were used. The contrast between the planet and the field was great enough to give a perfectly definite limb, but not sufficient to introduce any sensible irradiation. It is difficult to see how the conditions could have been better.

The observations of *Saturn* were taken by daylight, either shortly before, or immediately after the setting of the Sun. The process of measurement was the same as that employed in the case of *Jupiter* (*A.N.*, 3757). The limb of *Saturn* appeared perfectly sharp and distinct. I have never seen better definition on any object. The observations were so timed that on the whole the field illumination had just the right intensity. The concluded equatorial diameter is $17''.240 \pm 0''.006$. It is difficult to see how this value can be subject to an uncertainty of more than $\pm 0''.02$; which would mean that *Saturn's* diameter is accurate to $\frac{1}{862}$ part of the whole. The diameter now deduced from the night measures is $17''.804$. These observations were taken soon after dark and are strictly analogous to those made in daylight. Very great care was exercised to free the settings from systematic errors, and measures were taken only in perfect seeing.

Applying corrections for irradiation, the concluded dimensions of the entire ball and ring system of *Saturn* are as follows (*A.N.*, 3768):

	As Observed at Night	As Corrected for Irradiation	
	″	″	km
External diameter of the outer ring	40.274	39.971 =	276474
Internal diameter of the outer ring, or external diameter of CASSINI'S division	34.757	34.605 =	239358
Diameter of the centre of ENCKE'S division	37.747	37.747 =	261090
Width of ENCKE'S division	0.107	0.410 =	2836
Total width of the outer ring	2.758	2.455 =	16981
Width of the outer part of the ring	1.237	0.934 =	6460
Width of the inner part of the outer ring	1.414	1.111 =	7685
Width of CASSINI'S division	0.418	0.818 =	5658
External diameter of central ring	33.921	33.671 =	232898
Internal diameter of central ring, or external diameter of dusky ring	25.932	25.932 =	179368
Width of central ring	3.995	3.745 =	25904
Internal diameter of dusky ring	20.434	20.434 =	141339

Fig. *a*. Represents the Planet Mercury as Glimpsed by the Author with the 26-inch Refractor at Washington, in June, 1901.

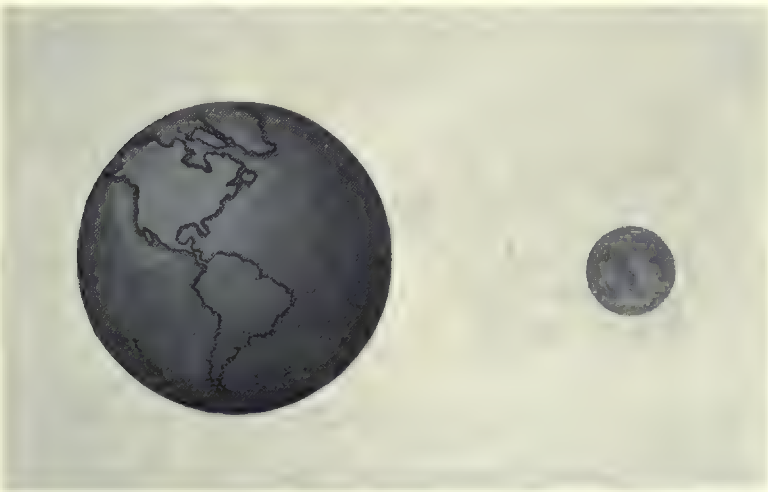

Fig. *b*. A General View of the Earth and Moon, as They Would Appear from a Point in Space.

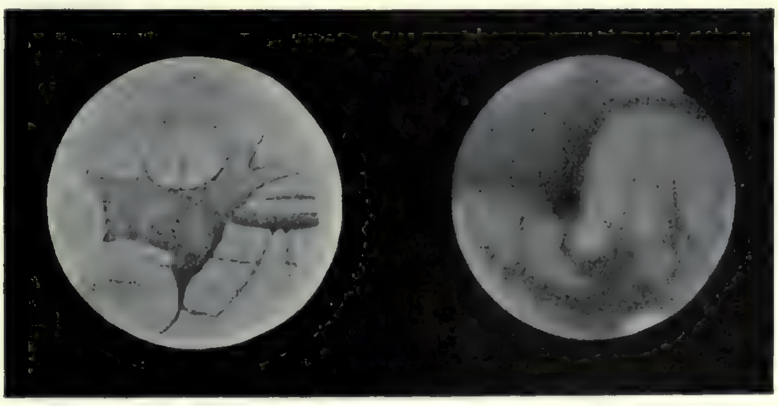

Fig. *c*. The Planet Mars, as Drawn and Photographed by Lowell. The Latter View is the Drawing Made from a Number of the Lowell Photographs by the Skillful Hand of Mr. W. H. Wesley, Assistant Secretary of the Royal Astronomical Society.

Plate XX. The Planets Mercury, the Earth and Moon, and Mars.

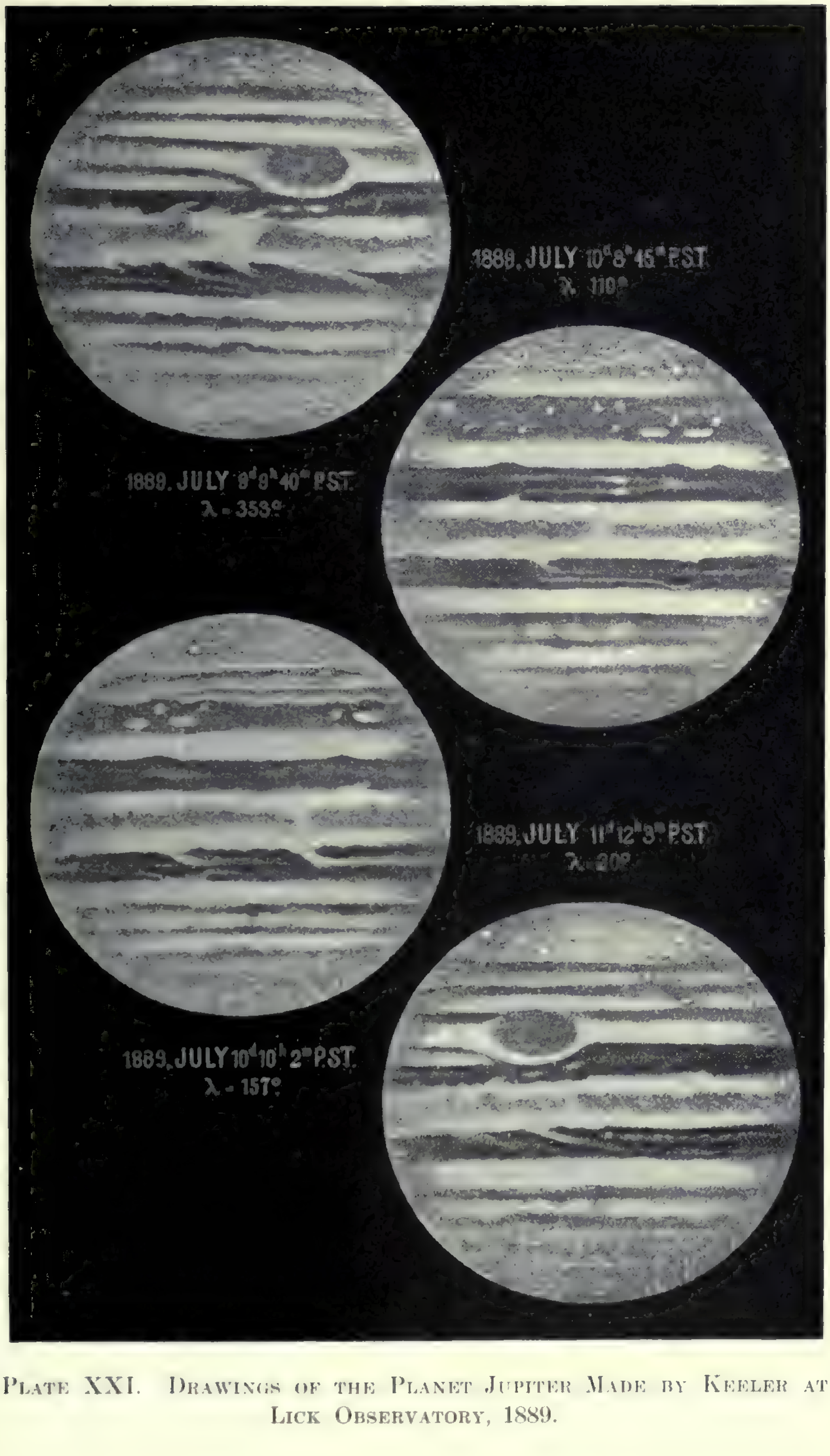

Plate XXI. Drawings of the Planet Jupiter Made by Keeler at Lick Observatory, 1889.

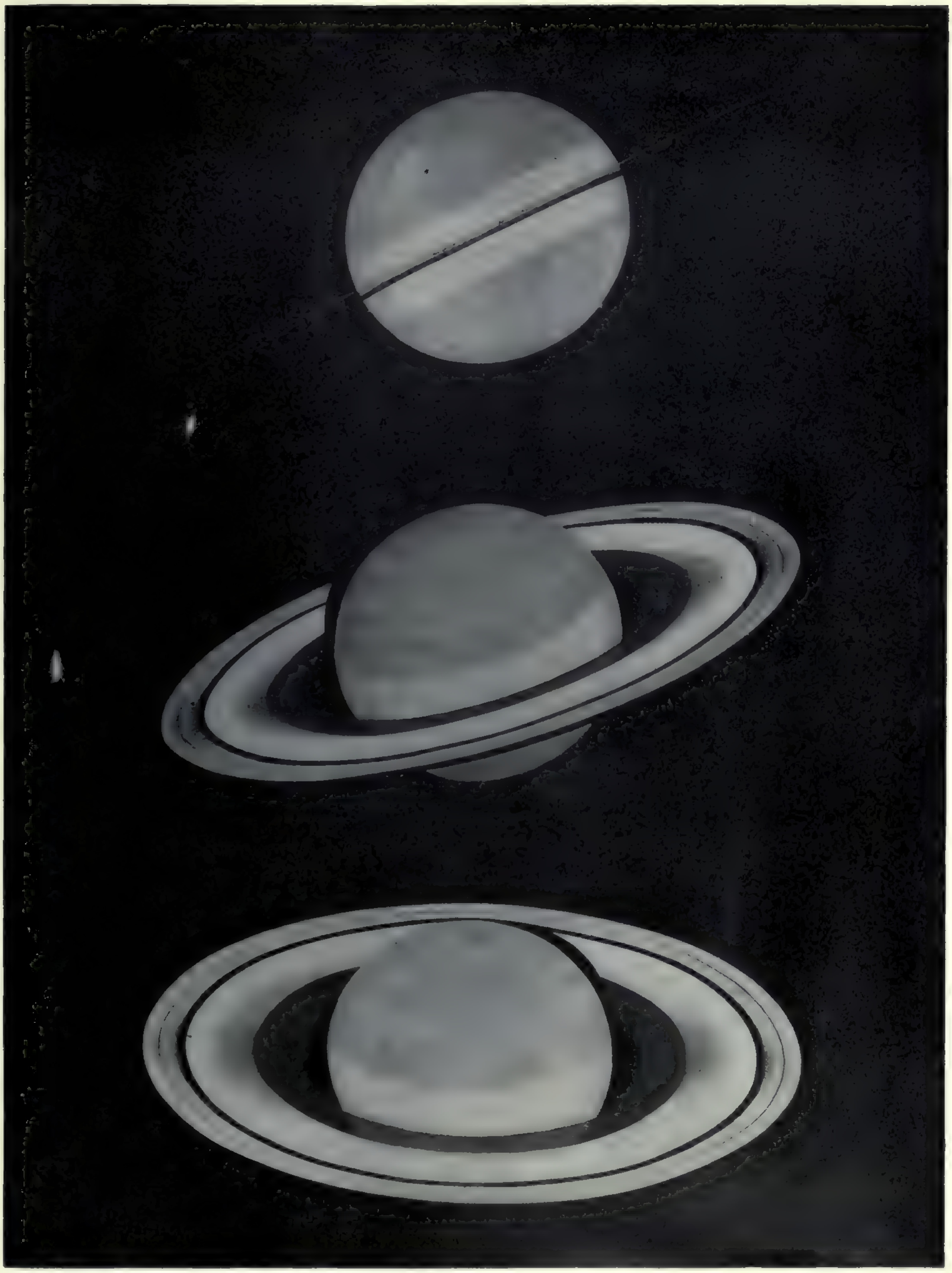

Plate XXII. The Planet Saturn, as Drawn by Proctor, but Modified to Take Account of the Extension of the Dusky Ring Observed by the Author at Washington in 1901 (*A.N.* 3768).

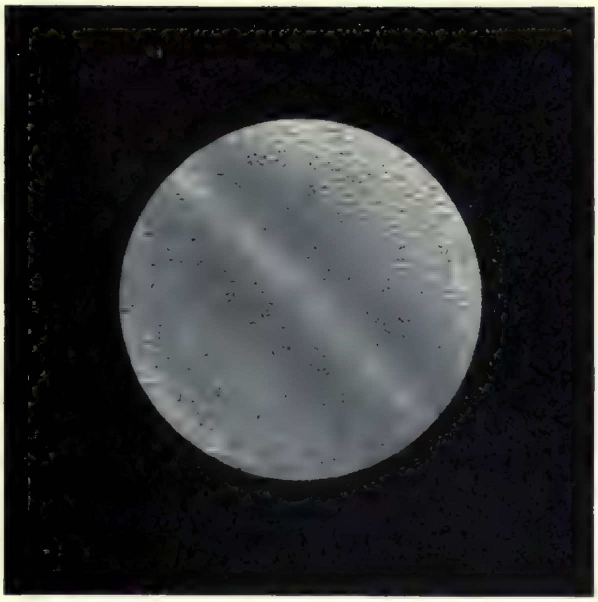

Fig. *a*. The Planet Uranus, with Equatorial Belts, as Drawn by the Henry Brothers at Paris, 1884.

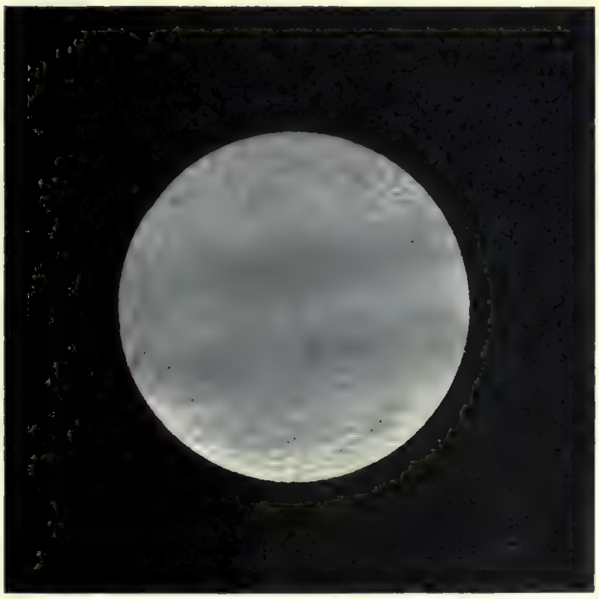

Fig. *b*. Drawing of the Planet Neptune, Showing the Faint Equatorial Belts Discovered by the Author, with the 26-inch Refractor at Washington, Oct. 10, 1899.

Plate XXIII. Views of the Planets Uranus and Neptune.

	As Observed at Night	As Corrected for Irradiation
	″	″ km
Width of dusky ring	2.749	2.749 = 19014
Black space between *Saturn's* globe and dusky ring	1.315	1.597 = 11046
Equatorial diameter of *Saturn*	17.804	17.240 = 119247
Assumed oblateness of *Saturn* (H. STRUVE's value)	0.1013	
Polar diameter with this oblateness	16.005	15.494 = 107167
Assumed mass of *Saturn* (H. STRUVE's value)	1 : 3495.3	
Resulting mean density of the planet	0.7105 (water = 1)	

SUMMARY OF DATA ADOPTED FOR THE PLANETS AND SATELLITES OF THE SOLAR SYSTEM* (*A.N.* 3992).

Planet	i	Mass $m_i \pm u_i$	u_i	Mass in Units of the Earth without Moon	log Δ	Absolute Equatorial Diam.	
						In Arc	In Km.
						″	
The Sun	0	1		332750	0.0000000	1920.00	1392196
Mercury	1	1:(14868548±743427)	1:20	0.02238	"	6.00	4350.62
Venus	2	1:(408134±8165)	1:50	0.815296	"	16.80	12181.72
The Earth & Moon	3	1:(328715±328)	1:1000	1.00000	"	17.592	12756.000
Mars	4	1:(3089967±10300)	1:300	0.107685	"	9.30	6743.44
Jupiter	5	1:(1047.35±0.10)	1:10000	317.701	0.7160033	37.646	141944
Saturn	6	1:(3500±2.0)	1:1750	95.072	0.9795000	17.240	119244
Uranus	7	1:(22780±76)	1:300	14.607	1.2831044	3.0738	42772
Neptune	8	1:(19313±96)	1:200	17.229	1.4781414	2.000	43608

Satellite	Mass in Units of the Earth without Moon	log Δ	Absolute Diameter	
			In Arc	In Km.
			″	
The Moon	1: 81.45	0.0000000	4.8000	3480.5
Jupiter's Satellite I	1: 111.2	0.7160033	0.834	3145
" " II	1: 135.5	"	0.747	2817
" " III	1: 38.75	"	1.265	4770
" " IV	1: 146.5	"	1.169	4408
Titan	1: 49.4	0.9795000	0.73	5049

The following discussion is taken from the *Astronomische Nachrichten*, 3764, and explains how the diameters of the satellites were measured.

As observed with the telescope the satellites attending the planets of the Solar system have measurable discs in three cases, viz: the terrestrial Moon, the four satellites of *Jupiter* discovered by GALILEO, and *Titan* and *Iapetus*, the two principal satellites of *Saturn*. In every other case arising among the nineteen remaining satellites, the discs presented are spurious or stellar in character, and no estimate of the diameters can be formed except on photometric hypotheses involving with the measured brightness assumed albedoes of the satellites. While the resulting diameters are the best that science affords, such hypothetical determinations are always more or less unsatisfactory.

* cf. *A.N.*, 3665, 3670, 3676, 3737, 3897, 3750, 3757, 3764, 3768, 3923; also Report of Superintendent U.S. Naval Observatory, Washington, 1902.

Even in the cases of those satellites of *Jupiter* and *Saturn* which present measurable discs, the difficulties of exact measurement are very great. Besides being extremely small, the images as seen in the telescope are easily disturbed by the stream-like movements of the atmosphere (cf. *A.N.*, 3455). The unsteadiness of the image augments the difficulty of precise measurement, and the same causes which produce a tremor of the image as a whole, render its limbs broken and indistinct, so that it is difficult for the eye to judge of its apparent diameter. A series of diameter measurements on the four large satellites of *Jupiter* and on *Titan* were made with the 26-inch refractor last year (1900). The observer's inexperience in the measurement of such small discs, combined with certain disturbing causes incident to the low altitude of the planet prevented the observations, taken at various hours of the night and in the midst of other work, from being entirely satisfactory. Observations during the present year (1901) show that the fringes surrounding the satellites are a source of greater embarrassment than was formerly suspected. Their motion about the limb of the satellite obscures its outline; and as some allowance must be made mentally for this effect of atmospheric blurring, it was easy to set the wires too close together by about their own thickness. A systematic correction of about $+0''.20$ to the diameters found last year would have given a better approximation to the truth. Under the circumstances it was decided to renew the observations this year, and I availed myself of a number of fairly good nights during the summer. Yet such is the effect of a slight unsteadiness on these small discs, that but few wholly satisfactory measures could be secured. The results of these observations are given below and require no further explanation (cf. *A.N.*, 3764).

Subsequently measures were begun on the diameters by daylight, in the brief period of stillness which precedes and follows immediately after the setting of the Sun. During these favorable moments the images of the satellites were beautifully round, wholly devoid of fringes, and affected by no sensible irradiation. The field illumination being supplied by the diffused light of the evening sky, was probably more steady and uniform than any light which could be furnished at night by artifical means. A color screen filled with Picric Acid and Chloride of Copper was used throughout. During the work on the absolute diameters the field was so strongly illuminated by skylight that as seen through the screen it appeared of a soft greenish yellow color. The limbs of the satellites, while perfectly smooth and distinct, presented very little contrast with their surroundings. The satellites differed adequately in brightness, but hardly at all in color, from the field in which they were projected. In the whole series of observations the measures were so timed with regard to the brightness of the skylight field

that the irradiation was, I think, well eliminated. The dark wires appeared sharp and could be set with the nicest accuracy. In fact every condition which could contribute to accuracy was fulfilled, and I have little hope of improving on the results.

The complete elimination of systematic errors in such delicate work is always extremely difficult. To exercise adequate precaution in this regard, the measures were taken by two methods: first, with the satellite discs placed behind the wires; and second, with the discs just adjacent to the wires, so that the illuminated space between the wires could be made equal to the diameters of the satellites. As the daylight illumination of the field made the space between the wires comparable in brightness with the satellites themselves, while the absorbing medium in the screen gave both the same color, the second method proved to be the most natural, and apparently gave the best results. In the later observations the second method was generally preferred, and its superior accuracy seemed to be confirmed by experience.

Accepting the results of the Washington observations for the absolute diameters (*A.N.*, 3764), it is easily seen that the irradiation constants for the several satellites are

	$D_n - D_d = I$
	$'' \quad '' \quad ''$
Satellite I	$1.08 - 0.83 = 0.25$
Satellite II	$0.98 - 0.75 = 0.23$
Satellite III	$1.60 - 1.26 = 0.34$
Satellite IV	$1.44 - 1.17 = 0.27$

These values are considerably smaller than the irradiation for *Jupiter* (0″.75). When projected on the planet's disc in transit all the satellites, as is well known, appear fainter than the body of *Jupiter*, except Satellite II, which appears as a white spot, and is thus brighter area for area. Mr. E. J. Spitta's albedoes: 0.656; 0.715; 0.405; 0.266; that of *Jupiter* being 0.624, conform accurately to my own estimates of the relative lustre of these several objects. But while Satellites I and II are slightly brighter area for area than the disc of *Jupiter*, their irradiations when off his disc nevertheless are smaller, on account of the great increase in the area of the satellite discs due to irradiation. Thus it appears from the figures given above that the diameters of the satellites are enlarged about 28 per cent. by irradiation, which gives 64 per cent. of increase in illuminated surface.

The average brightness at the limb of the satellite disc is therefore greatly reduced, and the reduction in brightness of the limb increases rapidly with the shrinkage of the radius.

In measuring the diameters by daylight, two opposite causes had to be considered: first, the encroachment of the daylight illumination upon the disc of the satellite, when the Sun was too bright; second, the expansion of the disc beyond its real size, on account of irradiation, which would begin as soon as the field illumination attained a certain faintness. The aim was to so time the observations that these two causes would destroy each other. And in the whole series of observations I believe that the desired end was well attained. It is doubtful if an uncertainty of $\pm 0''.05$ (188 kilometres) attaches to the concluded diameter of any one of these four satellites.

§ 191. *Certain Elements of the Terrestrial Spheroid Used in the Calculations on the Other Planets.*

The following account from *A.N.*, 3992, explains the elements of the Terrestrial Spheroid, employed in the calculations on the other bodies of the solar system and also the results adopted for the major planets.

The equatorial radius of the Earth adopted is 6378000m. This is the value which Professor Helmert recently communicated to Professor Newcomb, and subsequently announced to the International Geodetic Association in August, 1903 (cf. *Procès-verbeaux de la Quatorzième Conférence Générale de l'Association Géodésique Internationale, Rapport du Bureau Central, Août,* 1903). It is noticeable that the geodetic authorities of the Continent from the time of Bessel have used slightly smaller values than Clark and some other British writers. But Superintendent Tittman, of the U.S. Coast Survey, and others of large experience in geodetic work, agree that an uncertainty of at least 250m. still exists in the equatorial radius of the Earth, without great promise of early disappearance; yet the accuracy already attained represents about one part in 25000.

The masses of the planets here adopted are those used in the paper on the Invariable Plane of the Planetary System (*A.N.*, 3923), which contains also the methods and considerations leading to these masses, as well as the probable uncertainties (u_i) by which they are affected.

The material included in Table I, § 198, is that considered most accurate, and in general requires little explanation. The oblateness of the Earth is taken to be 1 : 297.7, in accordance with an unpublished investigation based upon all the available data for determining the figure of the Earth. The equatorial radius at distance unity subtends an angle of $8''.796$, and the polar radius $8''.766645$, making the mean radius $8''.78615$. The ratio of the centrifugal force to the force

of gravity at the equator here adopted is the value recently communicated to PROFESSOR E. W. BROWN by PROFESSOR HELMERT, namely: $\varphi = 0.00346768 = 1 : 288.3773$. With HELMERT's value of the equatorial radius, 6378000m. and a rotation period of 86164.09 mean solar seconds, the centrifugal force is 3.3915 cm. and the acceleration of gravity at the equator 9.780305m. This makes the acceleration of pure gravity at the equator 9.81422m. And using HELMERT's formula for the variation of gravity in different latitudes, the observed gravity in mean latitude 35° 15′ 52″ becomes 9.79762m. We adopt this value as mean gravity for the Earth, and shall use it hereafter in treating of the other planets. The values of the oblateness adopted in Table I for *Mars*, *Jupiter* and *Saturn* appear to be the most probable when all the available investigations are considered, and the resulting figures of these planets have been found to satisfy the equations of equilibrium (cf. STRUVE, *Beob. der Saturnstrabanten, Publicat. de l'Obs. Cent. Nicolas, Série* II, Vol. XI, p. 233). Further explanation of this material seems unnecessary.

In *A.N.*, 3768, it was estimated that the uncertainty in the absolute diameter of *Saturn* did not much exceed ±0″.02, or 1 : 862nd part of the whole. In this connection attention should be called to PROFESSOR KAISER's very elaborate investigation of the diameter of *Saturn* (*Leid. Ann.* III, 264), made from observations on 40 nights in 1862-3, by means of a double-image Micrometer, which is supposed to eliminate the irradiation quite perfectly. The diameter found at Washington is 17″.240, while KAISER found from his observations 17″.274. The difference is only 0″.034, clearly within the range of the two probable errors.

In *A.N.*, 3768, attention was also called to the nearly uniform difference of the Washington daylight diameters of *Jupiter*, *Saturn* and *Uranus*, from those found with the Heliometer, which were smaller by 0″.20, 0″.18 and 0″.19, respectively. One can hardly believe this uniform difference is accidental; and if it is due to a systematic tendency of the Heliometer, as seems probable, the results may perhaps justify the designation absolute given to the daylight diameters determined at Washington. The figures of *Uranus* and *Neptune* have heretofore remained so uncertain that it becomes advisable to treat the two outer planets with some detail.

§ 192. *The Theoretical Rotation Periods of Uranus and Neptune* (cf. *A.N.*, 3992).

The following measures of the oblateness of *Uranus* have been made by experienced observers:

Observer	Oblateness	Remarks
HERSCHEL, 1781–1792	Longer axis of planet extends in the plane of the orb. of the Sat.	No measures of oblateness. *Philos. Trans.*, 1798, p. 71
MÄDLER, 1842	0.092166	120 diameters on 5n., 24cm. refrac. at Dorpat.
MÄDLER, 1845	0.10080	150 diameters on 7n.
MÄDLER, 1845	0.10582	60 diameters on 6n.
SCHIAPARELLI, 1883	0.091477	25n., fine series of observations
SCHIAPARELLI, 1884	0.076394	18n., fine series of observations
YOUNG, 1883	0.071480	14n., with 58.4cm. refrac. at Princeton
BARNARD, 1894	0.069832	9n., with 91.44cm. refrac. at Lick

The best oblateness we have been able to deduce from this material is gotten by the following process:

	Obl.	Wt.
Observations of MÄDLER	0.099599	1
Observations of SCHIAPARELLI	0.083935	2
Observations of YOUNG	0.071480	2
Observations of BARNARD	0.069832	1
Mean	0.080042 = 1:12.4935	

In view of the claims of MEYER and MILLOSEVICH that the disc of *Uranus* was sensibly round in 1883, it is improbable that the oblateness is really so large as these figures indicate. Moreover we can learn something about the probable form of the disc of *Uranus* from the history of the oblateness of *Mars*, which also has a rather small disc.

SIR W. HERSCHEL made the oblateness of *Mars* 1: 16.3; SCHROETER made it 1: 81; ARAGO, 1: 45.7; ENCKE, 1: 41.2; GALLE, 1: 89.4; AIRY, 1: 50; while BESSEL's long series of 305 equatorial and polar diameters taken with the Königsberg Heliometer in 1830-37, and reduced by OUDEMANS in 1852 (*A.N.*, 838), indicate that the oblateness is insensible. The large value of the oblateness found by many good observers has not been verified by experience, and the recent proof that the oblateness is about 1:192, shows that among all the old observations, BESSEL's measures alone were remarkably accurate. It thus appears that the oblateness of *Mars* for many years was made out from two to four times larger than it really is. Under the circumstances there is very strong probability that the compression of the disc of *Uranus* has been similarly overestimated. It would certainly be moderate to conclude that the oblateness does not exceed 0.04, or 1 : 25, which is half of the mean afforded by the best observations.

In view of the almost unbroken testimony of observers from the time of HERSCHEL, there can be little doubt that the planet is sensibly oblate, especially when the planes of the orbits of the satellites pass through the Earth. An oblateness not larger than 1 : 25 makes the difference in the equatorial and polar axes

not greater than 0″.15, which is less than the thickness of the usual micrometer wire, and would be difficult to recognize in some of the telescopes employed in measuring the planet. We therefore adopt for *Uranus* an oblateness of 1 : 25.

With regard to the time of rotation of *Uranus*, such observations as have been made are inconclusive, and entitled to little confidence. Yet it seems to be established by the observations of YOUNG in 1883 and the HENRY BROTHERS at Paris in 1884, that equatorial belts exist on the planet, somewhat inclined to the planes of the orbits of the satellites. The equatorial regions are reported somewhat brighter than the dusky regions near the supposed poles; and if this be true the resulting differential irradiation might account for part of the large oblateness observed.

DR. ÖSTEN BERGSTRAND of Upsala has considered the probable rotation period from the effects of the oblateness upon the motion of the Periuranium of *Ariel* (*A.N.*, 3889). His work seems to show a secular motion of the Periuranium of *Ariel* amounting to about 14°.31 per annum. Much of the data employed in his paper appears to me rather too uncertain for safe use; and I have preferred to use only the best of his material. This is the Lick observations of 1897 by SCHAEBERLE and HUSSEY and those of AITKEN in 1899, to which BERGSTRAND assigns weights of 20.66 and 11.87, respectively. These two series make the motion of the Periuranium 9°.7 per annum. In the secular equation

$$\frac{d\pi}{dt} = -\frac{3}{8}\frac{n_0^2}{n}(1 + 2\cos\gamma) + n\left(\chi - \frac{\varphi}{2}\right)\frac{\alpha^2}{a^2} + n\sum_i m_i \frac{a\,B^{(1)}}{4}, \tag{402}$$

where n_0 = mean motion of *Uranus* = 4°.284931 per Julian year, n = mean motion of *Ariel* = 52170°.12, γ = the inclination of the satellite orbits to the plane of the planet's orbit, χ = oblateness, φ = ratio of centrifugal force to equatorial gravity, m_i = the masses of the disturbing satellites, $B^{(1)}$ the usual coefficient introduced by LEVERRIER, a = mean distance of *Ariel* = 13″.748, α = equatorial radius of planet, DR. BERGSTRAND has shown that the first term is less than 0°.0001, and that all the components of the last term together do not exceed 0°.5, so that the first and last terms may be neglected. This leaves the middle term for the right member of the equation, and with the values here adopted, we get by observation for the left member

$$\left.\begin{aligned} 9°.7 &= +\,652°.0\,(\chi - \tfrac{1}{2}\varphi) \\ \text{or approximately} \qquad \chi - \tfrac{1}{2}\varphi &= 0.015. \end{aligned}\right\} \tag{403}$$

To find the ratio $\frac{\chi}{\varphi}$ by BERGSTRAND's method, we must first decide upon a value for the function

$$\psi = \frac{\chi - \frac{1}{2}\varphi}{\chi}.$$

In the case of some of the other planets we have

Planet	ψ
Earth	0.49
Mars	0.56
Jupiter	0.31
Saturn	0.19

The mean value of ψ for these four planets is about 0.39, and we shall therefore be safe in taking for *Uranus* the value $\psi = \frac{\chi - \frac{1}{2}\varphi}{\chi} = 0.375$, which would accord somewhat better with the tendency in the two great planets *Jupiter* and *Saturn*, where the value of ψ is rather small. As the average density of *Uranus* is 2.21, the value of ψ for such a body should by analogy lie between those of *Jupiter* and *Saturn* on the one hand, and of the Earth and *Mars* on the other.

In case of a homogeneous body it is well known that χ is very nearly $\frac{5}{4}\varphi$ (cf. *Tisserand's Méc. Céleste*, Tome II, p. 90). Thus on the hypothesis of homogeneity and $\chi - \frac{1}{2}\varphi = 0.015$ by observation, the oblateness of the Uranian spheroid becomes $\chi = 0.025 = 1:40$.

If on the other hand we take $\psi = 0.375$, as concluded above, then $\chi = 0.04 = 1:25$ for the case of heterogeneity having apparently the greatest probability. This gives $\varphi = 0.05 = 1:20$. Substituting these values in the formula for the rotation of the planet

$$T_0 = \frac{T}{\sqrt{\varphi}}\left(\frac{\alpha}{a}\right)^{3/2}, \tag{404}$$

where T = periodic time of *Ariel* = 60.489192 hours, a = *Ariel's* mean distance = 13″.748, α = equatorial radius = 1″.5369, we find for the most probable rotation period

$$T_0 = 10^h\ 6^m\ 40^s.32.$$

It is somewhat remarkable that this time is within $4^m\ 18^s$ of the mean of the equatorial rotation periods of *Jupiter* and *Saturn*, and is, I think, likely to prove quite near the truth. The Henry Brothers of Paris, Young and McNeil of Princeton, Perrotin and Thollon of Nice, and perhaps others, recognized faint equatorial belts on *Uranus* in 1883 and 1884; and the Nice observers inferred a rotation period of about ten hours, from a lucid spot which appeared near the equator. The belts seen upon the disc were usually thought to deviate about

40° from the plane of the orbits of the satellites. Using the Nice 76cm. refractor in 1889, however, PERROTIN made the deviation only 10°. The reaction of the oblateness of the planet upon the orbit plane of *Ariel* would be larger than that of *Neptune* upon his satellite, which is quite sensible in a few years; and as all the Uranian satellites appear to lie in one plane, and no secular reaction on the orbit planes has been discovered, it seems most probable that the equator of *Uranus* practically coincides with the orbit plane of the satellites, as was originally remarked by HERSCHEL.

In the case of the planet *Neptune* the only known criterion for finding the rotation period, aside from observation, which has not yet yielded satisfactory results, is derived from the motion of the pole of the orbit of the satellite. This appears to be describing a precessional motion, due to the action of the oblateness of the planet, in a period of about 500 years; and it has been concluded that the plane of the orbit is inclined to the Neptunian equator by about 20°. If the planet were homogeneous TISSERAND concluded that an oblateness of 1 : 100 would account for the perturbation of the satellite's orbit plane, while others have required an oblateness as great as 1 : 85.

During a series of observations of *Neptune* made with 66cm. refractor at Washington between Oct. 6, 1899, and Feb. 27, 1900, faint equatorial belts were occasionally seen on the planet (cf. *A.N.*, 3663, 3665) at intervals when the seeing was extraordinarily sharp and steady, and the glare of the planet was reduced by dense smoke in the atmosphere. These belts were again seen by MR. DINWIDDIE and the writer in January, 1901, and their existence is beyond doubt.

On more than one occasion in October and November, 1899, when the seeing was at its best and the air smoky, so that the disc appeared small, faint and perfectly sharp at the limb, a slight oblateness was repeatedly suspected, and once or twice noted in the observations. The amount of the oblateness was very small, usually estimated at 1 : 20, or 1 : 25 (cf. *A.N.*, 3663). This appeared rather too small to measure with any confidence in the case of a disc only 2″.3 in apparent diameter. I am convinced that the suspected oblateness was real, though as in the case of *Uranus*, already discussed, the observer may have considerably overestimated the amount. If we apply the same reasoning employed in the case of *Uranus*, fixing the oblateness at 1 : 45, which corresponds to a difference in the equatorial and polar diameters of 0″.05, a difference just perceptible in very perfect seeing, and holding $\psi = \frac{\chi - \frac{1}{2}\varphi}{\chi} = 0.375$, as in the case of *Uranus*, we shall find $\varphi = 0.0278$. Taking for the equatorial radius of *Neptune*

$\alpha = 1''.00$, the distance of the satellite, $a = 16''.305$, $T = 141.04407$ hours, the rotation period of the planet becomes

$$T_0 = 12^h.84817 = 12^h\ 50^m\ 53^s.4.$$

The precessional action of such a heterogeneous planet would be equivalent to that of a corresponding homogeneous spheroid of oblateness 1 : 70. Probably this corresponds quite as closely with the oblateness predicted from the motion of the pole of the satellite's orbit, as the present state of our knowledge of that phenomenon will justify. It thus appears probable that *Neptune* has an oblateness of about 1 : 45, and rotates in about $12^h\ 50^m\ 53^s$, a period considerably longer than that of any other large planet.

§ 193. Laplace's *Law of Density Applied to the Earth and to the Other Encrusted Planets* (cf. *A.N.*, 3992).

The celebrated law of Laplace results from an hypothesis introduced into Clairaut's general differential equation for the equilibrium of a heterogeneous fluid mass endowed with a rotatory motion, which permits the integration of the equation in some particular cases, and thus enables the geometer to connect the oblateness of the layers of the fluid with the radius of the spheroid, of which the density is supposed to be a function. If x denote the radius of the spheroid, σ the density, and Q a constant, then the simplified integral equation expressing Laplace's law is

$$\sigma = \frac{Q \sin qx}{x}, \tag{405}$$

where q is a coefficient denoting the degree of compressibility of the matter of the globe (cf. Thomson and Tait's *Natural Philosophy*, Vol. I, Part II, p. 404; Tisserand's *Méc. Cél.*, Tome II, p. 233; or Pratt's *Figure of the Earth*, Third Edition, p. 100).

Lord Kelvin remarks that in default of knowledge Laplace assumed as an hypothesis that the law of compressibility of the matter of which, before its solidification, the Earth consisted, should be that the increase of the square of the density is proportional to the increase of pressure, or $dp = \kappa\sigma\, d\sigma$, the integration of which leads to the above equation.

The solution of (404) is effected by finding the angle $q\,x$ from the transcendental equation

$$f = \frac{5.50}{2.55} = \frac{3}{q^2x^2}(1 - qx \operatorname{cotg} qx), \tag{406}$$

where f is the ratio of the mean density of the Earth (5.50) to the density at the surface (2.55), and $q\,x$ is found to be 144° 53′ 55″.2 = 2.52896 radians. Then the constant Q is found by the condition

$$Q = \frac{2.55}{\sin q x} = 4.434595\,.$$

The density at the centre is found by multiplying Q by $q\,x$ in radians, and the result is $\sigma_0 = 11.215$.

In the case of the other inner planets, the Moon, the four Galilean satellites of *Jupiter* and *Titan*, the ratio of the mean to the surface density is found by solving an equation of the same form as (406), in which $q\,x$ is known from the ratio of the radius of the body in question to the mean radius of the Earth. If the mean radius of the Earth in latitude 35° 15′ 52″ be R and the radius of *Mercury* be r; then we shall have

$$q x \left(\frac{r}{R}\right) = [9.9362578] = 49° \; 28' \; 30''\,, \tag{407}$$

where the bracket denotes a logarithm. This value of $q\,x$ for *Mercury* leads to the value of f by an equation like (406); and as the mean density of the planet is taken as known, the surface density is thus determined. Q and the central density σ_0 are then easily determined as in the case of the Earth. The method here applied to *Mercury* is applicable in like manner to *Venus*, *Mars*, the Moon, *Jupiter's* satellites and *Titan*.

The detailed numerical results found by this application of LAPLACE's law of density to the encrusted bodies of the solar system are given in § 198. The data for a planet like *Venus* naturally are very similar to those found for the Earth, while those deduced for *Mars* and *Mercury* are considerably different. Yet notwithstanding some differences depending mainly on the masses and mean densities of these bodies, it cannot be doubted that there is a great similarity in the physical constitution of the terrestrial planets, just as there is also among the four major planets of the solar system, which constitute an analogous group of bodies of much larger size.

§ 194. LAPLACE'S *Law of Density Applied to the Major Planets* (cf. *A.N.*, 3992).

When we come to deal with the major planets, *Jupiter*, *Saturn*, *Uranus* and *Neptune* and the Sun, the above method of procedure must be considerably modified. Their surfaces are covered with clouds, and on *Jupiter* and *Saturn* changes are frequently noted by observers. This indicates that their visible

surfaces are cloud surfaces, made by the precipitation of water and other vapors in the upper layers of their atmospheres, where the cold is intense.

Upon the Earth clouds seldom form at a height exceeding 16kms. On the major planets, where the force of gravity is considerably greater than upon the Earth, atmospheres under the same conditions would terminate more suddenly than our own, in the inverse ratio of the intensity of the respective surface gravities. As the atmospheric circulation appears to be rather violent on *Jupiter* and *Saturn*, clouds might perhaps be forced up to regions of rarer atmosphere than upon the Earth, but yet not very much higher, because under their greater attractions the atmospheres terminate more suddenly. If we take one-tenth of the density of atmospheric air, corresponding to a barometric pressure of 7.6 cms. of quicksilver, and a terrestrial height of 21.6 kms., according to the barometric formula of LAPLACE, perhaps we shall have a satisfactory approximation to the surface densities of *Jupiter* and *Saturn;* and the same value will apply to the surface densities of *Uranus* and *Neptune* and the Sun.

REGNAULT found by experiment that the specific gravity of air under temperature of 0° C and 76.0cms. barometric pressure is 0.001293187.

Thus in the case of *Jupiter* the surface density would be 0.0001293187, and

$$f = \frac{1.35}{0.0001293187} = 10439.34 \; . \tag{408}$$

In the use of LAPLACE's law, where the surface density is extremely small, and $q\,x$ infinitely nearly equal to 180°, PROFESSOR G. H. DARWIN has shown that

$$\left.\begin{aligned} f &= \frac{3}{\pi\,(\pi - q\,x)} \\ \frac{5\,\varphi}{2\,\chi} &= \tfrac{1}{3}\,(\pi^2 - \pi\,(\pi - q\,x)\,(5 - \tfrac{1}{3}\,\pi^2)\,) \end{aligned}\right\} \tag{409}$$

(cf. THOMSON and TAIT's *Natural Philosophy*, Vol. I, Part II, p. 409). Applying the first formula to *Jupiter*, we find

$$qx = [0.4971372] = 179^\circ\; 59'\; 41''.132 \; ,$$

where the bracket is the logarithm of $q\,x$ in radians. The second equation of (409) supplies the important condition that $\frac{5\varphi}{2\chi}$ cannot surpass $\frac{\pi^2}{3}\cdot$ With $q\,x$ accurately determined, Q is found as usual, and also the central density. In the case of *Jupiter* these quantities are respectively

$$\log Q = 0.1503624 \qquad \sigma_0' = 4.444 \; .$$

The treatments of the densities of *Saturn, Uranus, Neptune* and the Sun are similar to that of *Jupiter*, and the results are given in Table I, §198.

It will be observed that LAPLACE's law does not make the density in the major planets and the Sun so large as in the cases of *Venus* and the Earth. The densities at the centres of *Uranus* and *Neptune*, however, are 7.27 and 7.96, and thus decidedly large; while the central density of *Saturn* is only 2.336. If such be the real distribution of the density within these bodies, there is no doubt also a high internal temperature, increasing like the density towards the centres of these masses. This inference accords very well with the evidence of observation. For it seems to be established that none of these bodies except the Sun gives a sensible amount of inherent light; nor should this be expected of non-incandescent planets covered with clouds precipitated from vapors in close contact with the cold of space.

§ 195. *The Curves of Density for the Planets and the Probable Physical Properties of Matter Under Planetary Pressure* (cf. *A.N.*, 3992).

The numbers deduced in Table I are graphically illustrated by the curves of density drawn in Plates XXVI–VII, Chapter XVIII. The pressure curves determined by the processes described in the next section are also drawn on these plates.

Some important features of the laws of density here brought out are worthy of careful consideration. In the first place it should be remarked that the law of LAPLACE appears to be fairly exact in the case of the Earth. The figure of the Earth determined from the precession by means of this law, and depending on the difference of the moments of inertia $\frac{C-A}{C}$, f and $q\,x$, as well as the observed annual precession and inclination of the lunar orbit at some chosen epoch, yields an oblateness of 1 : 297.1034. The method afforded by LAPLACE's law of finding the ellipticity of any stratum, and therefore of the outer stratum or surface of the globe, gives an ellipticity of 1 : 298.041. The mean of these two values is 1 : 297.572: and as all the different methods for finding the Earth's figure make the most probable oblateness about 1 : 297.7, we may conclude that such a close agreement indicates that LAPLACE's law is very approximately a law of nature. If this law applies so accurately to the Earth, it is natural to suppose that it must apply almost as accurately to all similar bodies of the solar system, such as the other inner planets and the satellites. Indeed this must necessarily be true if the coefficient of compressibility be the same in all these bodies.

As the Earth's precessional phenomena indicate the validity of LAPLACE's law, it becomes obvious that the increase in density towards the centre is very nearly what the curve of density indicates. Probably the matter in the interior of our globe is of the same general character as the lava which flows from our volçanoes, simply compressed by the enormous weight of the superincumbent matter surrounding it on all sides. PROFESSOR WIECHERT's hypothesis that the Earth has an iron core, covered by a superstructure of rock, while apparently a possibility, is ingenious rather than probable: for such discontinuity in the arrangement of the matter of the Earth seems highly improbable, and in view of the tremendous pressure shown to exist, such an hypothesis is inadmissible, because the rigidity is too high to permit of a separation of the elements.

The curve of density in the case of the Earth is such as to suggest great strength and power of resistance in the arrangement of our globe. The increase of pressure towards the centre is equally significant. It seems that the pressure is the cause of the density, and the density of the matter under great pressure in turn the cause of the high effective rigidity of our globe, which PROFESSOR G. H. DARWIN has shown to be somewhat greater than that of a corresponding globe of steel (cf. *The Tides and Kindred Phenomena of the Solar System*, p. 260). In view of these considerations, and the similiarity in the form of the density curve for *Venus*, would it be very rash to infer that the globe of *Venus* also is quite rigid, perhaps about as rigid as a corresponding globe of platinum or wrought iron?

On the other hand nearly homogeneous globes of rock such as those of *Mercury*, *Mars*, the Moon, and other satellites would be notably less rigid than glass, because such a lithosphere is nowhere vitrified or crystallized; and, moreover, would be almost equally compressible throughout. From this point of view we may perhaps approximate the rigidity of the smaller planets and satellites, though we can never hope to measure the stiffness of their matter by any direct experiments, such as are available in the case of the Earth.

In comparing the diagrams for the four inner planets it is worthy of note that as the mass increases, and therefore also the internal pressure, the curve of density assumes more and more the form of the curves of pressure in approximately homogeneous bodies like *Mercury* and *Mars*. In accordance with the assumptions underlying LAPLACE's law, this means of course that the condensation of matter augments with the pressure, and that small bodies under the feeble gravitation of their matter are nearly homogeneous, while large ones are greatly condensed and hardened near their centres. We have no present means of fixing the extent to which such condensation may be carried, and the unknown temperatures within the actual planets is a complicating circumstance very difficult to

eliminate. From their large central densities *Uranus* and *Neptune* appear to be approaching the stage of consolidation long since attained by *Venus* and the Earth.

Perhaps we can safely say that under planetary pressure molecular forces are insignificant, and the hardest natural bodies would behave as if perfectly porous. This has been distinctly indicated by numerous experiments since the days of the Florentine Academicians, of 1661, and still more obviously by the familiar pressure of the deep sea in forcing water into hollow glass balls with walls several centimeters thick. But as the depth of the sea does not much exceed 10,000m. the pressure on such balls is never greater than 1,000 atmospheres, which does not compare with the pressure at the centres of even the satellites.* The least pressure, 5,844 atmospheres, at the centre of *Jupiter's* Fourth Satellite, arises from the extremely rare state of that mass, which presents striking analogies with the planet *Saturn*.

If in addition to perfect interpenetrability of matter under planetary pressure, we imagine an enormously high temperature which would instantly vaporize the most refractory elements, we may conceive that most of the matter in the interior of the Earth and similar planets has the property of a *rigid fluid*, a gas rendered more rigid than steel by its confinement, but capable of expansion with a violence surpassing the eruption of *Krakatoa* if the pressure could only be removed. In those bodies which have cooled down by the secular dissipation of their primordial heat, as perhaps the Moon and other satellites may have done, the explosive force would be wanting, and the matter already in a solid condition. No large body in the solar system seems as yet to have reached this stage.

§ 196. *Internal Pressures of the Heavenly Bodies* (cf. *A.N.*, 3992).

We shall determine the pressures throughout the planets which follow from LAPLACE'S law.

If σ_0 denote the central density of the Earth, σ the density at any other point, we shall have

$$\frac{\sigma}{\sigma_0} = \frac{\sin qx}{qx}. \tag{410}$$

The constant q is determined by the density at the surface, and approaches the limit $q = \pi$ when the surface density becomes infinitely small. If dM denote the element of mass enclosed within a spherical shell of density σ, and radius ρ we shall have

* The greatest artificial pressure yet produced appears to be 4,000 atmospheres, which AMAGAT developed in some experiments described in *Ann. de Chimie et de Physique*, sixth series, Vol. 29 (1893).

$$dM = 4\pi\rho^2 d\rho\sigma_0 \frac{\sin qx}{qx}. \tag{411}$$

The variable $x = \frac{\rho}{r}$, where r is the radius of the planet, and hence $d\rho = r dx$, and we have

$$M = \frac{4\pi r^3 \sigma_0}{q} \int_0^x x \sin(qx)\, dx \tag{412}$$

$$= \frac{4\pi r^3 \sigma_0}{q^3} (\sin(qx) - qx\cos(qx)). \tag{413}$$

In the particular case of the Earth, we may write the mass

$$E = \frac{4\pi r^3 \sigma_0}{q^3} (\sin q - q\cos q). \tag{414}$$

Also

$$E = \tfrac{4}{3}\pi r^3 \sigma_1, \tag{415}$$

where σ_1 is the mean density of the Earth treated as a sphere of mean radius r. By equating (414) and (415) we get

$$\sigma_0 = \frac{q^3 \sigma_1}{3(\sin q - q\cos q)}. \tag{416}$$

If $\frac{1}{f}$ be the ratio of the specific gravity of the matter at the surface of the Earth to the average specific gravity of the whole Earth, so that $\frac{1}{f} = \frac{2.55}{5.50}$, we shall have

$$\frac{\sigma_1}{f\sigma_0} = \frac{\sin q}{q}. \tag{417}$$

Equating this expression with (416), we find

$$\frac{1}{q^2} - \frac{1}{q \operatorname{tang} q} = \frac{f}{3}, \tag{418}$$

which gives $q = 2.52896 = 144° 53' 55''.2$, as already found in the theory of the Earth's density.

Equation (417) now gives for the central density

$$\sigma_0 = \frac{q\sigma_1}{f \sin q} = 2.03894\, \sigma_1. \tag{419}$$

Suppose the gravitational acceleration at the surface of a mass M of radius ρ to be g', and that due to the whole Earth E at its surface to be g; and let the

ratio $\frac{g'}{g} = \nu = \frac{Mr^2}{E\rho^2} = \frac{M}{Ex^2}$, since $x = \frac{\rho}{r}$. Then if we substitute for M and E their values from (413) and (415), we shall get

$$\nu = \frac{g'}{g} = 3\sigma_0 \frac{(\sin qx - qx \cos qx)}{\sigma_1 q^3 x^2}. \tag{420}$$

But by LAPLACE'S law we have also

$$\frac{\sigma}{\sigma_0} = \frac{\sin (qx)}{qx}. \tag{421}$$

If we differentiate this expression for σ with respect to x, and substitute the value of $\frac{d\sigma}{dx}$ in (420), we shall have

$$\frac{g'}{g} = -\frac{3}{\sigma_1 q^2} \frac{d\sigma}{dx}. \tag{422}$$

In a gaseous sphere in equilibrium, the pressure at a distance $\rho = rx$ from the centre is given by the equation

$$dp = -\sigma g' d\rho = -\sigma g' r dx. \tag{423}$$

Substituting for g' its value from (422), and integrating, we have

$$\int_0^p dp = \frac{3gr}{\sigma_1 q^2} \int_\delta^\sigma \sigma d\sigma, \tag{424}$$

where the limit δ denotes the density at the surface. This equation gives

$$p = \frac{3gr}{2\sigma_1 q^2}(\sigma^2 - \delta^2) = \frac{3r}{2q^2(\sigma_1 g)}[(\sigma g)^2 - (\delta g)^2]. \tag{425}$$

In reducing this formula to numbers we take for the mean radius of the Earth $r = 6370521$ m., $\sigma_1 = 5.50$, as before; and hence, with the metre and the kilogram as the units of length and weight respectively, $(\sigma_1 g) = 5500$ kg. At the surface of the Earth the density is 2.55, so that $(\delta g) = 2550$ kg., and at the centre the density is 11.215, so that $(\sigma_0 g) = 11215$ kg. The logarithm of the coefficient $\frac{3r}{2q^2(\sigma_1 g)}$ thus becomes [2.4340199], and the pressure at the centre of the Earth is found to be

$$p = 32401470000 \text{ kg. per square m.},$$
$$= 3240147 \text{ kg. per square cm.}$$

The pressure in atmospheres is found by dividing the pressure in kilograms per square centimetre by 1.0333, and hence

$$p = 3135727 \text{ atmospheres.}$$

The height of a column of quicksilver, which under uniform gravity would give the equivalent pressure, is found by multiplying by 0.76m.; the result is

$$H = 2383.152 \text{ km.}$$

This pressure is so enormous that it is difficult of comprehension. The tallest quicksilver column of the kind ever built is one devised for physical experiments on the Eiffel Tower in Paris, about 305m. in height, and with a pressure equal to about 400 atmospheres. Thus the pressure at the centre of the Earth is 7,838 times that of a column of quicksilver as high as the Eiffel Tower.

To compute the pressure at any point of the Earth's radius, the process is the same as the above, the density at the point in question taking the place of $\sigma_0 = 11.215$. And to apply the formula (425) to any other planet or the Sun, we have merely to introduce a factor $\frac{G}{g}$, where G is mean gravity on the planet, and g mean gravity on the Earth, or 9.79762m.

In Table II of § 198 we give the pressures at each tenth of the radius for the planets and the Sun; and also the pressures at the centres of the principal satellites. As some of the planets are considerably oblate, we have used their mean radii, and assumed the attraction to be as if the whole mass were collected at the centres of their spheroids. This will not in all cases be strictly accurate, but in the present somewhat uncertain state of our knowledge, it seems sufficient, and preferably to a method which varies from one planet to another.

The method of computing the internal pressure here developed is essentially that given by RITTER, in his *Anwendungen der mechanischen Wärmetheorie auf kosmologische Probleme*, Leipzig, 1882. For the simple case of homogeneous bodies it suffices to use the formula given by TISSERAND, *Mécanique Céleste*, Tome II, p. 92, namely:

$$p = \varpi\left[1 + \frac{1}{2}\frac{\sigma_1 a}{DH}\left(1 - \frac{\beta'^2}{\beta^2}\right)\right], \tag{426}$$

where ϖ = pressure of one atmosphere; the mean density of the Earth $\sigma_1 = 5.50$; the density of quicksilver $D = 13.5959$; the equatorial radius of the Earth $a = 6378000$ metres; the normal height of the barometer $H = 0.76$m.; β is the polar radius of the Earth, and β' the polar radius of any internal layer. For the centre of the Earth, $\beta' = 0$, and this formula gives for the pressure

$$p = 1697445\,\varpi .$$

TISSERAND remarks that if the Earth were formed of only one substance, as lava in fusion, notwithstanding the small coefficient of compressibility of liquids, it would be necessary to take account of compression under such enormous pressure, and thus one cannot consider the Earth as homogeneous.

The process given above is applicable to heterogeneity following LAPLACE'S law for any body whatever, provided the density is numerically determinable, and is thus of very general interest for all the heavenly bodies.

The matter of the interior of the Earth probably is sufficiently heated to vaporize with terrific explosive force if the pressure could only be removed. The matter in a cooled solid globe like the Moon, however, has acquired a fixed density, and while the removal of pressure would cause some expansion, and therefore cooling, it would not vaporize. To all appearances, therefore, our Earth is still effectively a gaseous sphere, except that the increase of density under tremendous pressure has rendered it highly rigid like a ball of steel.

§ 197. *Moments of Inertia and Other Constants for the Principal Bodies of the Planetary System.*

The law of density is of great importance in giving the moment of inertia of a body with respect to its axis of rotation. If r, θ and ϕ be the usual polar cöordinates, the element of mass $\mathrm{d}m = \sigma r^2 \sin\theta \,\mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$, and its distance from the axis is $r \sin\theta$, so that the moment of inertia about the polar axis becomes

$$\boldsymbol{C} = 2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sigma r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \,.\, r^2\sin^2\theta\,. \tag{427}$$

The moment of inertia about another principal axis, as an equatorial radius, taken in the plane from which ϕ is measured is

$$\boldsymbol{A} = 2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sigma r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \,.\, r^2(1 - \sin^2\theta\sin^2\phi)\,. \tag{428}$$

In bodies made up or concentric spheroidal layers having a common axis with the density increasing and the ellipticity decreasing from the surface to the centre

$$r = r_0[1 + \epsilon(\tfrac{1}{3} - \cos^2\theta)]\,, \tag{429}$$

where ϵ is the oblateness of any layer, and r_0 denotes the mean radius of the surface of equal density passing through the point (r, θ, ϕ).

By substituting the value of r from (429) in (427) and (428) and integrating it is easily shown (cf. THOMSON and TAIT'S *Nat. Phil.*, Vol. I, § 827) that

$$\frac{\boldsymbol{C}-\boldsymbol{A}}{C}=\frac{\chi-\frac{1}{2}\varphi}{1-6(f-1):f(qx)^2}. \tag{430}$$

Also

$$\boldsymbol{C}-\boldsymbol{A}=\tfrac{2}{3}Mr_0^2(\chi-\tfrac{1}{2}\varphi), \tag{431}$$

and

$$C=\tfrac{2}{3}Mr_0^2\left[1-\frac{6(f-1)}{f(qx)^2}\right]. \tag{432}$$

The moment of inertia of a homogeneous sphere is $\frac{2}{5}Mr_0^2$. A heterogeneous spheroid of revolution following LAPLACE'S law of density will have a smaller moment of inertia, and it is important to know the ratio of its moment of inertia to that of a corresponding homogeneous sphere.

From the above equation (432), we find the ratio

$$\frac{C}{\frac{2}{5}Mr_0^2}=\frac{5}{3}\left[1-\frac{6(f-1)}{f(qx)^2}\right], \tag{433}$$

which is tabulated for all the principal bodies of the system in Table III.

Table I also includes the precessional constants for these bodies, $\frac{\boldsymbol{C}-\boldsymbol{A}}{C}$, found from (430), and also the attraction constants $\frac{k}{C}$ for the several planets, determined by the condition

$$\frac{k}{C_0}<\frac{k}{C}<\frac{k_0}{C_0}, \tag{434}$$

where

$$\left.\begin{aligned}&\frac{k_0}{C_0}=\frac{3}{4}\frac{\alpha^2-\beta^2}{\alpha^2};\quad \frac{k_0}{\alpha^2}=\frac{3}{10}\left(\frac{\alpha^2-\beta^2}{\alpha^2}\right);\\ &\frac{l_0}{\alpha^4}=\frac{9}{70}\left(\frac{\alpha^2-\beta^2}{\alpha^2}\right)^2;\quad \frac{k}{C_0}=\frac{5}{2}\left(\frac{k}{\alpha^2}\right);\\ &\frac{k}{C}=\frac{3}{2}\left(\frac{\boldsymbol{C}-\boldsymbol{A}}{C}\right);\end{aligned}\right\} \tag{435}$$

by LAPLACE'S law; and

$$\frac{k}{\alpha^2}=\frac{3}{5}\frac{k_0}{\alpha^2} \tag{436}$$

for the major planets.

The moment of inertia of the Sun and major planets is necessarily open to considerable uncertainty, as different hypotheses regarding the internal distribution of density would give different results. The values here given are based upon LAPLACE'S law. For reasons already stated, it is not improbable that in most cases these numbers represent fair approximations to the truth.

In conclusion it may be remarked that most of the planets are so situated that the determination of their internal distribution of density is very difficult. This is best found from the precession, where it can be observed, as in the case of the Earth; but observation of the precessional motion is not yet achieved for any other of the planetary spheroids.* In the case of *Mars*, at least, such observations ought to become possible in the course of time. For the orbits of the satellites can be found with great accuracy, and as their planes are inclined but little to the Martian equator, on which the nodes regress uniformly, the position of the Martian equator itself will become known with the same accuracy as the planes of the orbits of the satellites. When the periodic shifting of the position of the orbit planes of the satellites due to the regression of their nodes on the equator of *Mars*, is determined by observation and allowed for, the remaining secular motion of these planes should be the Martian precession, depending on the action of the Sun. PROFESSOR HERMANN STRUVE computes this precession at $-7''.07$. This is much smaller than the Luni-Solar precession in the case of the Earth, but as it amounts to about $12'$ in a century, it ought to become sensible before many years if the satellites continue to be observed with great care.

A comparison between the observed precession of *Mars*, with that computed on the basis of LAPLACE's law, will afford a criterion of the accuracy of that law. It will also enable the future investigator to judge whether the coefficient of compressibility is sensibly the same for the matter of *Mars* as for that of the Earth, which is assumed in all our present work, in accordance with the suggestion of PROFESSOR G. W. HILL (*A.J.*, 452). Thus in time perhaps investigators may be able to use the great pressure in the centre of *Mars* as a means of verifying and extending the laws of compressibility found to hold true in the interior of the Earth. If the results in the two cases prove to be accordant, it will become evident that the densities in the centres of the planets are due to pressure upon matter similar to that of the crust of the Earth; and no doubt will remain that ordinary matter may be so compressed by forces such as we have computed for the centres of the planets. These forces are so hopelessly beyond any at the command of the experimenter, that those properties of matter which come to light under enormous pressure can be best deduced from phenomena observed in the laboratory of the heavens.

* The precession of *Saturn's* equator has not yet been recognized with certainty. From a discussion of the disappearances of the rings between 1750 and 1833, BESSEL determined the precession on the fixed orbit of 1750 at $-3''.85$. PROFESSOR H. STRUVE, however, has shown that the observations are equally well satisfied without any precession of the equator of *Saturn*; and from theoretical grounds finds the combined precession due to *Titan* and the Sun (the action of the other satellites being insensible) to have an average value of $-0''.46$ per annum during the nineteenth century. And he shows that at the maximum the precession will not exceed $-0''.7$. The motion of *Saturn's* equator is therefore so slow that probably it cannot be accurately determined by observations until several additional centuries have elapsed (cf. STRUVE, *Public. de l'Obs. Cent. Nicolas*, Série II, Vol. XI, p. 167, 235-6).

§198. *Tables of Constants for the*

TABLE I. INTERNAL DENSITIES

Radius r	*Mercury* $\sigma_1 = 3.09$	*Venus* $\sigma_1 = 5.14$	*The Earth* $\sigma_1 = 5.50$	*Mars* $\sigma_1 = 4.00$	*The Moon* $\sigma_1 = 3.31$	*Jup.* Sat. I $\sigma_1 = 3.29$	*Jup.* Sat. II $\sigma_1 = 3.76$	*Jup.* Sat. III $\sigma_1 = 2.71$	*Jup.* Sat. IV $\sigma_1 = 0.90$
1.0	2.933	2.681	2.55	3.50	3.20	3.204	3.689	2.52	0.85
0.9	3.01	3.70	3.75	3.73	3.25	3.25	3.72	2.59	0.87
0.8	3.07	4.73	4.99	3.94	3.30	3.28	3.75	2.66	0.89
0.7	3.13	5.74	6.21	4.12	3.34	3.31	3.78	2.73	0.91
0.6	3.18	6.69	7.38	4.31	3.37	3.34	3.81	2.78	0.93
0.5	3.22	7.57	8.46	4.45	3.40	3.37	3.83	2.83	0.94
0.4	3.26	8.33	9.40	4.58	3.43	3.39	3.85	2.87	0.95
0.3	3.29	8.94	10.12	4.68	3.45	3.40	3.86	2.90	0.96
0.2	3.31	9.41	10.74	4.75	3.46	3.41	3.87	2.92	0.97
0.1	3.32	9.69	11.07	4.79	3.47	3.42	3.88	2.93	0.97
0.0	3.332	9.788	11.215	4.805	3.476	3.422	3.889	2.941	0.972

Body	log qx in Radians	qx in Arc ° ′ ″	log f	log Q	Equat. Radius a km
Mercury	9.9362645	49 28 30	0.02226378	0.5864371	2175.31
Venus	0.3834225	138 32 2.4	0.2826916	0.6072985	6090.86
The Earth	0.4029418	144 53 55.2	0.3338225	0.6468540	6378.000
Mars	0.1258642	76 33 26.3	0.0582917	0.5558379	3371.72
The Moon	9.8393544	39 34 48	0.0142309	0.7017452	1740.25
Jup. Sat. I	9.7953576	35 46 0	0.0115489	0.7388731	1572.5
" " II	9.7475238	32 2 12	0.0092398	0.8422939	1408.5
" " III	9.9762553	54 20 50	0.0309019	0.4922097	2385
" " IV	9.9419785	50 7 50	0.0232857	0.0458744	2204
Titan	0.0009423	57 25 12	0.0310298	0.2961809	2524.5
Jupiter	0.4971372	179 59 41.132	4.0186725	0.1503624	70972
Saturn	0.4971258	179 59 24.125	3.7395969	9.8712869	59622
Uranus	0.4971422	179 59 48.475	4.2327309	0.3644209	21386
Neptune	0.4971425	179 59 49.475	4.2721634	0.4038644	21804
The Sun	0.4971367	179 59 41.839	4.0352624	0.1669524	696098

Body	Centrif. Force $=\left(\frac{2\pi}{T_0}\right)^2 a$ cm	$\frac{\varphi}{2}$	Theor. Oblateness χ_t	$\frac{5}{4}\varphi$	$\frac{k}{C_0}$
Mercury	....	....	....	....	....
Venus	....	....	....	....	....
The Earth	3.3915	0.00173384	0.00336054 = 1: 297.572	0.0043346	0.00414255
Mars	1.69455	0.00224289	0.00530412	0.00560721	0.007416475
Jupiter	223.3313	0.04463256	0.06440	0.11158140	0.053625*
Saturn	173.2068	0.08104	0.10050	0.20260	0.06280*
Uranus	63.7216	0.02500	0.04000	0.06250	0.0375 / 0.03525*
Neptune	43.2361	0.01390	0.02222	0.03475	0.02080 / 0.019755*
The Sun	0.592	0.0000108258 = 1 : 92372	0.0000153349 = 1 : 65360	0.0000270645 = 1 : 36948	0.000011273

$^*\ \frac{k}{C_0} = \frac{5}{2}\frac{k}{a^2}$

Bodies of the Solar System.

ACCORDING TO LAPLACE'S LAW.

Titan $\sigma_1=1.79$	*Jupiter* $\sigma_1=1.35$	*Saturn* $\sigma_1=0.71$	*Uranus* $\sigma_1=2.21$	*Neptune* $\sigma_1=2.42$	*The Sun* $\sigma_1=1.4026$
1.67	0.0001293187	0.0001293187	0.0001293187	0.0001293187	0.0001293187
1.72	0.485	0.255	0.795	0.870	0.504
1.78	1.039	0.545	1.700	1.862	1.079
1.82	1.63	0.858	2.67	2.93	1.697
1.86	2.24	1.179	3.67	4.02	2.328
1.90	2.83	1.484	4.63	5.07	2.938
1.93	3.36	1.765	5.50	6.03	3.492
1.95	3.81	2.001	6.24	6.83	3.961
1.97	4.15	2.181	6.80	7.45	4.316
1.98	4.37	2.293	7.15	7.81	4.539
1.982	4.444	2.336	7.270	7.962	4.614

Polar Radius β	Concl. Oblateness χ_e	Equat. Rotation T_0	Equat. Gravity γ	Mean Radius R	Mean Gravity G
km		h m s	m	km	m
2175.31	....	....	....	2175.31	1.87944
6090.86	....	....	....	6090.86	8.7537
6356.576	$\begin{cases}0.003359086\\= 1:297.7\end{cases}$	23 56 4.09	9.780305	6370.521	9.79762
3354.16	$\begin{cases}0.00520948\\= 1:191.96\end{cases}$	24 37 22.66	3.7765	3365.87	3.7714
1740.25	....	....	....	1740.25	1.6106
1572.5	....	....	....	1572.5	1.4468
1408.5	....	....	....	1408.5	1.4809
2385	....	....	....	2385	1.80735
2204	....	....	....	2204	0.55467
2524.5	....	....	....	2524.5	1.26361
66403	0.06440	9 50 20	25.01765	69449	26.21704
53660	0.10000	10 14 23.8	10.6865	57635	11.4423
20531	0.04000	10 6 40	12.744	21101	13.0400
21020	0.02222	12 50 53	14.474	21643	14.6460
696098	0.00000	25.00 days	273.010	696098	273.016

$\frac{k}{C}$	$\frac{k_0}{C_0}$	$\frac{k}{a^2}$	$\frac{l}{a^4}$	$\frac{C-A}{C}$	Volume Earth = 1
....	....	....	....	....	0.039814
....	....	....	....	....	0.87400
0.00498935	0.0050301	....	....	$\begin{cases}0.00333528\\= 1:298.2582\end{cases}$	1.00000
0.00765522	0.00775547	0.029666	0.00001	0.00513375	0.14749
0.075631†	0.0934576	0.02145	0.00060	0.0504131	1295.7
0.072531†	0.142495	0.02512	0.00074	0.048354	740.516
0.057375†	0.058769	0.01411	0.0003	0.038255	36.340
0.031833†	0.032927	0.007902	0.0002	0.021229	39.213
0.000058713†	0.00004114	....	....	0.0000115	1304628

† $\frac{k}{C}=\frac{3}{2}\left(\frac{C-A}{C}\right)$.

TABLE II. PRESSURES WITHIN THE LARGER BODIES OF

Radius R	*Mercury* ϖ LAPLACE'S Law	*Venus* ϖ LAPLACE'S Law	*The Earth* ϖ LAPLACE'S Law	*The Earth* ϖ Homogeneous	*Mars* ϖ LAPLACE'S Law
1.0	. . . .	. . . .	1	1	. . . .
0.9	12031	170950	198760	320295	44343
0.8	22176	399232	483691	611081	86758
0.7	31402	677251	842921	865698	124963
0.6	39697	987708	1260966	1086365	167152
0.5	46428	1291337	1710730	1273085	199437
0.4	53242	1635330	2152114	1425855	230350
0.3	58408	1912295	2521620	1544676	254735
0.2	61879	2139042	2861507	1629548	272124
0.1	63622	2279646	3050870	1680471	282166
0.0	65720	2329833	3135727	1697445	285957

TABLE III. RATIOS OF MOMENTS OF INERTIA TO THOSE OF CORRE-

Body	$\frac{C}{\frac{2}{5} M r^2}$	ϖ_0 Pressures at Centres in Atmospheres	H_0 Height of Equivalent Mercurial Column	V Parab. Velocity About the Sun	v Parab. Velocity at Surfaces of Bodies
			km	km	km
Mercury	0.9855	65720	49.947	108.79	2.8637
Venus	0.8482	2329833	1770.673	58.2196	10.3295
The Earth	0.82803	3135727	2383.153	42.112125	11.18566
Mars	0.9631	285957	197.327	27.6382	5.0500
The Moon	0.99103	48443	36.817	. . . .	2.3714
Jup. Sat. I	0.9932	37906	28.808	. . . .	2.135
" " II	0.9933	39810	30.246	. . . .	2.0436
" " III	0.9005	60444	45.938	. . . .	2.9367
" " IV	0.9860	5844	4.441	. . . .	1.5710
Titan	0.9801	29274	22.247	. . . .	2.5281
Jupiter	0.6535	39988390	30391.176	8.0941	60.3846
Saturn	0.6535	7609885	5783.513	4.4148	36.2560
Uranus	0.6535	9881392	7509.858	2.1943	23.4904
Neptune	0.6535	12465740	9473.962	1.4004	25.1904
The Sun	0.6535	4330637000	3291284.12	. . . .	617.28285

§ 199. *The Physical Properties of Matter Under Terrestrial and Celestial Conditions* (cf. *A.N.*, 3992).

The physical state of the matter composing the bodies of the solar system is a subject of deep interest to the natural philosopher, and probably his chief hope of understanding the varied conditions of matter existing throughout the Universe. The Sun, planets and satellites are the only bodies sufficiently near us to admit of careful study; and even these neighboring masses, including the Earth upon which we live, are largely beyond our powers of direct investigation. But it will perhaps be possible to determine some of the properties of the matter composing the heavenly bodies by indirect processes, based upon analogy with

THE PLANETARY SYSTEM, EXPRESSED IN ATMOSPHERES.

Jupiter ϖ LAPLACE'S LAW	*Saturn* ϖ LAPLACE'S LAW	*Uranus* ϖ LAPLACE'S LAW	*Neptune* ϖ LAPLACE'S LAW	*The Sun* ϖ LAPLACE'S LAW
....	....	....	....	....
476288	90680	118164	148834	51672260
2185882	414214	540316	681761	236286700
5379735	1026612	1332823	1688141	585814600
10159720	1938468	2518152	3177796	1102459700
16216560	3071137	4007854	5054633	1755901600
22807800	4344311	5655557	7150033	2480533200
29324850	5583748	7279795	9173076	3191583000
34792200	6633503	8645054	10914060	3798017000
38578780	7332292	9557887	11994320	4190994000
39988390	7609885	9881392	12465740	4330637000

SPONDING HOMOGENEOUS SPHERES, WITH OTHER DATA OF INTEREST.

ROCHE'S Limit of Stability for a Fluid Satellite			Remarks
Mean Distance of Nearest Satellite a	ROCHE'S Limit f. Equal Densities $\lambda=2.44\ a$	Difference $a-\lambda$	
530″.128 *Moon*	21″.462	+508″.666	In the case of the Earth and Moon ROCHE'S limit, $\lambda = 2.44\sqrt[3]{\frac{\sigma_1}{\sigma_1'}}$, becomes 2.890024 radii = 18433$^{km.}$ For equal densities *Phobos* and *Jupiter's* Fifth Satellite are respectively 1154 and 7665$^{km.}$ outside the limit, while *Saturn's* rings are wholly within; the limit due to the action of *Saturn* lying 7249$^{km.}$ outside the outer ring. As now situated the satellites would be stable if fluid throughout, while the rings could not form a satellite owing to the disintegrating action of the planet (cf. G. H. DARWIN, "The Tides and Kindred Phenomena in the Solar System").
12.938 *Phobos*	11.346	+1.592 = 1154$^{km.}$	
....	....	....	
....	....	....	
....	....	....	
....	....	....	
....	....	....	
47.961 Sat. V	45.928	+2.033 = 7665$^{km.}$	
{ 26.814 *Mimas*	21.033	{ +5.781	
{ 19.985 Ring		{ −1.048 = 7249$^{km.}$	
13.748	3.750	+ 9.998	
16.305	2.44	+13.865	
....	2342.4	....	

the known physical properties of terrestrial matter, and the behavior of our globe as a whole under the tidal strains to which it is subjected. From his well-known researches on long-period oceanic tides, LORD KELVIN concluded that the Earth is "more rigid than glass, but perhaps not quite so rigid as steel." In subsequent investigations PROFESSOR G. H. DARWIN has shown that LORD KELVIN'S estimate of the Earth's rigidity should be somewhat increased, and his final conclusion is that our globe under tidal strain yields less than a corresponding globe of steel.

In the course of Chapter XVIII we shall give a method for approximating the rigidity of the other planetary bodies, based upon the conclusions of KELVIN and DARWIN respecting the Earth, and certain analogies drawn from the adopted laws of density as applied to the other planets. Under certain assumptions we

have concluded for example that *Venus* has a rigidity about equal to that of a corresponding globe of wrought iron, while the rigidity of nearly homogeneous spheres of rock like the Moon, *Mercury*, and *Mars* is decidedly inferior to that of the Earth and *Venus*, both of which have hard, unyielding nuclei.

In default of adequate experimental knowledge the physical properties of spheres of amorphous rock, such as *Mercury* and the Moon, must be inferred chiefly from analogy; but they are certainly very different from those of spheres of steel and glass investigated by LORD KELVIN. Amorphous rock is not only porous, non-crystalline, and of various degrees of hardness, but when amassed into a satellite the resulting lithosphere would usually be of inferior elasticity.* Compressibility alone would not cause the rigidity of such a sphere to depart materially from that of the Earth, but the low elasticity and amorphous character of the matter would render the properties of the sphere quite different from that of the Earth, which behaves as a metallic, vitreous or gelatinous elastic solid, with a stiffness about 20,000 times greater than that of pitch at freezing temperature, when it is hard and brittle (cf. G. H. DARWIN, article, "Tides," *Encycl. Britt.*, Vol. XXIII, p. 374; also THOMSON & TAIT'S *Nat. Phil.* Vol. I, Part II, § 838).

LORD KELVIN has shown that if the yielding under tidal strain of a sphere of the hardness of steel corresponding to the Earth be 1.00, that of a corresponding sphere of glass would be about 1.20. On this scale the yielding of an amorphous and nearly homogeneous lithosphere such as the Moon would be considerably larger yet, so that the rigidity would be decidedly inferior to that of the Earth.

If it be true, as we have already pointed out, that the matter of the larger heavenly bodies experiences great condensation in the centres of their globes, owing to the enormous pressure to which it is subjected; and if this pressure increases both the density and elasticity of the matter, whatever be its temperature, so as to render the matter, which without pressure might be gaseous, both highly elastic and effectively rigid, because of its confinement under the stupendous gravitational pressure of the superincumbent matter on all sides; then it becomes clear that the larger bodies have nuclei of great effective rigidity, and such planetary globes should yield but little under tidal strains; whereas small globes, which are always nearly homogeneous because but little condensed towards

* If we drop various kinds of smooth spherical marbles on a heavy slab of polished steel and observe their rebounds, we have a recognized measure of the elasticity of the material of which they are composed. All who have witnessed such experiments will readily understand that as a rule the harder and more vitreous the substance, the more perfect the rebound; and that marbles made of soft or yielding rock show very imperfect elasticity, rebounding to but a small fraction of the height from which they fall, and soon coming to rest. The planetary spheres are made up of various elements distributed in concentric layers arranged according to density, and are thus conglomerate globes. When not much condensed towards their centres by gravity, the rigidity will naturally be inferior to those of the harder homogeneous rocks familiar to us upon the Earth. No rock which is at all abundant in the crust of the Earth has an elasticity so perfect as that of glass or steel.

their centres, should yield to strain much more easily, and are of inferior rigidity. This general conclusion will, I think, hold for all bodies which have become consolidated like the Earth, Moon, inner planets and satellites, and apply also to all except the outer layers of *Jupiter* and *Saturn*, *Uranus* and *Neptune*, which are fluid or quasi-solid planets.

It appears from LORD KELVIN'S researches that:

> A globe of the rigidity of steel yields $\frac{1}{3}$ as much as a corresponding fluid globe,
> a globe of the rigidity of glass yields $\frac{3}{5}$ as much as a corresponding fluid globe.

Accordingly, so far as one can see, a globe of amorphous rock but little condensed towards the centre would probably yield approximately $\frac{4}{5}$ as much as a corresponding globe of fluid. If these considerations and others which may be developed hereafter shall be the means of improving our understanding of the planetary bodies, the results thus deduced from the laboratory of the heavens will supplement in a useful way the properties of matter derived from terrestrial experiments, under very restricted variations of pressure and temperature.

Another question requiring the consideration of philosophers is whether gravitational attraction undergoes any modification on account of the physical state of the matter within the Earth and other heavenly bodies. In the General Scholium to the *Principia*, NEWTON long ago remarked that: "This is certain, that gravitation must proceed from a cause that penetrates to the very centres of the Sun and planets, without suffering the least diminution of its force, and that operates, not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes usually do), but according to the quantity of solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances. Gravitation toward the Sun is made up out of the gravitations toward the several particles of which the body of the Sun is composed."

Gravitation appears to be the most remarkable of known forces, in that it seems to be in no way modified or influenced by the pressure or temperature to which the particles of matter are subjected. Yet our grasp of the great variety of conditions offered by the solar system is vastly improved by a knowledge of the densities, pressures and probable temperatures within the several bodies; and we have therefore investigated this question in accordance with LAPLACE'S theoretical law of density. The actual temperature within the planetary bodies remains very uncertain, and we can only be sure that it is very high in the larger masses, and rapidly increases towards their centres, while the smaller globes

like *Mercury*, the Moon and other satellites may have lost already much of their primordial heat.

On account of this uncertainty respecting the internal temperature and its distribution some uncertainty will attach to the internal densities and pressures, but so far as one can see, this approximation will not vitiate the general tendency of the results, and the conclusions are therefore likely to be of interest to astronomers.

§ 200. *On the Surface Density of the Major Planets, and on the Boundary Distinguishing the Surfaces of the Heavenly Bodies from the Indefinite Nebulosity by Which They Are Often Surrounded.*

As regards the internal conditions of the heavenly bodies generally, it may be remarked that when the masses are of any considerable size, the internal pressure becomes so great that any relative motion of the imprisoned matter must be nearly impossible. A pressure of 3135727 atmospheres at the centre of the Earth makes the molecular friction so enormous, that even if the matter be at a temperature which would volatilize every known substance, a relative motion of the particles could be produced only by tremendous forces, such as do not usually develop in the interior of the heavenly bodies. The result seems to be that the matter near the centres of the Sun and planets, whatever its temperature, is almost absolutely devoid of circulation, and effectively of the highest rigidity. From this it follows that most of the circulation of such planets as *Jupiter* and *Saturn* does not extend to any very great depth.

As regards the superficial density of *Jupiter* and *Saturn*, which we have assumed to be one-tenth of atmospheric air, it may be pointed out that CALLANDREAU has shown that at the upper limit the surface density of *Saturn* could not possibly surpass one-fifth of the density of water (cf. M. LOEWY in *Vierteljahrsschrift*, 1904, p. 5). CALLANDREAU'S conclusion respecting *Saturn* will apply equally well to *Jupiter*, though the numerical coefficient might be slightly altered. Writing on the surface density of *Jupiter* in 1876, PROFESSOR DARWIN said: "In all cases (of varying the data of LAPLACE'S law) the physical conclusion is that the superficial density of the visible disc of *Jupiter* is very small compared to the mean density — a conclusion which appears to agree well with the telescopic appearance of that planet. A similar application to the planet *Saturn* points to a similar result, but the conclusion is less certain on account of the great uncertainty in the data" (THOMSON & TAIT'S *Nat. Phil.*, Vol. I, Part II, p. 410).

Some observers, including the late PROFESSOR G. W. HOUGH of Chicago, who devoted many years to the study of the surface markings, have found it easiest to explain the behavior of spots on *Jupiter* by the hypothesis that the surface density of the planet is about equal to that of water. In view of what is shown in this chapter respecting the vast increase of pressure towards the centre, and the probably great compressibility of even liquid lava under such tremendous forces, a mean density for *Jupiter* of only 1.35 suggests that observers might well consider the theoretical possibilities in framing their hypotheses for the explanation of surface phenomena. As the surface circulation is not enormously rapid, and as the disc is devoid of sensible luminosity, it would seem as if no possible arrangement of internal temperature could preserve the equilibrium of the Jovian spheroid were its surface of any considerable density.

Undoubtedly the surface density of all the major planets is small, and on the whole the phenomena are best explained on the hypothesis that the physical state of these planets corresponds closely to that of simple globes of monatomic gas. This hypothesis is treated more at length in next two chapters of this work, and the results applied also to the Sun and stars. Even if the concluded laws do not hold accurately true, they will no doubt furnish approximate representations of the actual phenomena, and enable one to grasp the physical states of the major planets both at the surfaces and in the deep interiors which are forever beyond our power of direct observation, but may be studied from the known effects of gravitational pressure.

In previous investigations of the problems of Cosmical Evolution made since the time of LAPLACE it has always been assumed that the central nucleus is a figure of equilibrium under the pressure and attraction of its parts, and that the attendant bodies have been detached, by acceleration of rotation, from the now dominant central masses, which are thus imagined to exert hydrostatic pressure from the center outwards. In the present investigation this traditional position is definitely abandoned. The capture theory does not require the exertion of any hydrostatic pressure from the center, and the nucleus therefore is made essentially independent of the diffused nebulosity revolving about it. If this view be admissible, we should no longer consider a nebula as a continuous mass, but rather as made up of a comparatively small spheroidal nucleus, in a diffuse medium of vast extent but devoid of hydrostatic pressure.

Accordingly we may investigate the laws of density of the Sun and planets which have resulted from such nuclei, but not imagine any hydrostatic connection to exist with the bodies now revolving about them, as in the abandoned hypothesis of LAPLACE. The planetary bodies thus have definite bounding surfaces at every

stage of their history, though the density of the surrounding nebulosity is variable. The problem of the internal density of the Sun and planets is an important one, and our theory of the solar system would be very incomplete without some adequate treatment of this obscure question. The results here brought together have already been published in the *Astronomische Nachrichten* (Nos. 3992, 4053, 4104, 4152), and these chapters are therefore founded upon the work of 1904–6.

In investigations dealing with the interior conditions of the heavenly bodies it necessarily follows that the exact state of fact is unattainable by direct observation, and we are therefore obliged to base our reasoning upon deductions from known physical laws; yet if we thus obtain even a good approximation to the truth, it will be extremely valuable, as affording an approximate basis of calculation for estimating the forces operating in the interior of the heavenly bodies. The results here set forth are believed to be the best afforded by the present state of science; and although open to a certain margin of uncertainty, will be highly satisfactory in establishing a suitable distinction between the dense and often highly rigid nuclei, with definite bounding surfaces, and their surrounding nebulosity of indefinite extent. *True fluid pressure definitely ceases at the surface.* Obviously the results found for the solar system will apply without great changes to other systems of stars and nebulae observed in the sidereal universe.

29

Μόνον δὲ τὸ μαθηματικόν, εἴ τις ἐξεταστικῶς αὐτῷ προσέρχοιτο, βεβαίαν καὶ ἀμετάπιστον τοῖς μεταχειριζομένοις τὴν εἴδησιν παράσχοι ὡς ἂν τῆς ἀποδείξεως δι' ἀναμφισβητήτων ὁδῶν γιγνομένης, ἀριθμητικῆς τε καὶ γεωμετρίας, προήχθημεν ἐπιμεληθῆναι μάλιστα πάσης μὲν κατὰ δύναμιν τῆς τοιαύτης θεωρίας, ἐξαιρέτως δὲ τῆς περὶ τὰ θεῖα καὶ οὐράνια κατανοουμένης, ὡς μόνης ταύτης περὶ τὴν τῶν αἰεὶ καὶ ὡσαύτως ἐχόντων ἐπίσκεψιν ἀναστρεφομένης.

The Science of Mathematics alone gives to those who apply themselves to it with assiduity a knowledge solid and exempt from doubt, since the demonstration proceeds by rigorous methods of calculation and of measurement. We have been led to give particular attention to all these theories, according to our ability, but more especially to things of divine origin observed in the heavens, for these alone offer a subject of investigation which is immutable and eternal. — PTOLEMY, *Almagest,* Introduction.

CHAPTER XVII.

Researches on the Physical Constitution of the Heavenly Bodies.*

§ 201. *Introductory Remarks.*

The physical state of the matter composing the heavenly bodies depends primarily upon their internal temperatures, in connection with the corresponding densities and pressures, all of which have arisen from the secular action of universal gravitation. These three effects of one primitive cause conjointly determine the forces to which matter is subjected in the depths of cosmical globes; but to analyze these forces we require a theory of the constitution of celestial bodies which remains valid for enormous ranges of temperature. In the case of the principal bodies of the planetary system, we have already investigated the internal densities and pressures which result from Laplace's celebrated law, on the hypothesis that the temperature is adequate to sustain the equilibrium with the given distribution of density. The theory thus developed in the preceding chapter affords a first approximation to the actual state of these masses; and in the case of encrusted planets like the Earth, *Venus*, *Mars* and *Mercury*, there are physical grounds for supposing that the law of Laplace becomes essentially a law of nature. But as the major planets and the Sun are still in a very primitive state of development, a second approximation becomes necessary to discover their true condition; and the subject is therefore treated with considerable care in the present chapter in the hope that the results thus established may be applicable not only to the larger masses of our solar system, but also to innumerable fixed stars scattered throughout the immensity of space.

Since we can never penetrate the heavenly bodies deeper than the outside layers, which are imperfectly disclosed in our telescopes, it becomes evident that our chief means of exploration must be the theory of gravitation in connection with the mechanical theory of gases. If we proceed upon this basis, and follow Newton's first rule of Philosophy: "To admit no more causes of natural things

* Reprinted with slight changes from *Astronomische Nachrichten*, No. 4053.

than are both true and sufficient to explain the phenomena," we shall probably reach safe conclusions.

In the following investigations we have adopted the simplest possible causes, and admitted no departure from these fundamental principles. Whether one should prefer to vary the results for somewhat different conditions, depends upon the judgment of the investigator; but in the present state of our knowledge of the heavenly bodies, we believe that the best results will be obtained by the simple hypothesis of a monatomic gas, which postulates nothing but conditions known to result from temperatures recognized to exist in most of the larger heavenly bodies. Yet even if others should prefer somewhat different processes from those here adopted, the present results are still likely to be of interest to astronomers.

In considering the state of the heavenly bodies it is evident that nothing could be simpler or more natural than the gaseous constitution of masses which are of small average density and high temperature, like the fixed stars, the Sun and major planets. Our next step then is to inquire into the nature of the gas, whatever the chemical elements involved, when the temperature is very high, as in the interior of these bodies. A slight consideration of the state of the Sun must convince us that the matter of the whole interior of that immense globe, everywhere at temperatures ranging from ten thousand to more than ten millions of degrees centigrade, must be dissociated and reduced to its simplest possible state. This is obviously a monatomic gas, in which the theoretical ratio of the specific heat under constant pressure to that under constant volume would be $k = 1\frac{2}{3}$.

More than twenty years ago Professors Kundt and Warburg (*Poggend. Ann.*, Bd. CLVII) found that the vapor of *Mercury*, which on chemical grounds was known to be monatomic, actually has this ratio; and in more recent times the discovery of the new elements Argon, Helium, Neon, Xenon, Crypton, and the investigation of their properties by Lord Rayleigh and Professor Sir Wm. Ramsay (*Phil. Trans. and Proceedings of the Roy. Soc.*, 1895, *et seq.*) have shown that the same value of k holds for these simple elements.

Now experiments show that gases with very complex molecules like vapor of oil of turpentine give values of k no larger than 1.03; and that the value of this constant rises with the simplicity of the molecule, attaining a maximum of $1\frac{2}{3}$ when the gas is monatomic.

Although the actual mechanism of a compound molecule is somewhat obscure, and Physicists are not yet fully agreed as to the relation between k and the number of atoms in a molecule, Boltzmann using the formula

$$k = 1 + \frac{2}{3(1+\beta)}, \tag{437}$$

where β is a function of the interaction between the constituent atoms of the molecule, and therefore zero when the gas is monatomic, the factor 3 specifying the degrees of freedom (cf. *Gas-Theorie*, Teil I, Equ. 56, p. 57), while other authorities employ somewhat different expressions, none of which seem as yet to explain all of the observed phenomena perfectly*; yet it is plain from experimental data alone that k augments with the simplicity of the molecule, attaining the value $1\frac{2}{3}$ for a monatomic gas, and sinking to unity for gases of the most complex structure in which β is very large.

This is sufficiently illustrated in the following table†:

Chemical Symbol	Name of Substance	k	No. of Atoms in Molecule
$C_{10}H_{16}$	Oil of Turpentine	1.03	26
$HCOOC_2H_5$	Ethyl Formate	1.125	11
CH_3COOCH_3	Methyl Acetate	1.137	11
CH_2ClCH_2Cl	Ethylene Chloride	1.137	8
$CHCl_3$	Chloroform	1.154	5
C_2H_3Br	Vinyl Bromide	1.198	6
CH_2Cl_2	Methylene Chloride	1.219	5
CS_2	Carbon Bisulphide	1.239	3
C_2H_4	Ethylene	1.250	6
CO_2	Carbon Dioxide	1.308	3
H_2S	Sulphuretted Hydrogen	1.321	3
NH_3	Ammonia	1.33	4
O_2	Oxygen	1.408	2
N_2	Nitrogen	1.408	2
H_2	Hydrogen	1.408	2
	Air	1.408	2
Hg_1	Mercury	1.66	1
He_1	Helium	1.66	1
A_1	Argon	1.66	1

On the other hand MR. J. H. JEANS has shown (*Phil. Trans.*, 1901, Part I, p. 404) that for excessively low temperatures (below a critical value) all gases tend to behave under experiment as if monatomic, while at higher temperatures the value of k in nearly all cases becomes smaller, and "may have any value between $1\frac{2}{3}$ and $1\frac{2}{5}$." Thus if the temperature is so low that the interaction of the constituent atoms exercises no considerable effect, each molecule behaves sensibly as an atom, but the tendency diminishes with rising temperature; and where

* cf. Correspondence of LORD RAYLEIGH and MR. J. H. JEANS in *Nature*, April 13, 1905, *et seq.*

† cf. DR. J. W. CAPSTICK, "On the Ratio of the Specific Heats of Some Compound Gases," *Phil. Trans., Roy. Soc.*, 1895, Part I.

the temperature is excessively high general dissociation follows, and the atoms act as molecules.

Not only is the value of k small for gases with complex molecules, but in general such gases are the more easily decomposed the smaller the value of k, so that a slight rise in temperature often suffices to break up the molecules into their component atoms. If such dissociation can be produced for the more volatile elements under the low temperatures available to the experimenter in the laboratory, it must be perfectly accomplished for the metallic and other vapors in the fixed stars and the Sun. And if this dissociation is not complete in the layers of the photosphere exposed to our view, it would seem that it must become so at a slight depth below. For in spite of the difficulty* of dealing experimentally with heated vapors in the laboratory, it has been remarked by Chemists that most of the metals in vapor probably are monatomic: and they seem certain to be so at a short distance below the Sun's visible photosphere, where the temperature decidedly exceeds 10,000° C. If these analogies be sound, it follows therefore that we may take the whole interior of the Sun's globe as monatomic gas, in which $k = 1\frac{2}{3}$, and the laws of internal density, pressure and temperature should be computed upon this basis.

Simple calculations show that while the major planets are much less violently heated than the Sun, yet the larger portion of their matter in each case is at a temperature of more than 10,000° C., and may thus be taken as composed chiefly of dissociated vapors in the monatomic state. This law will fail only in the outer layers of the planets where the density, pressure and temperature are greatly diminished. Thus we may conceive the density of the major planets to follow the monatomic law except in the outer layers, where compounds develop, and k becomes about the same as in atmospheric air. In these surface layers the exact value of this constant will depend upon the proportion in which the separate gases are mixed as well as upon their individual properties; and since it is generally recognized that the lighter gases such as Hydrogen have a tendency to escape to the surface, it is natural to take k as about 1.41 in the outer layers, and imagine it to increase in value as we descend into the planets, attaining the maximum value of $1\frac{2}{3}$ when a few hundredths or possibly a tenth of the radius has been traversed, and the temperature risen to several thousands degree centigrade.

Accordingly the curves of density and also of pressure and temperature should experience some discontinuity as we pass through the outer layers of the planet's

* cf. *Über die spezifische Wärme der Gase in höherer Temperatur, von* Prof. L. Holborn *und* Prof. L. Austin, *Sitzungsberichte der Kgl. Preuss. Akad. d. Wiss.*, Berlin, February 2, 1905. The authors attained temperatures of 800°, and thus approximated the conditions in explosive experiments beyond 1,000°.

atmosphere, corresponding in the interior part to $k = 1\frac{2}{3}$, and in the outer layers changing gradually so as to conform to $k = 1.4$.*

§ 202. *Theory of the Sun's Physical Constitution.*

(1) *Historical.* In the *American Journal of Science* for July, 1870, the late MR. J. HOMER LANE, for many years connected with the U.S. Coast and Geodetic Survey at Washington, has published the earliest important paper on the gaseous constitution of the Sun. It had been read before the National Academy of Sciences at the meeting of April 13-16, 1869, and in many respects is classic and justly celebrated; but the obscurity of some of the processes is recognized, and has been noted by LORD KELVIN in his address on the "Sun's Heat" (*Popular Lectures and Addresses*, Vol. I, p. 406). Referring to the central density of about twenty times the average found by LANE, on the hypothesis that $k = 1.4$, as in common air, LORD KELVIN says: "Working out LANE's problem independently, I find $22\frac{1}{2}$ as very nearly the exact number."

By supposing the Sun's atmosphere to extend above the photosphere by $\frac{1}{22}$ part of the radius, and putting $k = 1.4$, LANE found the specific gravity at the centre 28.16, "nearly one-third greater than that of metal platinum." He assumed the Sun's average density within the photosphere to be one-fourth that of the Earth, or 1.375; but in the calculation of the central density this value is multiplied by the factor, $(\frac{22}{23})^3$ and thus reduced to 1.20334, which is the mean density used in deriving the central density 28.16. Failing to recognize the process of calculation employed, owing to the undue conciseness of LANE's expressions, LORD KELVIN and others have assumed that LANE's theory made the central density about 20 times the average density of the whole mass; whereas, neglecting the effects of the supposed atmosphere, which is now known to be practically insensible, the true central density becomes 23.4016 times the mean, and is thus decidedly larger than has been heretofore recognized. Accordingly it is very satisfactory to find that the results of LANE, RITTER and LORD KELVIN, all found by different processes, agree quite perfectly.

In RITTER's investigations, originally published in WIEDEMANN's *Annalen*, 1878-1882, and partially reprinted in his interesting and suggestive pamphlet: "*Anwendungen der mechanischen Wärmetheorie auf kosmologische Probleme,*"

* Theory indicates that gases with molecules composed of two fixed atoms as Hydrogen, Oxygen, Nitrogen, Air, should give for k a value of exactly 1.4, but as experiments make the value about 0.01 larger, it is natural to conclude that the deviation of the observed from the theoretical value of k is due to the influence of the new monatomic gases, Argon, Helium, Neon, Xenon, Crypton, all of which alone would give $k = 1\frac{2}{3}$. Traces of these gases in the atmospheric air used in the experiments would have a tendency to raise k slightly above the theoretical value $1\frac{2}{5}$.

Leipzig, 1882, the process employed for finding the central density depends upon the celebrated equation of POISSON connecting the density and temperature,

$$\frac{\sigma}{\sigma_0} = \left(\frac{T}{T_0}\right)^{2.44}. \tag{438}$$

The method employed by RITTER is essentially one of successive approximations, and the value finally obtained for the central density is 23, as against 23.4016 found by LANE, and 22.5 by LORD KELVIN. It is evident that differences in the assumed data, and in the degree of approximation attained is the cause of the slight differences in these results. For if LANE had used $k = 1.41$ his central density would have agreed almost perfectly with RITTER's value, 23.0; and we have therefore no hesitation in taking 23.0 as about the true value when $k = 1.41$, as in common Air, Oxygen, Nitrogen and Hydrogen. This case, however, does not represent the phenomena of nature when the temperature is high, and we shall therefore consider more in detail the theory required in dealing with bodies like the Sun and self-luminous fixed stars, which are made up of gases reduced to the monatomic state by enormously high temperatures.

(2) LANE's *Theory*. The mathematical process employed by LANE depends essentially on the integration by successive approximations of the equations

$$\mu = \int_0^x \frac{\sigma}{\sigma_0} x^2 \,\mathrm{d}x, \tag*{[6] (439)}$$

$$1 - \left(\frac{\sigma}{\sigma_0}\right)^{k-1} = \int_0^x \frac{\mu \,\mathrm{d}x}{x^2}, \tag*{[7] (440)}$$

where x is a certain function, directly proportional to the radius, and σ the density at any point, while σ_0 is the value of σ at the Sun's centre, where $x = 0$.

In the first approximation, applying to the region at the centre of the Sun, $\frac{\sigma}{\sigma_0}$ is taken as unity, and then the first terms of the expansion give

$$\mu = \frac{x^3}{3}, \quad \text{and} \quad 1 - \left(\frac{\sigma}{\sigma^0}\right)^{k-1} = \frac{x^2}{6}. \tag{441}$$

Successive approximations add terms to each series, the expansions and integrations being of the form

$$\frac{\sigma}{\sigma_0} = \left\{ 1 - \int_0^x \frac{\mathrm{d}x}{x^2} \int_0^x \left\{ 1 - \int_0^x \frac{\mathrm{d}x}{x^2} \int_0^x \left[1 - \int_0^x \frac{\mathrm{d}x}{x^2} \int_0^x \right.\right.\right. \\ \left.\left.\left. \left(1 - \int_0^x \frac{\mathrm{d}x}{x^2} \int_0^x \ldots \frac{\sigma}{\sigma_0} x^2 \,\mathrm{d}x \ldots \right)^{\frac{1}{k-1}} x^2 \,\mathrm{d}x \right]^{\frac{1}{k-1}} x^2 \,\mathrm{d}x \right\}^{\frac{1}{k-1}} x^2 \,\mathrm{d}x \right\}^{\frac{1}{k-1}} \ldots \tag*{[α] (442)}$$

When the value of $\frac{\sigma}{\sigma_0}$ is thus obtained, the value of μ is found by one additional integration as in equation [6]. The convergence of the series obtained by this process depends on the value of k, the ratio of the specific heat of the gas under constant pressure to that under constant volume; and becomes less and less satisfactory, as x increases in value. When $k = 1\frac{2}{3}$ the value of x corresponding to the surface of the Sun is considerably smaller than when $k = 1.4$, but in both cases the satisfactory treatment of the series is very difficult.

LANE bridged over the difficulty by the use of interpolation formulae and numerical differences calculated for the regions of undoubted convergence. The theoretical soundness of this method is embarrassed in practice by the considerable effects introduced into the final result by small errors in the higher orders of differences; and with the number of terms in the series given by LANE great accuracy seems unattainable. He remarks that "there is no need of great precision in these calculations," and does not seem to have aimed at it for the objects in view when his original paper was prepared thirty-six years ago; yet on examining his results with care I find that a moderate degree of precision was usually obtained. It may have been the uncertainty in LANE'S process which led LORD KELVIN to adopt a different method for finding the central density of the Sun (*Philosophical Magazine*, Vol. 35, 1887, p. 287).

Considerable experience in the use of LANE'S method led the writer to the conviction that it is not capable of extreme accuracy, unless the expansions are carried to terms considerably higher than those published. To make sure that no misprints had occurred in that paper, the original manuscript, in LANE'S handwriting, preserved in the Library of the U. S. Coast Survey, and kindly placed at my disposal by SUPERINTENDENT TITTMAN, was carefully compared with the printed paper. No material errors were discovered, nor was it found on the other hand that the published paper had been abridged in the slightest degree. We may therefore assume that LANE did not carry the approximations to terms of higher order than those given in the *American Journal of Science* for July, 1870.

(3) *Determination of the Required Series.* The importance of extreme accuracy in this theory seemed so great, and LANE'S method of differences so unsatisfactory, that I finally carried the approximations to terms of the 20th and 21st order in x. It was then noticed that from the differences in the logarithms of the resulting coefficients the logarithms of several more of the higher coefficients could be easily found without calculating the enormous numbers involved in the actual expansions. The series actually calculated extends six terms beyond the first four terms calculated by LANE; and by taking the logarithms of the higher terms from the small and steadily varying logarithmic differ-

ences it was possible to calculate the final results without the use of LANE's original process. Our series as found by actual calculation and checked with the greatest care is as follows:

$$\left.\begin{aligned}\mu = \frac{x^3}{3} - \frac{x^5}{20} + \frac{x^7}{240} - \frac{x^9}{3888} + \frac{19x^{11}}{1425600} - \frac{2719x^{13}}{4447872000} + \frac{20621x^{15}}{800616960000} \\ - \frac{193328x^{17}}{190546836480000} + \frac{39667364x^{19}}{104267228921856000 0} - \frac{8078124341x^{21}}{591195187986923520000 0} + \dots\end{aligned}\right\} \quad [\beta] \quad (443)$$

$$\left.\begin{aligned}1 - \left(\frac{\sigma}{\sigma_0}\right)^{2/3} = \frac{x^2}{6} - \frac{x^4}{80} + \frac{x^6}{1440} - \frac{x^8}{31104} + \frac{19x^{10}}{14256000} - \frac{2719x^{12}}{53374464000} + \frac{20621x^{14}}{11208637440000} \\ - \frac{193328x^{16}}{3048749383680000} + \frac{39667364x^{18}}{1876810120593408000 0} - \frac{8078124341x^{20}}{11823903759738470400000 0} + \dots\end{aligned}\right\} \quad [\gamma] \quad (444)$$

It will be seen that the divisors of the highest coefficients, running into the hundreds of sextillions, become so immense as to be almost unmanageable, but the forms of the fractions here adopted were found to be the most practicable; and after several repetitions of the calculation I think the accuracy of these coefficients may be absolutely depended upon.

The logarithms of the coefficients in the series finally used, including the higher terms based on logarithmic differences, are as follows:

$$\left.\begin{aligned}\mu = \underset{-10}{[9.5228787]}x^3 - \underset{-10}{[8.6989700]}x^5 + \underset{-10}{[7.6197888]}x^7 - \underset{-10}{[6.4102738]}x^9 + \underset{-10}{[5.1247559]}x^{11} \\ - \underset{-10}{[3.7862569]}x^{13} + \underset{-10}{[2.4108849]}x^{15} - \underset{-10}{[1.0062931]}x^{17} + \underset{-20}{[9.5802856]}x^{19} \\ - \underset{-20}{[8.1355797]}x^{21} + \underset{-20}{[6.6744]}x^{23} - \underset{-20}{[5.1992]}x^{25} + \underset{-20}{[3.7110]}x^{27} - \underset{-20}{[2.2102]}x^{29} \\ + \underset{-20}{[0.6981]}x^{31} - \underset{-30}{[9.1751]}x^{33} + \underset{-30}{[7.6425]}x^{35} - \underset{-30}{[6.1003]}x^{37} + \underset{-30}{[4.5488]}x^{39} - \underset{-30}{[2.9891]}x^{41} \\ + \underset{-30}{[1.4212]}x^{43} - \underset{-40}{[9.8455]}x^{45} + \underset{-40}{[8.2628]}x^{47} - \underset{-40}{[6.6732]}x^{49} + \underset{-40}{[5.0970]}x^{51} - \dots\end{aligned}\right\} [\delta] (445)$$

$$\left.\begin{aligned}\left(\frac{\sigma}{\sigma_0}\right)^{2/3} = [0.0000000] - \underset{-10}{[9.2218487]}x^2 + \underset{-10}{[8.0969100]}x^4 - \underset{-10}{[6.8416375]}x^6 + \underset{-10}{[5.5071838]}x^8 \\ - \underset{-10}{[4.1247559]}x^{10} + \underset{-10}{[2.7070757]}x^{12} - \underset{-10}{[1.2647569]}x^{14} + \underset{-20}{[9.8021731]}x^{16} \\ - \underset{-20}{[8.3250131]}x^{18} + \underset{-20}{[6.8345497]}x^{20} - \underset{-20}{[5.3320]}x^{22} + \underset{-20}{[3.8190]}x^{24} - \underset{-20}{[2.2960]}x^{26} \\ + \underset{-20}{[0.7630]}x^{28} - \underset{-30}{[9.2210]}x^{30} + \underset{-30}{[7.6700]}x^{32} - \underset{-30}{[6.1110]}x^{34} + \underset{-30}{[4.5440]}x^{36} - \underset{-30}{[2.9690]}x^{38} \\ + \underset{-30}{[1.3870]}x^{40} - \underset{-40}{[9.7980]}x^{42} + \underset{-40}{[8.2020]}x^{44} - \underset{-40}{[6.6000]}x^{46} + \underset{-40}{[4.9920]}x^{48} - \underset{-40}{[3.3980]}x^{50} \\ + \dots\end{aligned}\right\} [\epsilon] (446)$$

$$\left.\begin{aligned}\frac{\sigma}{\sigma_0} &= \frac{1}{x^2}\frac{\mathrm{d}\mu}{\mathrm{d}x}\\ &= [0.0000000] - \underset{-10}{[9.3979400]}x^2 + \underset{-10}{[8.4648868]}x^4 - \underset{-10}{[7.3645163]}x^6 + \underset{-10}{[6.1661486]}x^8\\ &\quad - \underset{-10}{[4.9002003]}x^{10} + \underset{-10}{[3.5869762]}x^{12} - \underset{-10}{[2.2367420]}x^{14} + \underset{-10}{[0.8590392]}x^{16}\\ &\quad - \underset{-20}{[9.4577990]}x^{18} + \underset{-20}{[8.0361]}x^{20} - \underset{-20}{[6.5971]}x^{22} + \underset{-20}{[5.1424]}x^{24} - \underset{-20}{[3.6724]}x^{26}\\ &\quad + \underset{-20}{[2.1895]}x^{28} - \underset{-20}{[0.6936]}x^{30} + \underset{-30}{[9.1866]}x^{32} - \underset{-30}{[7.6685]}x^{34} + \underset{-30}{[6.1399]}x^{36} - \underset{-30}{[4.6019]}x^{38}\\ &\quad + \underset{-30}{[3.0547]}x^{40} - \underset{-30}{[1.4987]}x^{42} + \underset{-40}{[9.9349]}x^{44} - \underset{-40}{[8.3634]}x^{46} + \underset{-40}{[6.8046]}x^{48} - \ldots\end{aligned}\right\}[\zeta]\ (447)$$

The logarithms of the coefficients of the higher terms are given only to the fourth decimal place, but the attainment of greater accuracy is difficult; and as the expansions seldom need to be extended beyond x^{30}, the accuracy resulting from the series is quite sufficient for all cases which can arise even for the regions near the Sun's surface, where x is a maximum, and the convergence is slowest.

(4) *Limits of x at the Sun's Surface, and Density at the Centre.* To find the upper limit of the Sun's atmosphere, we determine the value of x for which the series $[\epsilon]$ vanishes. To the sixth decimal place, corresponding to a density less than one one-hundredth that of atmospheric air under normal pressure at sea level, this maximum value of x is found to be

$$x' = 3.653962, \qquad [\eta]\ (448)$$

terms higher than x^{34} being not required in the calculation.

The value of μ corresponding to the same limit must be found, and as the series for μ converges much more slowly than that for $1 - \left(\frac{\sigma}{\sigma_0}\right)^{2/3}$, it is necessary to take account of all the terms in equation $[\delta]$. The result is

$$\mu' = 2.709691. \qquad [\theta]\ (449)$$

LANE found $x' = 3.656$, $\mu' = 2.741$. Accordingly it will be seen that his value of x' is nearly correct, while that of μ' is decidedly too large.* The value of x' given in $[\eta]$ appears to be accurate to the fifth decimal place inclusive. That of μ' in $[\theta]$ is somewhat more uncertain, but careful consideration of the extent of this difficulty leads to the conviction that the uncertainty could not affect more than the three last decimal places. Using these values in LANE's equation (14), namely,

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* In a letter to SIR NORMAN LOCKYER, published in *Nature*, July 13, 1899, PROFESSOR J. PERRY, F. R. S., discusses the difficulty of attaining accuracy in these investigations, adding: "LORD KELVIN from $x = 0$, and MR. LANE for values beyond $x = 1$, obtained their results by methods such that errors may have increased as the work proceeded."

$$\sigma_0 = \frac{m'x'^3}{4\pi\mu' r'^3} = \frac{\sigma_1 x'^3}{3\mu'}, \qquad [\iota]\ (450)$$

we find $\quad \sigma_0 = 6.001377\,\sigma_1$

and $\quad \sigma_0 = 8.417531\,, \qquad [\kappa]\ (451)$

when $\quad \sigma_1 = 1.4026$ (cf. *A.N.* 3992).

It thus appears that considered as a sphere of monatomic gas the central density of the Sun is almost exactly six times the mean density; and taking $\sigma_1 = 1.4026$, the density of the matter at the Sun's centre is about 8.42 times that of water. This exceeds the specific gravity of steel (7.816) and even brass (8.383), and proves to be practically identical with that of German silver (8.432).

The uncertainty attaching to the calculated value of μ' affects the value of $\frac{\sigma}{\sigma_0}$, but the amount of this probably does not much exceed one unit in the third decimal place. Thus for most purposes we adopt for a sphere of monatomic gas

$$\left.\begin{aligned} &\sigma_0 = 6.000\,\sigma_1 \\ \text{corresponding to}\quad &\mu' = 2.710312\,; \end{aligned}\right\} \qquad [\lambda]\ (452)$$

and are led to the remarkable conclusion that in such a mass the central density probably is exactly six times the mean density.*

Using LANE's values $x' = 3.656$, $\mu' = 2.741$, and putting with him $\sigma_1 = 1.20334$, the results are

$$\sigma_0 = 5.94276\,\sigma_1 = 7.15115\,;$$

instead of 7.11 given in his published paper and also in his original manuscript. If we use $\sigma_1 = 1.4026$, the result will be $\sigma_0 = 8.3353$, in fair accord with our value in equation [κ].

LANE finished some of his approximations by means of two equations numbered (13) and (14), which are not rigorous.

On substituting his adopted values of x' and μ' in these equations, it is found that they are not satisfied. The outstanding difference probably arises both from the defective character of the approximate equations, and from the inaccurate values of x' and μ' found by the method of differences.

Under the circumstances the necessity for a sufficient expansion of the series to permit of a direct calculation of $\frac{\sigma}{\sigma_0}$ and μ' is obvious; and accordingly in the

* In his address on the "Sun's Heat," p. 407, LORD KELVIN remarks concerning the central density: "We may assume that it is in all probability much less than this (thirty-one times that of water, which he finds when $k = 1.4$), though considerably greater than the mean density, 1.4. This is a wide range of uncertainty, but it would be unwise at present to narrow it, ignorant as we are of the main ingredients of the Sun's whole mass, and of the laws of pressure, density and temperature, even for known kinds of matter, at very great pressures and very high temperatures."

Fig. 1. Sun's Disc in "K" Light as Photographed on 1904, September 20, at 10^h 42^m, A.M., G.M.T. Exposure 73^s. Enlarged about One and One-Half Times.

Fig. 2. Composite Photographs of Sun's Disc and Limb in "K" Light as Photographed on 1904, July 19, at 11^h 45^m, A.M., G.M.T. Exposure for Disc 18^s, and for Limb 18^m. Enlarged about One and One-Half Times.

Plate XXIV. Solar Photographs Made with the New Spectroheliograph of the Solar Physics Observatory, South Kensington, London, by Dr. W. J. S. Lockyer (From *Monthly Notices* of the Royal Astronomical Society, March, 1905).

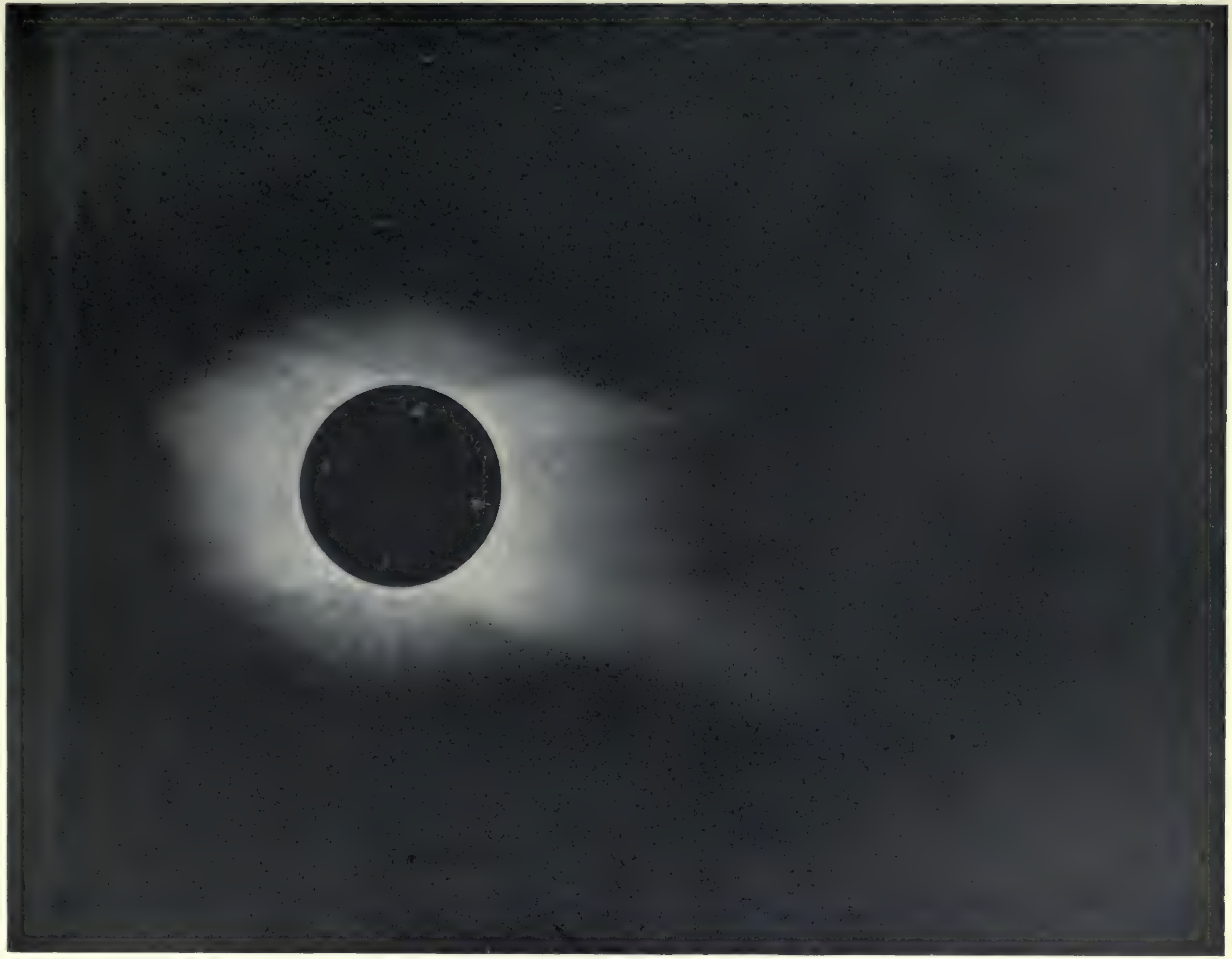

Plate XXV. General View of the Corona, Obtained by 82-Second Exposure, with 11-Foot Focus Camera, by Smithsonian Observatory Solar Eclipse Expedition, May 28, 1900 (Langley).

work of this paper we have used the rigorous formulae throughout, and made use of differences only in deriving the logarithms of the higher terms of the series, where a slight variation in the admissible values does not sensibly affect the final result.

Using the formula [ϵ] or [ζ] we find that the distribution of density in the Sun is as indicated in Table A. The values are computed for each tenth of the radius, and also for each hundredth in the outer tenth, which is the only region that could be affected by the Sun spots, prominences, and other surface phenomena witnessed by observers.

In view of the enormous pressure of over 21,000,000 atmospheres shown to exist at a depth of $\frac{1}{10}$ of the Sun's radius, the power of resistance of the Sun's globe, even near the surface, becomes so great that evidently not the slightest effect is produced at that depth by the most violent explosions witnessed externally in the ejection of prominences. For at a depth of $\frac{1}{10}$ of the radius the pressure is about seven times that at the centre of the Earth, which alone is effective in giving the nucleus of our globe a rigidity much greater than that of steel; and as no visible depressions are ever seen in the Sun's disc, it is evident that all surface disturbances are quite shallow and very effectively resisted by the pressure and intense radiation from below. Much of the ejected material seen in prominences is extremely rare, like a tenuous spray, and carried upward partly by explosive forces, and partly by the powerful pressure of the Sun's radiation, which is known to be occasionally augmented in certain local areas about the spots where the prominences develop. In fact MR. MAUNDER'S recent researches have shown that the development of intense local action is the cause of the disturbances of the Earth's magnetism.

(5) *Rise of Temperature Towards the Sun's Center.* This is calculated from the formula for adiabatic circulation

$$T = T'\left(\frac{\sigma}{\sigma'}\right)^{2/3}. \qquad [\mu]\ (453)$$

in which σ' and T' are the surface density and temperature respectively, and σ and T the density and temperature at any point within the Sun's mass. The Sun's atmosphere terminates quite suddenly, owing to the great intensity of solar gravity; but as the corona extends to a great height, and is maintained by the repulsion of fine particles under the pressure of the Sun's radiation, there is no absolutely definite boundary to the atmosphere which underlies the corona. Perhaps a sufficient approximation to the true density at the visible surface where solar clouds cease to form will be obtained by taking the density as lying between

one-tenth and one one-hundredth that of atmospheric air. According to recent experimental determinations the surface temperature will almost certainly lie between 6000° C. and 12000° C. Accordingly a density of 0.1 of atmospheric air, and a surface temperature of 6000° C. will give the minimum internal temperature at any point; while a surface density of 0.01 and a temperature of 12000° C. will give the maximum admissible temperature.

If the true temperature lies outside of the limits thus established, it would seem that it cannot be much outside. ARRHENIUS has recently shown (*Bulletin of the Lick Observatory*, No. 58) that the temperature of the corona high above the photosphere does not fall off very rapidly, even when the density approaches that of a vacuum of the best air pump; and judging by this analogy the smaller density and the higher temperature appears to have fully as great probability as those values which fix the minimum limit. It has been customary of late to reduce the Sun's temperature to figures but little above the highest available temperatures upon the Earth; but such a general tendency will naturally be resisted by the cautious investigator who is aware of the oscillatory movements of thought usually witnessed in successive ages. On the whole we think it practically certain that the Sun's temperature is above 6000° C., while 12000° C. is a figure by no means improbable. The following table shows the results of these calculations.

TABLE A. DENSITIES, PRESSURES AND TEMPERATURES CALCULATED FROM THE MONATOMIC THEORY.*

R Radius	$\frac{\sigma}{\sigma_0}$	σ Water = 1	σ Air = 1	ϖ Pressure in Atmospheres	T Hypothesis I = Min. $T' = 6000°$ C. $\sigma' = 0.0001293187$	T Hypothesis II = Max. $T' = 1200°$ C. $\sigma' = 0.00001293187$
1.00	0.0000015	0.0000129	0.01	....	6000° C.	12000° C.
0.99	0.000651	0.005472	4.233227	54635	72856	676338
0.98	0.001864	0.015700	12.140670	316531	144870	1365662
0.97	0.003476	0.029286	22.646390	894699	222921	2069413
0.96	0.005435	0.045786	35.405630	1884224	300284	2787593
0.95	0.007713	0.064977	50.245406	3376805	379212	3520293
0.94	0.010296	0.086735	67.070500	5464675	459733	4267780
0.93	0.013173	0.110978	85.817600	8240856	541837	5029970
0.92	0.016343	0.137680	106.466000	11804019	625593	5807490
0.91	0.019799	0.166792	128.681100	16250623	710932	6599703
0.90	0.023541	0.198320	153.357800	21636565	797912	7407160
0.80	0.077049	0.649096	....	156467430	1758935	16328500
0.70	0.161316	1.359001	....	536137160	2878659	26723100
0.60	0.276802	2.331913	....	1318551200	4125900	38301450
0.50	0.419776	3.536397	....	2639437700	5446090	50556970
0.40	0.579211	4.879548	....	4513802000	6747892	62660430
0.30	0.737977	6.217069	....	6759055500	7932940	73642880
0.20	0.874463	7.366897	....	8968448000	8883145	82463830
0.10	0.967136	8.147617	....	10607851000	9500156	88191620
0.00	1.000000	8.424480	....	11215403000	9714170	90178370

* The method employed in calculating the pressures tabulated in this chapter are explained in the next chapter, § 219.

§ 203. *On the Dissociation of the Elements by High Temperatures, the Probable Average Specific Heat of Solar Matter, and on the Sun's Radiation.*

(1) *Dissociation.* The effects of high temperature and of powerful electric currents in producing dissociation of bodies have long been studied by Physicists and Chemists, but the phenomena observed have always been limited by the feeble forces available to the experimenter in the laboratory. Almost all known compound molecules may be thus decomposed, and the recognized *experimentum crucis* of an elementary substance is its ability to withstand various degrees of molecular and atomic agitation due to heat and electricity. The forces accessible to the Chemist, however, are as nothing compared to those at work in the interior of the Sun*; yet in default of knowledge of the behavior of matter under such stupendous forces, the tendencies observed upon the Earth may guide us to correct conclusions in regard to what takes place in the depths of the Sun's flaming globe.

Recent experiments of PROF. SIR WM. RAMSAY showed that Radium, with an atomic weight of 225, was unstable under ordinary conditions, and slowly producing Helium by its decomposition. As Radium has one of the highest atomic weights, while Helium has one of the lowest (3.96), this observation of RAMSAY seems to have been accepted as a genuine case of the transmutation of an element. What has been observed in the case of Radium is also suspected if not demonstrated in the cases of Thorium and Uranium, the atomic weights of which are respectively 233.4 and 239, the very highest yet known. Recent researches indicate that elements with such high atomic weights are all unstable, and the conclusion is drawn that higher atomic weights are practically impossible under ordinary conditions. If an element such as Radium by disintegration is slowly evolving Helium under terrestrial conditions, its decomposition in the Sun would no doubt be very rapid. Indeed it would never develop there because of the immense temperatures to which the elements are subjected at great depths in the Sun's mass.

Accordingly we may with considerable probability assume that all the more complex elements would be reduced to the most primitive constituents in the interior of the Sun; and it may be that nothing less elementary than Hydrogen and Helium could withstand the enormous temperature of millions of degrees.

Abundant terrestrial experiments show that the so-called atomic heat of our ordinary elements is about 6.4; and as the specific heat, by the law of DULONG

* Speaking of the effects of the increasing temperature as we descend into the Sun's interior, NEWCOMB describes it as "an inconceivable degree of heat, such that were matter exposed to it on the surface of the Earth, it would explode with a violence to which nothing within our experience can be compared" (Article "Sun," *Encyclopaedia Americana*). Note added June 5, 1905.

and PETIT* is equal to this constant divided by the atomic weight (or the molecular weight in case of some of the non-metals), it follows that if the primitive elements in the interior of the Sun have an average atomic weight not larger than 6.4, the resulting specific heat of the solar matter will be equal to that of water. If the atomic weight be equal to that of Helium it will give a specific heat of 1.62, and if equal to that of Hydrogen, 3.409, and so on.

Most of the terrestrial elements exist in the photosphere and chromosphere, but we have no means of knowing at what depth their stable development is possible. In view of the rapid rise of the temperature downward it seems most likely that partial dissociation occurs within the sight of the observer, and augments from layer to layer as we descend below the photosphere.†

It probably is not without significance that observation has failed to establish the undoubted existence in the Sun of elements with such high atomic weights as Iridium (196.7), Platinum (196.7), Osmium (198.6), Gold (196.2), Mercury (199.8), Thorium (233.4), and Uranium (239). The absence of these elements probably indicates partial dissociation in the solar photosphere, and thus we may with great probability adopt the view that whatever elements exist in the interior of the Sun must be of extreme simplicity and small atomic weight.

(2) *The Sun's Average Specific Heat.* The resulting specific heat of the Sun's matter may thus have any value between 0.5 and 6.8, to which it would rise if all the elements were as simple as monatomic Hydrogen. Whether under the terrific heat operating in the Sun's interior the elements might disintegrate into the yet more elementary electrons especially investigated by J. J. THOMSON we pass over as too extreme an hypothesis to be considered at present.

LANE seems to have attached high importance to the theory of dissociation and the production in this way of small atomic weights for the gases in the Sun's interior and consequently great specific heat for the solar matter; but it is remarkable that the subject has been so completely overlooked by subsequent writers that very few investigators during the past thirty-six years have even touched upon the monatomic constitution of the Sun.

LANE says: "In forming his theory CLAUSIUS found that the known specific heats of the gases are all much too great for free simple atoms impinging on one another, and he therefore introduced the hypothesis of compound molecules, each

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* Confirmed by PROF. W. A. TILDEN'S recent researches, *Phil. Trans.*, 1900, Part I, p. 250.

† After this paper was finished, attention was drawn to the article "Sun," in the new *Encyclopaedia Americana*, where NEWCOMB discusses briefly the effects of great heat and pressure upon solar matter. Though he adopts no definite theory of the Sun's internal density, merely remarking that it is smaller than the mean, 1.4, at the surface, and increases towards the centre, he inclines to the theory of dissociation: "One thing which we can say with confidence as to the effect of these causes is that no chemical combinations can take place in matter so circumstanced." Note added June 5, 1905.

compound molecule being a system of atoms oscillating among each other under forces of mutual attraction. Now if this were accepted as the actual constitution of the gases it is of course easy enough to conceive that in the fierce collisions of these compound molecules with each other at the temperatures supposed to exist in the Sun's body, their component atoms might be torn asunder, and might thenceforth move as free, simple molecules. In this case, still retaining the hypothesis of CLAUSIUS' theory, that the average length of the path described by each between collisions is large compared with the diameter of the sphere of effective attraction or repulsion of atom for atom, the value of k would reach its maximum of $1\frac{2}{3}$. Experiment has not shown us any gas in this condition, and for the present it is hypothetical. Even in Hydrogen the value of k does not materially, if any, exceed the value of 1.4 which it has in air. But if it were found that the hydrogen molecule is compound, and that in the body of the Sun the heat splits this molecule into two equal simple atoms, and in fact that all the matter in the Sun's body is split into simple atoms equally as small, then, while the value of k would be $1\frac{2}{3}$, the value of σ* would be about 1600 feet. If with these values we repeat the calculation of the density of the layer of 54,000° Fahr., we find its specific gravity to be 0.000363 of that of water, or 4.35 times that of Hydrogen gas at common temperature and pressure and in its known condition, or 8.7 times that which the Hydrogen in the hypothetic condition would have if it retained that condition at common temperature and pressure. We find also that the mechanical equivalent of all the heat that a cubic foot of the layer would give out in cooling down, under pressure, to zero, would be no less than 13,500,000 foot pounds. Instead, therefore, of a layer ten miles thick, it would now require only a thickness of 38 feet to give out, in cooling down to zero, twice the heat emitted by the Sun in one minute. It will be seen [equations (17) and (19)], that this thickness, retaining the constant value $k = 1\frac{2}{3}$, would diminish with the $2\frac{1}{2}$ power of the masses of the atoms into the Sun's body is hypothetically resolved (the reciprocal of the value of σ*) and I leave each to form his own impression how far this view leads towards verisimilitude."

From the above passage it will be seen that LANE appreciated the advantage of the monatomic theory in explaining the radiation at the Sun's surface, since on this theory a shallow layer might contain a great amount of heat.

(3) *Solar Radiation.* No aim is made in this paper to treat mathematically of the problem of the Sun's surface radiation, but we may remark that the commonly accepted theory of enormous convective currents, one set ascending with highly heated matter and the other descending with the material cooled by radi-

* LANE's σ is denoted by h in the present paper.

ation, does not seem to be well founded. When the true character of the Sun's radiation is made out, it will be found that the heat and light are supplied in the main not by convective currents, which would necessarily encounter great resistance and constantly antagonize one another by the contrary motions of the neighboring streams of fluid, but by direct radiation — the internal heat being so intense and the overlying medium of gases so thin and transparent to radiation of that intensity that the heat is driven bodily through the Sun's mass with a velocity smaller but not immeasurably smaller than that of light and electricity. In this way the outside surface never cools, nor are great convective currents required to maintain the surface luminosity.

We may conceive that with the enormously rapid rise of temperature below the photosphere, amounting to from 100,000° to 500,000° C. at a depth of $\frac{1}{100}$ of the radius or 7000 km., most of the necessary supply of heat is radiated through the flimsy overlying layers of gas, with a velocity but little inferior to that of light. Nor can we conceive of any material of this small density which could resist the terrific heat beneath sufficiently to cause a great fall in the external temperature, even if there were no convective currents whatever. Currents developing at the Sun's surface probably are comparatively shallow and irregular in their character, while the supply of heat from beneath is radiated through the intervening medium with great velocity, thus maintaining the surface in a state of dazzling brilliancy.

This rapid propagation of heat is not difficult to understand when we remember that by its nature heat is the energy of vibrating molecules of perfect elasticity moving with enormous velocities* in a rare medium, admitting by its abundant vacancies of a vast direct radiation from beneath with the actual velocity of light. The propagation of light in the Earth's atmosphere on a clear day is perhaps nearly analogous to that in the Sun's outer layers just below the photosphere.

The illumination of the Earth's surface, which enables one to view objects hundreds of kilometres away with but a very slight absorption of their light due to the intervening atmosphere, illustrates the penetrating power of sunlight here upon the Earth, where the intensity is as nothing compared to that in the Sun. Yet the outer layers of the Sun's photosphere are of the same order of density as our atmosphere; and while the carbon and perhaps other particles in the photosphere form a cloud which obstructs the light, no doubt the interior gases are very transparent. The radiating particles in the photosphere thus derive their heat and light from two sources:

(*a*) Collisions with neighboring particles;

* cf. The mean molecular velocities of the Sun's particles calculated in Table D.

(*b*) Direct radiation from the intensely heated body of the Sun beneath, filling the solid angle of a whole hemisphere.

The latter source furnishes incomparably the larger part of the radiant energy emitted by the photosphere, and as it is transmitted from the interior with sensibly the velocity of light, there is no opportunity for the cloud-surface to cool, and little or no necessity for convective currents.*

The views here expressed depart widely from those heretofore held by leading authorities, and it is perhaps worthy of inquiry how far the theory of convective currents has been critically examined from the theoretical or observational standpoint.†

In his perplexity LANE expresses the opinion that something quite beyond our present experimental knowledge is required to explain the Sun's radiation. Proceeding on the hypothesis that $k = 1.4$, he says: "The heat emitted each minute would therefore be fully half of all that a layer ten miles thick would give out in cooling down to zero, and a circulation that would dispose of volumes of cooled atmosphere at such a rate seems inconceivable."

LORD KELVIN expresses himself thus: "Gigantic currents throughout the Sun's liquid mass are continually maintained by fluid, slightly cooled by radiation, falling down from the surface, and hot fluid rushing up to take its place" (Address, p. 392).

In response to a request of DR. VOGEL, made recently in revising ENGELMANN's translation of NEWCOMB's *Popular Astronomy*, YOUNG has expressed his views very clearly. As given authoritatively in *Popular Astronomy*, April, 1904, they are as follows: "From the under surface of this cloud shell (the photosphere), if it really exists, there must necessarily be a continual precipitation into the gaseous nucleus below with a corresponding ascent of vapors from beneath — a vertical circulation of great activity and violence, one effect of which must be a constricting pressure upon the nucleus much like that of the liquid skin of a bubble upon the enclosed air. With this difference, however, that the photospheric cloud shell is not a continuous sheet, but 'porous,' so to speak, and permeated by vents through which the ascending vapors and gases can force their way into the region above."

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* The absence, however, of such rapid currents for conveying the heat to the photosphere obviously constitutes no denial of the convective arrangement of the Sun's internal density.

† In the article "Sun," *Encyclopaedia Americana*, NEWCOMB says: "It follows that the heat radiated from the surface must be continually supplied by the rising up of hot material from the interior, which again falls back as it cools off. It is difficult to suppose that even a liquid could rise and fall back rapidly enough to keep up the supply of heat constantly radiated. We therefore conclude that the photosphere is really a mass of gas in which, however, solid particles of very refractory substances may be suspended." He says it is probable that the "rice grains" are produced by currents of heated matter from the interior, which are constantly rising to the surface, there to radiate their heat, and then fall back again. Note added June 5, 1905.

The difficulty which LANE and others encounter in explaining the rapid supply of hot and the disposition of cool matter by means of currents, we meet by substituting direct radiation, which ARRHENIUS found adequate to explain the corona, the isolated particles of which were calculated to have a temperature of 4620° C. (*Bulletin of Lick Observatory*, No. 58). There seems little doubt that if this theory is examined in the light of the methods used by ARRHENIUS, it will be found to afford a simple and natural explanation of the Sun's photospheric radiation.

§ 204. *Calculation of the Total Amount of Heat Stored Up in the Sun's Globe.*

Suppose the Sun's globe composed of monatomic gas with the elements so mixed as to be essentially homogeneous in all parts, or made up of a series of concentric spherical layers each of uniform quality throughout. An element of the Sun's mass conceived as an infinitely thin shell of density σ will be

$$\mathrm{d}m = 4\pi\sigma x^2 \mathrm{d}x.$$

And if T be the temperature above absolute zero and ζ the average specific heat of the matter, the element of heat in a shell will be

$$\mathrm{d}H = 4\pi\zeta\sigma x^2 \mathrm{d}x\, T.$$

And since throughout the Sun's mass $T = T'\left(\frac{\sigma}{\sigma'}\right)^{\frac{2}{3}}$, we obtain by integration

$$H = 4\pi\sigma_0 T' \int_0^{x'} \zeta \frac{\sigma}{\sigma_0} x^2 \mathrm{d}x \left(\frac{\sigma}{\sigma'}\right)^{\frac{2}{3}} = 4\pi\sigma_0 \left(\frac{\sigma_0}{\sigma'}\right)^{\frac{2}{3}} \zeta_0 \int_0^{x'} \left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}} \left(\frac{\sigma}{\sigma_0}\right) \left(\frac{\zeta}{\zeta_0}\right) x^2 \mathrm{d}x, \qquad [\nu] \quad (454)$$

ζ_0 being the specific heat of the fluid at the Sun's centre. The two series $\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}}$ and $\frac{\sigma}{\sigma_0}$ given in equations [ϵ] and [ζ] must be multiplied together, the product by x^2 taken and then integrated, and the result will be the total amount of heat stored up in the Sun's globe.

As the specific heat ζ probably is a function of the temperature, $\frac{\zeta}{\zeta_0}$ can not properly be removed from under the integral sign. But as there is no experimental or theoretical method of discovering the form of this function, it becomes necessary to assume an average value for the entire mass of solar matter, as indicated in the last section.

If the Sun were a homogeneous globe of density σ_1, and temperature T_1 corresponding to that of the outside surface, as found by observation, and ζ_1

were the specific heat of the homogeneous fluid, the amount of heat stored up in such a globe would be

$$H_1 = \tfrac{4}{3}\pi\sigma_1\zeta_1 T_1 x'^3 .$$

Accordingly the ratio of the amount of heat in the actual heterogeneous Sun to that in the corresponding hypothetical homogeneous globe is found to be

$$\frac{H}{H_1} = \frac{3\sigma_0}{\sigma_1}\cdot\frac{T'}{T_1}\cdot\frac{\left(\frac{\sigma_0}{\sigma'}\right)^{\frac{2}{3}}\zeta_0'}{\zeta_1 x'^3}\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x , \qquad [\xi]\ (455)$$

where ζ_0' is a value such that the product by the integral

$$\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x$$

is rigorously the same as the expression

$$\zeta_0\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}}\left(\frac{\sigma}{\sigma_0}\right)\left(\frac{\zeta}{\zeta_0}\right)x^2\,\mathrm{d}x .$$

We found that $\frac{\sigma_0}{\sigma_1} = 6.000$, and $x' = 3.653962$; and as limits of the Sun's external temperature we took 6000° C., and 12000° C. respectively. The value of the integral in $[\xi]$, found by the numerical evaluation of more than 450 sensible terms in a square of 676 elements, was found to be

$$U' = \int_0^{x}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{3}}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x = 1.438226 .$$

The ratio $\frac{\zeta_0'}{\zeta_1}$ is known to be greater than unity, but if we assume that value for the moment, and take the surface density

$$\sigma' = 0.0001293187 ,$$

putting also $T' = T_1$ we shall find the ratio

$$\frac{H}{H_1} = 596.9387 .$$

If we use the other limit for the surface density

$$\sigma' = 0.00001293187 ,$$

the ratio becomes

$$\frac{H}{H_1} = 2770.1064 .$$

These numbers represent the ratios of the amounts of heat stored up in the actual heterogeneous Sun to those contained in corresponding homogeneous globes, under the hypotheses indicated. Thus by varying the surface density from 0.1 to 0.01 that of atmospheric air, the ratio increases from 596.9387 to 2770.1064; whence it is clear that the surface density is an important element in determining the amount of heat accumulated in the interior of the Sun's mass.

If we use the outside limits of the surface temperature, multiplying the smaller ratio by 6000°, and the larger ratio by 12000°, we shall find that the heat stored up in the Sun, on these two hypotheses, would elevate the mean temperature of the corresponding homogeneous globes to

5151200° C.,

and 47808450° C. respectively.

Thus taking the ratio $\frac{\zeta_0'}{\zeta_1}$ as unity, the amount of heat stored up in the Sun would raise a homogeneous globe of the same mass and constitution to a temperature somewhere between these limits.

Let us now consider the probable value of the ratio $\frac{\zeta_0'}{\zeta_1}$. As the temperature near the Sun's centre probably lies between 10,000,000° C. and 100,000,000° C., all experimental knowledge would lead us to infer that the specific heat at the Sun's centre is not much inferior to that of monatomic Hydrogen, which may be taken at 6.8. At the Sun's surface, on the other hand, the specific heat may not surpass that of water (1.00), or even water vapor (0.4805), or carbon (0.4589). MR. W. E. WILSON's valuable researches seem to prove that carbon is the chief constituent of the photosphere, but obviously its specific heat under solar conditions might be larger than that found in laboratory experiments.

Assuming that the specific heat augments towards the Sun's centre as some function of the temperature, it appears probable that the ratio $\frac{\zeta_0'}{\zeta_1}$ is not likely to be less than 2, nor more than 6. If this multiplier were 3 or 4, the total heat stored up in the Sun on the hypothesis which assumes the minimum surface temperature of 6000° C., and the maximum surface density of one-tenth that of

atmospheric air, giving the minimum heat-accumulation, would furnish enough heat to raise the temperature of an equal mass of water to a temperature of some 20,000,000° C. This accumulated heat alone would supply radiation at the present rate for 10,000,000 years, if the Sun did not shrink another millimetre during that period.

On the other hand, if the higher figures be adopted for the surface temperature, and the ratio $\frac{\zeta_0'}{\zeta_1}$ be reduced to 1.0 or 2, there will be more than ample heat to maintain radiation for 10,000,000 years without additional shrinkage.

It is generally agreed that the Sun will continue to shrink until the radius is less than one-half of its present dimensions, which would have the effect of multiplying the mean density by 8, and doubling the total development of heat. Instead of a quantity of heat that would elevate an equal mass of water about 40,000,000° C., shown elsewhere in this paper to be the total heat generated up to the present time, the Sun's total heat-producing capacity is therefore adequate to raise an equal mass of water through at least 80,000,000° C. Since up to the present time only one-fourth of this supply has been radiated away (cf. § 208), the future duration of the Sun's activity will probably be at least three times that of the past, or not less than 30,000,000 years of uniform radiation at the present rate. This great prolongation of the Sun's future activity is not the least interesting conclusion of astronomical science, and it ought to dispose once for all of the suggestions occasionally heard that already the Sun's activity is beginning to wane.*

§ 205. *Potential of the Sun Upon Itself, or Amount of Heat Produced by the Condensation of the Sun Considered as a Sphere of Monatomic Gas.*

As the Sun is made up of concentric spherical layers of uniform density throughout, it is sufficient to consider the action of an enclosed sphere upon its surface layer, and then extend the integration to the whole mass of the Sun.

The element of the potential of the enclosed sphere of density σ_1 upon its surface layers of density σ is

$$d\boldsymbol{Y} = \frac{4}{3}\pi\sigma_1\frac{x^3}{x}\cdot 4\pi\sigma x^2 dx = \frac{16}{3}\pi^2\sigma x^4 dx\,\sigma_1 \,.$$

Now the mass of the enclosed sphere

$$m = \frac{4}{3}\pi\sigma_1 x^3 = 4\pi\sigma_0\int_0^x \frac{\sigma}{\sigma_0}\cdot x^2 dx \,,$$

* The calculations here given include the effects of gravitation alone; the energy arising from radio-active forces probably would enormously prolong the period of the sun's future activity, and the downpour of meteoric matter considered in §158 would tend in the same direction. Note added, May 17, 1910.

and

$$\sigma = \frac{3\,m}{4\,\pi\,x^3} = \frac{3\sigma_0}{x^3}\int\limits_0^x \frac{\sigma}{\sigma_0}\,x^2\,\mathrm{d}x\,;$$

and with this value the above differential equation becomes

$$\mathrm{d}\boldsymbol{Y} = 4\,\pi\,\sigma\,x\,\mathrm{d}x\,\cdot\,4\,\pi\,\sigma_0\int\limits_0^x \frac{\sigma}{\sigma_0}\,x^2\,\mathrm{d}x\,.$$

Therefore

$$\boldsymbol{Y} = 16\,\pi^2\sigma_0^{\,2}\int\limits_0^{x'} \frac{\sigma}{\sigma_0}\,x\,\mathrm{d}x\int\limits_0^{x'} \frac{\sigma}{\sigma_0}\,x^2\,\mathrm{d}x\,. \qquad [\circ]\ (456)$$

The evaluation of the integrals

$$U = \int\limits_0^{x'} \frac{\sigma}{\sigma_0}\,x\,\mathrm{d}x\int\limits_0^{x'} \frac{\sigma}{\sigma_0}\,x^2\,\mathrm{d}x$$

involves the calculation of a square of 625 terms, of which 470 are sensible in units of the sixth place of decimals, and the result is $U = 1.725720$.

The potential of a homogeneous sphere upon itself considered by Helmholtz (cf. *Phil. Mag.*, for 1856, p. 516; or the author's paper in *Transactions Academy of Sciences* of St. Louis, Vol. X, No. 1) is easily shown to be

$$V = \frac{16\,\pi^2\sigma_1^{\,2}x'^5}{15}\,.$$

Therefore the ratio of the potential upon itself of the heterogeneous sphere to the homogeneous sphere becomes

$$\frac{\boldsymbol{Y}}{V} = \frac{15\,\sigma_0^{\,2}\,U}{\sigma_1^{\,2}x'^5} = \frac{540\;U}{x'^5} = 1.430685\,.$$

Thus we see that in condensing from a state of infinite expansion to its present state, the Sun, considered as a sphere of monatomic gas, has produced a little over forty-three per cent. more heat than the corresponding homogeneous sphere.

§ 206. *The Moment of Inertia of the Sun.*

This is one of the most important mechanical constants of the solar system. Heretofore it has remained uncertain to a considerable degree, because there has been no adequate theory of the Sun's physical constitution. The methods

already developed, however, enable us to derive the moment of inertia of the Sun with great accuracy. The mass of any spherical shell of density σ is

$$dm = 4\pi\sigma x^2 dx$$

and the element of the moment of inertia due to such a shell is

$$dN = \frac{2}{3} r^2 dm = \frac{8}{3}\pi\sigma x^4 dx .$$

Therefore for the whole of the Sun's globe

$$N = \frac{8}{3}\pi\sigma_0 \int_0^{x'} \frac{\sigma}{\sigma_0} x^4 dx . \qquad [\pi]\ (457)$$

For a homogeneous sphere we have

$$N_1 = \frac{8}{3}\pi\sigma_1 \int_0^{x'} x^4 dx = \frac{8}{15}\pi\sigma_1 x'^5 .$$

Accordingly the ratio of the moment of inertia of the heterogeneous to the homogeneous sphere is

$$\left.\begin{aligned} \frac{N}{N_1} &= \frac{5\sigma_0}{\sigma_1 x'^5} \int_0^{x'} \frac{\sigma}{\sigma_0} x^4 dx = \frac{30}{x'^5} \int_0^{x'} \frac{\sigma}{\sigma_0} x^4 dx \\ &= 6 \left\{ 1 - \frac{5}{7}[9.3979400]x'^2 + \frac{5}{9}[8.4648868]x'^4 - \frac{5}{11}[7.3645163]x'^6 \right. \\ &\quad \left. + \frac{5}{13}[6.1661486]x'^8 - \frac{5}{15}[4.9002003]x'^{10} + \ldots \right\} . \end{aligned}\right\} \qquad [\rho]\ (458)$$

This integral is easily evaluated numerically, and the result is found to be

$$\frac{N}{N_1} = + 0.506208 . \qquad [\sigma]\ (459)$$

Thus it follows that the moment of inertia of a sphere of monatomic gas is almost exactly one-half that of the corresponding homogeneous sphere.

LAPLACE's law as applied to the Sun in *A.N.*, 3992, made this ratio 0.6535. The value there assigned for the gaseous theory by quadrature (0.5863) is seen to be too large when the integration is rigorous. The accuracy attained by the series used in the integral of equation $[\rho]$ is such that probably no uncertainty attaches to the fourth decimal place in the final result.

If the radius of inertia be denoted by ι, then

$$\iota = r\sqrt{\tfrac{2}{5}}$$

for a homogeneous sphere, and for the monatomic Sun

$$\iota = r\sqrt{\frac{1.012416}{5}} = 0.4499814\,r\,.$$

Thus it follows that if the whole mass of the Sun, viewed as made up of monatomic gas, were placed at a distance of $0.45r$ from the axis, the moment of inertia would be the same as that of the actual Sun.

These constants obviously apply also to *Jupiter* and *Saturn*, *Uranus* and *Neptune*, and a multitude of fixed stars constituting the visible universe.

§ 207. *The Annual Shrinkage and the Secular Equation of the Sun's Diameter.*

Suppose the value of the Sun's gravity at the surface be G, and at any point below the surface G'; then

$$\frac{G'}{G} = \nu = \frac{\dfrac{4\pi\sigma_0}{x^2}\displaystyle\int_0^x \frac{\sigma}{\sigma_0}x^2\,\mathrm{d}x}{\dfrac{M}{x'^2}} = \frac{\dfrac{4\pi\sigma_0}{x^2}\displaystyle\int_0^x \frac{\sigma}{\sigma_0}x^2\,\mathrm{d}x}{\dfrac{4}{3}\pi\dfrac{\sigma_1 x'^3}{x'^2}} = \frac{3\sigma_0}{x'x^2\sigma_1}\int_0^x \frac{\sigma}{\sigma_0}x^2\,\mathrm{d}x = \frac{18}{x'x^2}\int_0^x \frac{\sigma}{\sigma_0}x^2\,\mathrm{d}x\,. \quad [\tau]\ (460)$$

In shrinkage any element of the mass distant x from the centre will be displaced by $\frac{x}{x'}\cdot o$, when the surface displacement due to shrinkage is o. The work will be

$$\mathrm{d}W = \mathrm{d}m\,n\,g\,\frac{x}{x'}\,o\,\nu\,,$$

where $G = ng$, or n is the number of times the solar gravity surpasses terrestrial gravity, and

$$\mathrm{d}m = 4\pi\sigma x^2\,\mathrm{d}x\,.$$

Since we have

$$M = \frac{4}{3}\pi\sigma_1 x'^3\,,$$

the above expression may be written

$$\mathrm{d}m = \frac{3\,M\sigma x^2\,\mathrm{d}x}{\sigma_1 x'^3}\,,$$

and the differential equation for the work becomes

$$dW = \frac{3M \cdot ng \cdot o}{x'^4} \cdot \frac{\sigma}{\sigma_1} \cdot x^3 dx \cdot \nu . \qquad [\upsilon]\ (461)$$

Integrating we have

$$W = \frac{3M \cdot ng \cdot o}{x'^4} \frac{\sigma_0}{\sigma_1} \int_0^{x'} \frac{\sigma}{\sigma_0} x^3 dx \cdot \frac{18}{x' x^2} \int_0^{x} \frac{\sigma}{\sigma_0} x^2 dx = \frac{324 n \cdot Mg \cdot o}{x'^5} \int_0^{x'} \frac{\sigma}{\sigma_0} x\, dx \int_0^{x} \frac{\sigma}{\sigma_0} x^2 dx . \qquad [\varphi]\ (462)$$

where Mg is the weight of the Sun's mass at the Earth's surface expressed in kilograms. The evaluation of these integrals has already given us

$$U = 1.725720 .$$

Therefore

$$W = \frac{324 n \cdot Mg \cdot o \cdot U}{x'^5} .$$

The mechanical equivalent of this work is

$$W' = \frac{324 n \cdot Mg \cdot o \cdot U \cdot A}{x'^5} ,$$

where $A = \frac{1}{424}$, the unit being the metre.

We shall see later that only 50 per cent. of all the heat produced in a sphere of monatomic gas is radiated away. Calling the radiation Q, we have

$$\frac{W'}{2} = Q = \frac{324 \cdot n \cdot Mg \cdot o \cdot U \cdot A}{2 x'^5} . \qquad [\chi]\ (463)$$

According to recent researches, especially those of Langley, the amount of heat annually lost by the Sun is sufficient to raise an equal mass of water through about 2.00° C., so that $Q = 2.00\, Mg$; and we get

$$2 Mg = \frac{324 n \cdot Mg \cdot o \cdot U \cdot A}{2 x'^5} ,$$

and

$$o = \frac{4 \cdot 424 \cdot x'^5}{324 n U} = \frac{1696 x'^5}{324 (27.86554)(1.725720)} = 70.902770 \text{m} . \qquad [\psi]\ (464)$$

Hence it follows that the Sun's annual shrinkage on the monatomic theory is only 70.902770 m.*

* The shrinkage originally found by Helmholtz for a homogeneous Sun with Pouillet's smaller radiation (1.25 instead of 2.00 used above), on the hypothesis that no heat is stored up for raising the temperature, was about 35m., or almost exactly one-half of that here calculated.

This gives a daily shrinkage of 19.4 cm., or an hourly shrinkage of about 8 mm. In 2000 years the shrinkage would amount to 141.8 km., or almost exactly $\frac{1}{9800}$ part of the Sun's radius (139.2196 km). The Sun's angular diameter is 1920″.00, and thus the diminution on this basis is as follows:

″ 1.92	in	10000	years
0.96	"	5000	"
0.192	"	1000	"
0.096	"	500	"
0.0192	"	100	"
0.000192	"	1	"

Thus the secular equation of the Sun's diameter becomes

$$D = 1920''.00 - 0''.000192\,(t - 1900)\,. \qquad [\omega]\ (465)$$

Assuming the precision of the concluded solar radiation this expression will be rigorously exact for a period of at least 1,000,000 years from the present epoch. Such a period would give a shrinkage of 192″, or $\frac{1}{10}$ of the Sun's diameter, which could just be perceived with the naked eye. Accordingly if an immortal with perfect vision and memory could compare the Sun's angular diameter now with what it was 1,000,000 years ago, he would find that in the mean time it has shrunk to $\frac{10}{11}$ of its former value.

§ 208. *Total Amount of Heat Developed in the Condensation of the Planets Rigorously Determined.*

The formula originally developed by Helmholtz (*Phil. Mag.*, 1856, p. 516) for finding the temperature to which an equal mass of water would be raised by all the heat developed in the condensation of the Sun, considered as homogeneous, namely,

$$\theta = \frac{3}{5}\,\frac{M}{m_3}\cdot\frac{r^2}{R}\cdot\frac{A}{\zeta}\,, \qquad (466)$$

where M is the Sun's mass, R its radius, r the Earth's mean radius, and m_3 the Earth's mass, $A = \frac{1}{424}$, the unit of length being the metre, and ζ the specific heat of water taken as unity, may be applied directly to homogeneous planets including the Earth itself. In this case M becomes the mass of the planet, or of the hypothetical aqueous globe of equivalent mass, while R is the planet's actual radius. Assuming the planets and satellites to be homogeneous we obtain the results given in the second column of the following table:

TABLE B. TOTAL HEAT OF CONDENSATION OF BODIES OF PLANETARY SYSTEM.

Body	θ Temperature of Equal Mass of Water, Body Supposed Homogeneous	$\frac{Y}{V}$ Heterogeneity Factor for LAPLACE'S Law	Θ Heterogeneous According to LAPLACE'S Law	$\Theta-\theta$ Effect of Laplacean Heterogeneity on Equal Mass of Water	$\frac{Y}{V}$ Heterogeneity Factor for Sphere of Monatomic Gas	Θ' Heterogeneous According to Monatomic Law	$\Theta'-\theta$ Effect of Monatomic Heterogeneity on Equal Mass of Water
	C °		C °	C °		C °	C °
Mercury	590.8	1.0075	595.3	4.5	....	....	....
Venus	7687.3	1.08981	8377.7	690.4	....	....	....
Earth	9014.9	1.104214	9954.4	939.5	....	....	....
Mars	1837.3	1.019318	1872.8	35.5	....	....	....
Moon	405.2	1.007343	408.1	2.9	....	....	....
*Jup.*Sat.I	328.4	1.0065	331.4	3.0	....	....	....
" " II	300.9	1.0060	302.7	1.8	....	....	....
" " III	621.4	1.0092	627.1	5.7	....	....	....
" " IV	177.9	1.0067	179.1	1.2	....	....	....
Titan	460.5	1.0102	465.2	4.7	....	....	....
Jupiter	262717	1.251654	328831	66114	1.430685	375865	113148
Saturn	94733	1.250485	118462	23729	1.430685	135533	40800
Uranus	39755	1.249879	49689	9934	1.430685	56877	17122
Neptune	45717	1.250248	57157	11440	1.430685	65406	19689
Sun	27452563	1.2490604	34289908	6837345	1.430685	39276000	11823437

It will be seen that on the hypothesis of homogeneity the condensation of the smaller planets and the satellites has not produced sufficient heat to raise them to high temperatures, if the specific heat of their matter is at all comparable to that of water here taken as the unit. Before considering what their actual average specific heats may be, we shall first consider the effects of heterogeneity in augmenting the amounts of heat above computed.

The case of the Sun and major planets, assumed to be essentially monatomic gas, has already been considered, and we need therefore consider only the inner planets and satellites, which have attained the encrusted state, and obey LAPLACE'S law of density. We have elsewhere found that the potential of a heterogeneous sphere upon itself is

$$\boldsymbol{Y} = 16\pi^2\sigma_0{}^2 \int_0^x \frac{\sigma}{\sigma_0} x \, dx \int_0^x \frac{\sigma}{\sigma_0} x^2 \, dx . \tag{467}$$

To apply this formula to planets in which the density follows LAPLACE'S law, we should substitute for $\frac{\sigma}{\sigma_0}$ the expression required by that law (cf. *A.N.*, 3992), namely

$$\frac{\sigma}{\sigma_0} = \frac{\sin(qx)}{qx} .$$

For a planet of this type the expression thus becomes

$$\boldsymbol{Y} = 16\,\pi^2\sigma_0^2 \int_0^x \frac{\sin(q\,x)}{q\,x}\,x\,\mathrm{d}x \int_0^x \frac{\sin(q\,x)}{q\,x}\,x^2\,\mathrm{d}x = \frac{16\,\pi^2\sigma_0^2}{q^2}\int_0^x \sin(q\,x)\,\mathrm{d}x \int_0^x x\sin(q\,x)\,\mathrm{d}x\ . \tag{468}$$

The evaluation of the second integral is easily found to be

$$\int_0^x x\sin(q\,x)\,\mathrm{d}\,x = \frac{\sin(q\,x) - q\,x\cos(q\,x)}{q^2}\ ;$$

and our expression thus reduces to

$$\boldsymbol{Y} - \frac{16\,\pi^2\sigma_0^2}{q^4}\int_0^x \left[\sin^2(q\,x)\,\mathrm{d}x - \cos(q\,x)\sin(q\,x)\,q\,x\,\mathrm{d}x\right]\ .$$

The integral of the first term is found to be

$$\frac{q\,x - \sin(q\,x)\cos(q\,x)}{2\,q}\ ,$$

and that of the second

$$-\frac{x\sin^2(q\,x)}{2} + \frac{q\,x - \sin(q\,x)\cos(q\,x)}{4\,q}\ ;$$

so that the complete expression becomes

$$\boldsymbol{Y} = \frac{4\,\pi^2\sigma_0^2}{q^6}\left(3\left[q\,x - \sin(q\,x)\cos(q\,x)\right] - 2\,q\,x\sin^2(q\,x)\right)\ . \tag{469}$$

The potential upon itself of a homogeneous sphere was shown by HELMHOLTZ to be

$$V = \frac{16\,\pi^2\sigma_1^2 x^6}{15}\ ;$$

and the ratio of the two potentials thus becomes

$$\frac{\boldsymbol{Y}}{V} = \frac{15\,\sigma_0^2}{4\sigma_1^2\,(q\,x)^6}\left(3\left[q\,x - \sin(q\,x)\cos q\,x\right] - 2\,q\,x\sin^2(q\,x)\right)\ . \qquad [a'] \tag{470}$$

Using the constants for the Earth given in *A.N.*, 3992, this ratio on calculation proves to be

$$\frac{\boldsymbol{Y}}{V} = 1.104214\ ; \tag{471}$$

so that the effect of heterogeneity following LAPLACE's law is to increase the amount of heat developed by the hypothetical homogeneous Earth by a little over 10 per cent. Thus the total heat developed by the gravitational shrinkage

of the Earth from a state of infinite expansion to its present condition would raise an equal mass of water through 9954.365° C. The effect of heterogeneity is thus small in the case of the Earth, and naturally smaller still in the case of the other interior planets and satellites. In fact it is nearly insensible except in the case of *Venus*, but in the table we give the results as computed by the rigorous formula [a'] for each case*.

In the present state of science it is difficult to determine an appropriate value for the specific heat of the terrestrial globe as a whole. Surface rock gives a specific heat not far from 0.2, while most of the metals give smaller values with coefficients increasing with the temperature (cf. PROF. W. A. TILDEN, *Phil. Trans. Roy. Society*, 1904, p. 143). Under conditions existing in the Earth the average specific heat would probably lie between 0.3 and 1.0; perhaps 0.5 would be a rough approximation sufficient to afford us some idea of the temperature to which the nucleus of the Earth may have been raised in the condensation of the globe. It thus appears improbable that the heat of the Earth at any point has ever much exceeded 20,000° C. As the process of condensation was very slow, the surface layers, exposed to constant radiation into space, may never have attained any considerable elevation of temperature, and it is doubtful if the heat was ever such as to vaporize the surface material and cause the unclouded planet to appear self-luminous.

Nevertheless, even now the Earth's temperature increases downward at the rate of about 1° C. for 30m. of descent, and, at a depth of 32km., the effect of this primordial heat alone would probably cause the material to vaporize if the pressure were relieved so as to expose it directly to the air. Moreover, by considering the amount of heat developed in the crushing at the base of a column of granite 64km. high under its own gravity, it is easily calculated by the mechanical equivalent of heat with the known specific heat of granite, taken at 0.15, that a temperature of 1020° C. is produced; and this would not only melt the material, but reduce it to a state of flaming vapor. Hence it follows that the matter of the Earth at that depth is fluid, and instantly flows when the pressure is relieved.

Accordingly it appears that the Geological theories of the Earth's interior after all have a certain foundation; and the rigidity of the globe, found from the tidal researches of KELVIN and DARWIN and long held to prove the solidity, shows principally the effects of pressure in imparting rigidity to material which is essentially fluid (cf. *A.N.*, 3992; and also "The Physical Cause of the Earth's Rigidity,"

* In the case of masses but slightly heterogeneous this formula furnishes a very delicate criterion for the accuracy of the constant qx. In applying it to *Mars* it was found that the value of qx given for that planet in *A.N.*, 3992 is slightly erroneous. The correct value of qx for *Mars* is $\log(qx) = 0.1258642$, $qx = 76° \; 33' \; 26''.3$. This error does not seem to have vitiated any of the other data given in *A.N.*, 3992.

in *Nature*, April 13, 1905). For many years it has been somewhat customary to explain the rigidity of the Earth by an iron nucleus of great stiffness, but as iron and steel lose all rigidity at high temperatures, the observed rigidity of the Earth must depend on some cause other than the nature of the material composing the nucleus; and this can be nothing else than the pressure acting throughout the interior, which everywhere produces strains for surpassing the strength of any known substance. For this reason the rigidity of the nucleus exceeds that of any known material, but there is no good ground for thinking that the Earth has such a great preponderance of iron towards the centre as has been generally supposed, the increase of density being most naturally explained by the condensation of ordinary matter under pressure.

In the case of the Moon and other satellites the primordial heat is scarcely sufficient to produce fusion of rocks, unless we suppose their specific heats to be very small; in which case the temperature attained would be proportionally higher. However rapid the development of the Moon by gradual accretion, it is impossible to see how the surface temperature can have exceeded the melting point of rock, unless very considerable heat was derived from the primitive condition of the elements, by chemical transformation, as in radio-activity.* These considerations, in connection with the insignificant amounts of heat developed in the condensation of the other small masses, fix limits to the assignable temperatures of these bodies in past ages which seem likely to be useful in future cosmical investigations.

In the case of the major planets and the Sun heterogeneity following LAPLACE's law has a much more considerable effect, increasing the total heat in each case about 25 per cent. over that of the corresponding homogeneous globes; while heterogeneity following the law of a monatomic gas gives an effect which is much larger yet, or about 43 per cent. over that due to homogeneity.†

§ 209. *Precise Evaluation of the Moments of Momentum of the Solar System.*

In the theory of the solar system certain mechanical constants are of high importance, and as the preceding methods give the moments of inertia of the several bodies with great accuracy, it becomes advisable to determine also the moments of momentum, upon which the ultimate constitution of the system so essentially depends. If two bodies, such as the Sun and a planet, with masses M and m_i be revolving about their common centre of inertia in an elliptic orbit

* As the Moon is now shown to be a captured body this conclusion is natural. It seems certain that the Moon never had a temperature that would produce fusion of rock, for the reasons indicated above.

† The smaller value of about 29 per cent. found in the author's paper published by the Academy of Sciences of St. Louis (*Transactions*, Vol. X, No. 1, p. 19) was due partly to the method of quadrature there employed, but more especially to misinterpretation of LANE's result (similar to that made by LORD KELVIN, as already pointed out) and also to the inaccuracy of LANE's curves which give too great a density near the surface.

with semi-axis major a_i, eccentricity e_i, and angular velocity Ω_i, the moment of momentum of orbital motion becomes

$$O_i = M\left(\frac{m_i a_i}{M + m_i}\right)^2 \Omega_i \sqrt{1 - e_i^2} + m_i\left(\frac{M a_i}{M + m_i}\right)^2 \Omega_i \sqrt{1 - e_i^2} = \frac{M m_i}{M + m_i} a_i^2 \sqrt{1 - e_i^2}\, \Omega_i \,. \quad [\beta'] \ (472)$$

Applying this formula to the several planets by means of the data given in the paper on the invariable plane (cf. *A.N.*, 3923), we find the results given in the accompanying table.*

From the definitively adopted moments of inertia given in the table, it is easy to compare the moments of momentum of axial rotation of the several bodies by the familiar formula

$$\frac{C_i n_i}{C_3 n_3} = \frac{\frac{2}{5}\rho_i r_i^2 m_i}{\frac{2}{5}\rho_3 r_3^2 m_3}, \qquad [\gamma'] \ (473)$$

where

$$\rho_i = \frac{C_i}{\frac{2}{5} m_i r_i^2}$$

given in the table, and the symbols with the subscript 3 denote the quantities appropriate to the Earth without the Moon (cf. *A.N.*, 3992). The results are thus found in units of the Earth's moment of momentum of axial rotation. As the Sun's moment of momentum of axial rotation is directly compared to the moments of momentum of orbital motion of the several planets, it was thought advisable to give the ratio of moment of momentum of axial rotation to that of orbital motion in the case of each planet; and also the moments of momentum of orbital motion of the satellites about the several planets, where the masses of the satellites are sensible.

TABLE C. MOMENTS OF INERTIA AND OF MOMENTUM OF THE PLANETARY SYSTEM.†

Body	$\frac{C}{\frac{2}{5} M r^2}$ (Definitive Value)	log ι (ι = Radius of Inertia in km.)	M. of m. of Axial Rotation, that of the Earth being Unity	O_i M. of m. of Orbital Motion, that of the Earth and Moon being Unity	M. of m. of Orbital Motion, that of Sun's Axial Rotation being Unity	Ratio of m. of m. of Orbital Motion to that of Axial Motion
Sun	0.506208	[5.4958649]	88886522	19.32832	1.0000000	....
Mercury	0.9855	[3.1353775]	....	0.013463	0.00069654	....
Venus	0.8482	[3.5499577]	....	0.68507	0.035444	....
Earth	0.82803	[3.5642279]	1.0000000	1.00000	0.0517385	4598770 : 1
Mars	0.96315	[3.3199742]	0.034067	0.13076	0.00676526	17651760 : 1
Jupiter	0.506208	[4.4948605]	55711.06	714.8170	36.98288	59006 : 1
Saturn	0.506208	[4.4138808]	11125.25	289.6105	14.98374	119704 : 1
Uranus	0.506208	[3.9774975]	231.91	63.1958	3.26959	1253163 : 1
Neptune	0.506208	[3.9885120]	226.49	93.4060	4.83260	1897315 : 1
			$\Sigma = 88953817.744$	$\Sigma = 1182.186913$	$\Sigma = 61.1634542$	

* In his celebrated address on the "Sun's Heat" (*Popular Lectures and Addresses*, Vol. I, p. 420) LORD KELVIN says: "The moment of momentum of the whole solar system is about eighteen times that of the Sun's rotation; seventeen-eighteenths of this being *Jupiter's* and one-eighteenth the Sun's, the other bodies being not worth taking into account in the reckoning of moment of momentum." If the Sun be taken as homogeneous this statement is fairly accurate as respects *Jupiter*, but in assigning such small importance to the other major planets the illustrious mathematician was evidently misled by some numerical error.

† Since the substance of this chapter was published in *A.N.*, 4053, LORD KELVIN has used the data here derived, in a paper "On the Formation of Concrete Matter from Atomic Origins," *Philosophical Magazine*, April, 1908, p. 407; also in another paper on "The Problem of a Spherical Gaseous Nebula," *Proc. Roy. Soc. of Edinburgh*, Vol. XXVIII, Part IV, No. 16, 1908, and a similar article in *Nature* of Feb. 14, 1907. Note added May 21, 1910.

MOMENTS OF MOMENTUM OF SATELLITE SYSTEMS.

Planet	Satellite	Distance in km.	Mass of Satellite (that of Planet being Unity)	M. of m. of Satellite's Orbital Motion, that of Planet's Axial Rotation being Unity
		km		
Earth	*Moon*	384418	1:81.45	4.859716
Mars	*Phobos*	9377	1:176583000	0.000000367
	Deimos	23475	1:594823000	0.000000261
Jupiter	Satellite V	180936	1:327115000	0.000000085
	" I	421632	1:35322	0.0012036
	" II	670859	1:43048	0.0012456
	" III	1070067	1:12309	0.005501
	" IV	1882150	1:46538	0.001930
Saturn	Ring System	103750	1:11209	0.000227
	Mimas	185465	1:13610000	0.0000017
	Enceladus	237942	1:4000000	0.00000655
	Tethys	294555	1:907600	0.0000321
	Dione	377258	1:536000	0.0000615
	Rhea	526847	1:250000	0.0001558
	Titan	1221340	1:4700	0.012618
	Hyperion	1479622	1:18800000	0.0000035
	Iapetus	3559253	1:100000	0.0010124
	Phœbe	12886600	1:18000000	−0.00001026
Uranus	*Ariel*	191312	1:83058	0.000817
	Umbriel	266526	1:155830	0.000514
	Titania	437174	1:36830	0.002785
	Oberon	584625	1:41705	0.002845
Neptune	Satellite	355518	1:4114	0.0295067

§ 210. *Mechanical Equivalent of the Heat in a Cubic Metre of Solar Matter, and Mean Velocity of Translation of the Molecules.*

LANE shows that the so-called atmospheric subtangent, referred to the force of gravity at the Earth's surface, or height of column of homogeneous gas whose terrestrial gravitating force would equal its elasticity, is given by the expression

$$ht = \frac{k-1}{k}\,\frac{M}{m_3}\,\frac{r^2}{R}\,\frac{x'}{\mu'}\left(\frac{\sigma}{\sigma_0}\right)^{k-1}, \qquad [16]\ (474)$$

where r is the Earth's mean radius, R that of the Sun, M the Sun's mass and m_3 that of the Earth. The function $\left(\frac{\sigma}{\sigma_0}\right)^{k-1}$ has already been calculated for various parts of the Sun's radius, so that the corresponding subtangent is easily found.

The mechanical equivalent of the heat in a mass σ of a cubic metre volume is also shown to be

$$W_0 = 1000\,\frac{k}{k-1}\,\sigma\,.\,ht\,, \qquad [19]\ (475)$$

where the metre kilogram is the unit.

From LANE's equation (20) it follows that when the metre is the unit of length the mean velocity of translation of the molecules is

$$\bar{V} = 4.4277\sqrt{\tfrac{8}{3}ht}\,. \tag{476}$$

The results of these calculations are given in the following table. The enormous velocities of the molecules in the Sun afford impressive illustrations of the violence of the collisions to which they are subjected; and any one can form his own impression as to the probability of the dissociation of the atoms under such conditions.

The Sun's radiation per minute on a square metre of surface at the mean distance of the Earth has been found by measurement to be about 27.2 calories, or very nearly the value reached by FORBES (28.5) in 1842. At the Sun's surface this corresponds to 1,255,675 calories per square metre per minute, equivalent to the heat required to melt a layer of ice 15.84 metres thick per minute, or 532,406,000 metre kilograms. This enormous radiation from a single square metre of the Sun's surface is mechanically capable of raising a metric ton upon the Earth's surface each minute through a vertical height of 532.4 kilometres. With such an outflow of radiant energy is it any wonder that the particles of the corona are projected to great heights and there suspended under the pressure of the Sun's radiation?

TABLE D. MECHANICAL EQUIVALENT OF HEAT OF SOLAR MATTER, AND MEAN VELOCITIES OF MOLECULES IN SUN AND MAJOR PLANETS.

R Radius	*Sun*			*Jupiter*	*Saturn*	*Uranus*	*Neptune*
	log (ht) = Atmos. Subtangent in Metres	W_0 Mechanical Equivalent of Cubic Metre of Gas in Metre Kilograms	$\bar{V}$ Mean Molecular velocity in km	$\bar{V}$	$\bar{V}$	$\bar{V}$	$\bar{V}$
			km	km	km	km	km
1.00	6.4704292	95.27	9.320	0.912	0.547	0.355	0.380
0.99	7.8947630	1073589	48.040	4.700	2.822	1.828	1.960
0.98	8.1999423	6219941	68.264	6.678	4.010	2.598	2.786
0.97	8.3804465	17581130	84.032	8.220	4.936	3.198	3.429
0.96	8.5098288	37051220	97.530	9.541	5.729	3.711	3.980
0.95	8.6111781	66355330	109.601	10.722	6.438	4.171	4.473
0.94	8.6948015	107382700	120.677	11.805	7.089	4.592	4.925
0.93	8.7661646	161935800	131.010	12.816	7.696	4.985	5.346
0.92	8.8285876	231952900	140.772	13.771	8.269	5.357	5.745
0.91	8.8841238	319330200	150.067	14.680	8.815	5.711	6.124
0.90	8.9342511	426148000	158.983	15.552	9.340	6.050	6.488
0.80	9.2775458	3074645000	236.046	23.093	13.866	8.983	10.808
0.70	9.4914861	10535250000	301.972	29.541	17.739	11.492	12.323
0.60	9.6478146	25910024000	361.519	35.366	21.237	13.758	14.753
0.50	9.7683803	51865820000	415.350	40.632	24.399	15.807	16.950
0.40	9.8615928	88697715000	462.402	45.235	27.163	17.597	18.870
0.30	9.9317302	132817700000	501.290	49.039	29.447	19.077	20.457
0.20	9.9808626	176233000000	530.463	51.893	31.161	20.187	21.647
0.10	10.0100267	208447800000	548.576	53.665	32.225	20.876	22.387
0.00	10.0197017	220386350000	554.721	54.266	32.586	21.111	22.637

§ 211. *Internal Conditions of the Major Planets.*

The major planets, especially *Jupiter* and *Saturn*, are still in an early stage of development, and obviously gaseous, so that most of the preceding results are immediately applicable to their internal conditions. *Uranus* and *Neptune* are either mainly gaseous, or have just passed beyond that stage, and begun to consolidate externally; while internally they still have high temperatures.

The gaseous theory, with $k = 1\frac{2}{3}$, will therefore apply with moderate accuracy to all of these bodies. If we consider an element of the gas at the centre of the Sun, and another at the centre of one of the major planets, as *Jupiter*, it is evident that whatever be the conditions so long as the matter is gaseous the formulae

$$pv = RT \qquad p'v' = RT'$$

will hold true.

Now we have
$$\frac{v}{v'} = \frac{1}{\sigma} : \frac{1}{\sigma'},$$

and hence we get from the last equations

$$\frac{p}{p'} = \frac{\sigma T}{\sigma' T'}. \tag{477}$$

On the theory of a monatomic gas the pressures at the centres of the Sun and *Jupiter* are known, and thus we find $T' = 478{,}529°$ C., which is the temperature at the centre of *Jupiter* on the theory of a monatomic constitution.*

Applying the same method to the other major planets, we find their central temperatures, and may then calculate the distribution of internal heat by the formula

$$T = T'\left(\frac{\sigma}{\sigma'}\right)^{k-1} \qquad k = 1\frac{2}{3}. \tag{478}$$

The results of these calculations are given in the following table:

TABLE E. PRESSURES IN THE MAJOR PLANETS CALCULATED ON THE MONATOMIC THEORY.†

R Radius	logarithm of $\left(\frac{\sigma}{\sigma_0}\right)^{1\frac{2}{3}}$	*Jupiter* ϖ in Atmospheres	*Saturn* ϖ in Atmospheres	*Uranus* ϖ in Atmospheres	*Neptune* ϖ in Atmospheres
1.00	....	....	....	....	....
0.99	4.6876531 − 10	503	96	125	157
0.98	5.4506015 − 10	2918	556	722	910
0.97	5.9018620 − 10	8246	1572	2040	2574
0.96	6.2253176 − 10	17367	3308	4296	5420
0.95	6.4786910 − 10	31124	5929	7700	9713
0.94	6.6877495 − 10	50368	9595	12461	15719
0.93	6.8661574 − 10	75957	14469	18791	23705
0.92	7.0222149 − 10	108799	20725	26916	33954
0.91	7.1610551 − 10	149784	28532	37056	46745
0.90	7.2863735 − 10	199886	38077	49451	62381
0.80	8.1446101 − 10	1442153	275356	356787	450078
0.70	8.6794610 − 10	4941631	941337	1222533	1542196
0.60	9.0702821 − 10	12153220	2315083	3006640	3792806
0.50	9.3716965 − 10	24327970	4634265	6018605	7592330
0.40	9.6047276 − 10	41604170	7925235	10292650	12983930
0.30	9.7800711 − 10	62298900	11867400	15412410	19442380
0.20	9.9029024 − 10	82663100	15746600	20450400	25797700
0.10	9.9758125 − 10	97773660	18625030	24188660	30513420
0.00	0.0000000	103373500	19691755	25574040	32261040

* In this calculation we take the mean value for the temperature at the Sun's centre, or $T = 50{,}000{,}000°$ C.

† A review of the author's researches on the physical constitution of the Sun and planets was prepared in 1908 for the Memoirs of the Italian Spectroscopic Society at the request of PROFESSOR RICCO, of Catania (cf. *Memorie della Società degli Spettroscopisti italiani*, Vol. XXXVII, anno, 1908). Some improvements in the tables there noted are included here. Note added May 21, 1910.

TABLE F. TEMPERATURES AND DENSITIES OF THE MAJOR PLANETS CALCULATED BY THE MONATOMIC THEORY.

R	*Jupiter*			*Saturn*			*Uranus*			*Neptune*		
	σ Water = 1	σ Air = 1	T	σ Water = 1	σ Air = 1	T	σ Water = 1	σ Air = 1	T	σ Water = 1	σ Air = 1	T
			°C			°C			°C			°C
1.00	0.0000121	0.00937	62.7	0.0000064	0.00493	22.7	0.0000199	0.01538	...	0.0000218	0.01684	.....
0.99	0.005261	4.0683	3589*	0.002767	2.2396	1300	0.0086126	6.6600	543	0.009431	7.2928	625
0.98	0.015095	11.673	7247	0.007939	6.1391	2626	0.024712	19.109	1096	0.027060	20.925	1263
0.97	0.028158	21.774	10981	0.014809	11.451	3979	0.046096	35.640	1661	0.050476	39.032	1913
0.96	0.044023	34.042	14792	0.023153	17.903	5360	0.072067	55.728	2237	0.078915	61.023	2577
0.95	0.062474	48.310	18680	0.032857	25.407	6769	0.10227	79.085	2825	0.111990	86.600	3255
0.94	0.083394	64.487	22647	0.04386	33.915	8206	0.13652	105.57	3425	0.14949	115.60	3946
0.93	0.106704	82.512	26691	0.056118	43.395	9672	0.174678	135.08	4037	0.19128	147.91	4651
0.92	0.132378	102.37	30817	0.069620	53.836	11167	0.21671	167.57	4661	0.23730	183.50	5370
0.91	0.160368	124.01	35021	0.084342	65.220	12690	0.26253	203.01	5297	0.28746	222.31	6103
0.90	0.19068	147.45	39306	0.10028	77.548	14243	0.31215	241.38	5945	0.34181	264.32	6849
0.80	0.624	....	86647	0.328	......	31398	1.02167	....	13105	1.11875	....	15098
0.70	1.307	....	141805	0.687	......	51386	2.1390	....	21448	2.3423	....	24710
0.60	2.242	....	203245	1.179	......	73650	3.670	....	30741	4.0192	....	35334
0.50	3.400	....	268279	1.788	......	97216	5.566	....	40576	6.095	....	46748
0.40	4.692	....	332506	2.467	......	120490	7.680	....	50291	8.410	....	57940
0.30	5.978	....	390784	3.144	......	141608	9.786	....	59105	10.715	....	68095
0.20	7.083	....	437592	3.725	......	158569	11.595	....	66185	12.697	....	76251
0.10	7.834	....	467986	4.120	......	169584	12.824	....	70782	14.043	....	81548
0.00	8.100	....	478529	4.260	......	173404	13.260	....	72343	14.519	....	83193

It will be seen that the pressures throughout the interior of the major planets are several times larger on the theory of a monatomic gas than they were on LAPLACE's law (Table II, § 198). The temperatures follow from the pressures and densities, and are such as to appear highly probable. On the gaseous theory, with $k = 1.41$, the central densities of *Jupiter*, *Saturn*, *Uranus* and *Neptune* would be 31.05, 16.33, 50.83, 55.66, respectively. LAPLACE's law as treated in *A.N.*, 3992, gave 4.444, 2.336, 7.270, 7.962. The values resulting from theory of a monatomic gas, namely, 8.10, 4.26, 13.26, 14.52, seems therefore inherently probable, since the true density will certainly lie between the values given by LAPLACE's law on the one hand, and the gaseous theory, with $k = 1.41$, on the other. If it be imagined that *Uranus* and *Neptune* are too dense to admit of the application† of the gaseous theory, with $k = 1\frac{2}{3}$, it may yet be supposed that the results deduced on this basis will give a close approximation to the truth, and are sufficient for all purposes in the present state of science.

The densities, pressures and temperatures of these bodies are illustrated by the curves in Plate XXVII, in which the density curves resulting from LAPLACE's law

* Such a high temperature only 694 km. below the surface supports the view that the Great Red Spot and other reddish colored markings seen on *Jupiter* furnish indications of the planet's internal heat. If the clouds were largely dissolved over such areas, the internal glow might show through the semi-transparent atmosphere. Layers with corresponding temperatures in *Saturn* are much more deep seated. This accords well with the eruptive phenomena noted on the two planets, disturbances in the interior of *Saturn* seldom reaching the surface, while in *Jupiter* they are continually producing conspicuous external effects.

† Circulation must be greatly impeded when the density is so large, as in *Uranus* and *Neptune*, but the internal distribution of density and temperature probably would undergo but little change on this account.

are indicated by the pointed lines. The surface temperatures calculated in the above table for the major planets are seen to be quite small, so that the surfaces of the planets under these conditions would probably be frozen solid, whereas they all appear to be gaseous. We have already pointed out that a discontinuity would arise in the outer layers, where the gas ceases to be monatomic, by the development of compound molecules; and we have shown that in such compound gas the temperature near the surface would fall much less rapidly than in monatomic gas. It thus appears probable that the law of monatomic gas holds up to within less than a tenth of the radius of the surface, after which compounds develop; and the result is surface temperatures probably lying between 300° and 800° C. absolute. These limits cannot be fixed very accurately, and we can only be sure that the surface is neither frozen nor self-luminous; but observation may sometimes throw additional light upon this subject.

Apparently the only escape from these conclusions is to overthrow the premises on which they are based. Some have attempted to do this by claiming that the Sun is not gaseous throughout its mass. But as the photosphere is wholly gaseous in all the parts exposed to our view, and the temperature is shown to increase with enormous rapidity as we descend into the Sun's globe, it seems to be impossible to avoid the conclusion that the mass is purely gaseous throughout. Because the gases obey the laws of BOYLE and GAY-LUSSAC only within narrow ranges of temperature and pressure under the feeble forces available in laboratory experiments, it is held that a similar deviation from these laws would develop in the mass of the Sun. If the temperature of millions of degrees reduces every element to the monatomic state, it would seem that the great pressure ought to secure a very perfect working of the gaseous laws, except perhaps as respects rapid and perfect circulation which would be somewhat hindered by the great pressure acting throughout the interior of the mass. The heat is so great that there can be no possible liquefaction or solidification, so that purely gaseous matter must make up the whole body of the Sun, and it would seem that the density should follow the law for a monatomic gas.

In the same way the gaseous constitution of *Jupiter* and *Saturn* seems natural and inevitable. Following ancient traditions long since abandoned some persons have supposed that the great planets were formerly hot and are now cooling down.* To show the untenability of this view we shall consider the present state of *Jupiter* and *Saturn*.

* PROFESSOR POYNTING is led in this way to suggest that *Saturn's* rings may have arisen from particles driven away by the pressure of the planet's radiation when the temperature was higher than at present. The fatal objection to such a view is that nearly all the heat is developed at a late stage of the contraction, when the force of gravity is large; and thus *Saturn's* maximum heat lies in the future, not in the past.

(1) Let us first suppose that *Saturn* has passed its maximum temperature long ago and that cooling is well advanced. Then as *Jupiter* is already about twice as dense as *Saturn*, it must have passed its maximum temperature still further, and be still further advanced in its cooling. The planet which is hottest will evidently exhibit it externally by markings denoting ebullition from within. Now it appears by observation that *Jupiter* exhibits vast multitudes of spots constantly changing their appearances, while *Saturn* scarcely exhibits a spot once in a quarter of a century. Thus *Saturn* is almost altogether quiescent, while *Jupiter* is violently agitated from within, indicating enormous internal heat. The facts do not therefore accord with our first hypothesis.

(2) Let us next assume that both planets are rising in temperature, and that *Jupiter* is the most advanced in its development. Then *Jupiter* should be most agitated from within, while *Saturn* should exhibit a quiescent disc, which has not yet begun its ebullition, if we may use that concise expression. This accords well with the facts of observation, and appears to be the true hypothesis. If we observe two kettles on a fire, one violently bubbling, while the other scarcely simmers, we conclude that the former is the hotter, just as in the case of *Jupiter* and *Saturn*.

It would seem that the only alternative to this conclusion is to hold that as *Jupiter* is much larger than *Saturn* it became much more violently heated, when at its maximum temperature; and although both are cooling, *Saturn* has already cooled more than *Jupiter*. Any one can judge for himself which of these suppositions is likely to be correct. According to the monatomic law each mass still retains one half of its primordial heat; and this inference accords well with known facts, and with the analogies which follow from the theory of the Sun.

§ 212. *On the Theoretical Accumulation of Heat and the Rise of Temperature in the Sun and Stars.*

While it has long been recognized that condensing masses rise in temperature and afterwards cool down, when a considerable average density has been attained, the accumulation of heat within the Sun and stars appear to have been regarded as arbitrary and indeterminate, rather than as following exact laws, which would enable the investigator to determine from the known conditions the percentage of energy radiated away to that stored up in the mass for raising the temperature. At least if such views recognizing definite laws of heat-accumulation are current, they find little or no expression in contemporaneous scientific literature. It seems appropriate therefore to consider in some detail the relations which RITTER has

shown to exist between the heat out-put and work of gravitation, and to give an extension of his work which seems likely to accord more closely with the actual conditions of the Sun and stars.

It will be proved, for example, that all stars which are still in the state of monatomic gas have $50\%+\gamma$ (where γ is probably less than 10%) of all the heat developed in their condensation from eternity still stored up in their globes. It is this accumulation of heat which gives luminosity to the visible universe; and the brilliant light of the stars is the best proof of the generality of the law. Moreover, it will be shown not only that the temperature rises steadily according to the law

$$T = \frac{K}{R} \tag{479}$$

(cf. *A.N.*, 3586; *A. J.*, 455; *Transactions of the Academy of Sciences of St. Louis*, Vol. X, No. 1), for a gaseous star, but also that a fall in temperature cannot take place until the work of gravitation is so resisted by molecular repulsion that the radiation exceeds its usual intensity by more than 100 per cent., and thus neutralizes the ordinary accumulation of heat, and permits the beginning of secular cooling.*

But as cooling is accompanied by contraction producing more heat, and a vast amount of heat is produced for a small shrinkage when the mass is much condensed, owing to the intensity of gravity at that stage of contraction, actual cooling must come about with extreme slowness. Encrustation probably arises from the impeded circulation due to the increase of density and pressure as the mass passes into the last stages of contraction.

The conditions thus required for the fall of temperature would imply that the cooling mass has long passed the gaseous stage; and as the stars in the main are gaseous, there would seem to be a strong probability that most of these bodies are rising in temperature.† MR. W. E. WILSON has recently pointed out, in the learned and original paper (cf. *Monthly Notices, Roy. Astron. Soc.*, Jan., 1905) some excellent reasons for thinking that but very few stars are at the highest temperature, so that conclusions on this point must be drawn with great care.

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* Will any one believe that the inactive radiation of *Jupiter* and *Saturn* exceeds the work of gravitation upon their masses by more than 100 per cent.?

† If we suppose, in accordance with STEFAN'S law, that the radiation is proportional to the fourth power of the absolute temperature, which is itself inversely as the radius, and remember that the radiation is also proportional to the area of the Sun's disc; we shall have for the radiations Q_0 and Q at the epochs t_0 and t

$$\frac{Q}{Q_0} = \frac{r^2}{r_0^2} \cdot \frac{r_0^4}{r^4} = \frac{r_0^2}{r^2}.$$

Thus the warming power of the Sun increases from age to age in the inverse ratio of the square of its diameter, so long as the mass is gaseous; and this rise in the radiation seems to indicate that the Sun gave less heat in past ages, and not more, as concluded in *A.J.*, 455, from NEWTON'S law of cooling.

RITTER has given two independent proofs of the accumulation of heat in the stars, both of which, according to the recognized laws of Thermodynamics, we find to be entirely rigorous. The first is based upon familiar thermodynamical methods, and leads to the following equation between the heat-output and the work of gravitation:

$$\frac{Q}{W} = \frac{k - \epsilon}{k - 1}, \qquad [\delta'] \ (480)$$

where ϵ is the exponent $\frac{4}{3}$, as in the adiabatic formula $p\,v^{4/3} = C$. If in the expression $[\delta']$ we put with RITTER $k = 1.41$, the ratio turns out 0.187; so that a body made up of gases such as Hydrogen, Oxygen, Nitrogen, and common air, would radiate only 18.7 per cent. of its heat, while 81.3 per cent. would go to raising the temperature. If however the gas be monatomic, $k = 1\frac{2}{3}$, the ratio

$$\frac{Q}{W} = 0.500 \; ,$$

so that exactly one-half of the energy is radiated away, and the other half goes to elevating the temperature.

On account of the importance of the above theorem, it seems desirable to give a generalized proof from another point of view. The work done on a unit mass in falling from infinity to the surface of the Earth, of mass m_3 and radius r, is

$$W = -\int_{\infty}^{r} \frac{m_3}{r^2}\,dr = \frac{m_3}{r} \; ;$$

which is the ordinary expression for the potential. This gives the relation

$$\frac{W}{r} = \frac{m_3}{r^2},$$

and for the whole Earth the work done would be

$$W = m_3 g r \; ; \qquad [\epsilon'] \ (481)$$

so that if gravity were constant over an interval equal to the radius, the work on the mass m_3 from infinity would be equal to $m_3 \,.\, g \,.\, r$, or the work done on the mass in falling through a space equal to the radius.

In descending within the Earth, the concentric layers exert no attraction on masses within; and to get the total potential of the Earth upon itself we have only to integrate the expression

$$Y = \int_{\rho=0}^{\rho=r} \mathrm{d}M g \rho \,, \qquad [\zeta'] \ (482)$$

where ρ is the distance from the centre.

But $\mathrm{d}M = 4\pi\rho^2\sigma\mathrm{d}\rho$, and hence

$$Y = 4\pi \int_{\rho=0}^{\rho=r} \sigma g \rho^3 \mathrm{d}\rho \,. \qquad [\eta'] \ (483)$$

Now $\sigma = \varphi(\rho)$, and the relation of g to ρ within the sphere is known, and the law of the pressure due to any element is obviously

$$\mathrm{d}p = - \sigma g \mathrm{d}\sigma \,; \qquad [\theta'] \ (484)$$

so that we get

$$Y = - 4\pi \int \rho^3 \mathrm{d}p \,. \qquad [\iota'] \ (485)$$

Integrating by parts with respect to the two variables, observing that at the surface $p = 0$, and at the centre $\rho = 0$, we find

$$Y = - 4\pi \left\{ [\rho^3 p]_{\rho=0}^{\rho=r} - 3\int_{\rho=0}^{\rho=r} p\rho^2 \mathrm{d}\rho \right\} = 12\pi \int_{\rho=0}^{\rho=r} p\rho^2 \mathrm{d}\rho \,. \qquad [\kappa'] \ (486)$$

The internal heat which the sphere of gas would possess, expressed in metre kilograms may be calculated from the equation

$$U = \frac{c_v}{A} \int \mathrm{d}M g T = \frac{4\pi c_v}{A} \int_{\rho=0}^{\rho=r} \sigma g T \rho^2 \mathrm{d}\rho \,, \qquad [\lambda'] \ (487)$$

in which T is the absolute temperature at the distance ρ from the centre. By the laws of Boyle and Gay-Lussac, $pv = RT$, $v = \frac{1}{\sigma g}$, so that we have $\sigma g T = \frac{p}{R}$; and

$$U = \frac{4\pi c_v}{AR} \int_{\rho=0}^{\rho=r} p \rho^2 \mathrm{d}\rho \,. \qquad [\mu'] \ (488)$$

But by the mechanical theory of heat

$$AR = c_p - c_v = (k - 1)\, c_v\,,$$

and hence

$$U = \frac{4\pi}{k-1}\int_{\rho=0}^{\rho=r} p\,\rho^2 d\rho\,. \qquad [\nu'] \ (489)$$

Dividing equation $[\kappa']$ by this expression, we have the relation

$$\frac{Y}{U} = 3\,(k-1)\,. \qquad [\xi'] \ (490)$$

When $k = 1\tfrac{2}{3}$, as in a monatomic gas, and $k = 1.41$, as in common air, respectively, this gives

$$\frac{U}{Y} = \frac{1}{3\,(k-1)} = \left\{ \begin{matrix} 0.500, \ k = 1\tfrac{2}{3} \\ 0.813, \ k = 1.41 \end{matrix} \right\}. \qquad [o'] \ (491)$$

It follows therefore that the internal heat possessed by the sphere of monatomic gas which has condensed under gravity is exactly one-half of the potential of the sphere upon itself, or of the total heat developed from eternity.* If the gas be of such a nature that $k = 1.41$, as in common air, the ratio rises to 81.3 per cent. of the potential of the sphere upon itself. If a body in its early stages of condensation were made up of ordinary gases, $k = 1.41$, and in its later stages of high temperature the gases should become monatomic, through dissociation, it would follow that in the earlier stages, over a considerable period, more than one-half of the energy of condensation would be stored up, while in the later stages exactly one-half of the energy would be conserved; so that at a later period the total energy stored up in the condensing mass would be

$$\sum = 0.500 + \gamma' = 0.500 + 0.313\,\frac{T_2 - T_1}{T_p - T_1}\,, \qquad [\pi'] \ (492)$$

where T_1, denotes the epoch at which the mass began as a compound gas, and T_2 that at which it finally became a monatomic gas through dissociation, and T_p the present epoch. If for considerable periods the mass were made up of compound and monatomic gases combined, average effects would have to be taken; and the value of γ' may be judged by the relation between $\tau = T_p - T_1$; the whole gaseous period, both compound and monatomic, and $\tau_0 = T_2 - T_1$, the period of the compound gas alone. For large masses τ_0 would appear to

* The formula $T = \frac{K}{R}$ exhibits the rise of temperature due to condensation, but does not show the relation between the energy radiated away and that stored up.

be small in comparison with τ, while for small masses the reverse probably would be true.*

Concluding Considerations on Cosmical Evolution.

It only remains to call attention to what appears to be the order of development of planetary density, as exemplified in the laws found to hold for some of the individual planets. On the usual hypothesis that bodies condense from nebulae, it will be seen that when made up of compound gases such as Air, Oxygen, Nitrogen, Hydrogen, the central density is about twenty-three times the mean. As the mass develops high temperature by condensation under gravity, the outer parts of the shrinking nebula become relatively denser, and when at length the gas is reduced to the monatomic condition, the central density is only six times the mean density. The body continues to follow this law till the mean density is considerable, and the temperature begins to fall, as contraction gradually ceases, owing to the increase of molecular resistance. The density thenceforth appears to adapt itself to the form of curve given by LAPLACE's law in the case of the Earth (cf. *A.N.*, 3992). This last change is effected chiefly by the increase of the surface density, as the process of encrustation advances.

If in accordance with these views the heavenly bodies develop from nebulae† made up at first of compound gases, afterwards becoming monatomic when high temperature develops, it follows that condensing masses are always heterogeneous to a very considerable degree. The condensation of the matter towards the centre is diminished by the high temperature developed, when the mass becomes monatomic, but the central density still remains six times the mean.

This seems to indicate that the figures of equilibrium of rotating masses of fluid investigated by DARWIN and POINCARÉ on the hypothesis of homogeneity would give only a rough approximation to the conditions arising in nature. Yet if the condensing masses are monatomic through most of their life-history, the hypothesis of homogeneity is much more accurate than it would be if the masses were made up of compound gases, such that the central density is twenty-three

* In § 204 we have effected a rigorous integration of the total amount of heat stored up in the Sun's globe; and on what appears to be the most probable hypothesis that could be adopted, have found that it actually exceeds one-half of all the heat developed in the condensation of the Sun from eternity. Accordingly the proof of the theoretical law of heat accumulation within the Sun and stars is supplemented by a concrete illustration of its practical operation, obtained through the integration of the amount of heat actually stored up in the Sun's mass, and thus made available for radiation through future ages.

When we contemplate the myriads of stars which stud our firmament it would seem natural to expect that various stages of development would be represented; yet as nearly all are shown by their spectra to be of the Sirian and Solar types, and thus gaseous masses at enormous temperatures, the law that the heat within their flaming globes is 50 per cent. $+\gamma$ of all the energy developed in their condensation from eternity would seem to be valid for the great multitude of stars composing the sidereal universe.

† The nebulae are only partially gaseous, being largely made up of solid non-luminous matter, of such vast extent and extreme tenuity as to be devoid of hydrostatic pressure. Note added May 21, 1910.

times the mean, as has been assumed by LANE, KELVIN, RITTER, PERRY, and other previous investigators.

The present inquiry therefore is not without interest in connection with the dynamical investigations attempted by DARWIN and POINCARÉ, but it indicates how far we are from a complete solution of the problems of stellar evolution.

In general the lines of thought carried out in this chapter are based very largely upon the analogies of nature as made known by the experiments of the past two centuries. Under the very narrow ranges of temperature hitherto accessible to the experimenter in the laboratory this seems at present the only course open to the investigator. For the specific heats and other properties of gases become difficult to determine when the temperature is very high, on account of the expansion of the cubical contents of the containing apparatus, and other obstacles, such as leakage, absorption and occlusion, and lastly the difficulty of measuring high temperatures with great accuracy.

The analogies used, however, seem to be of the utmost generality, and it is scarcely conceivable that they can fail to point the way to truth. On this point SIR ISAAC NEWTON lays down the rule that we are not to "recede from the analogies of nature, which uses to be simple and always consonant to itself" (*Principia*, Lib., III, Reg. Phil. III).

The problem under consideration has been described as a branch of Transcendental Physics, in which for the present judgment and sound analogies rather than actual experiments must be our chief guides. Laboratory experiments extending our knowledge along some of the lines here suggested is an urgent desideratum of science. Perhaps in no other way could the importance of such researches be so forcibly impressed upon the investigator as by a realization of the deficiency of our knowledge of the state of the gases in the Sun.

If when LANE wrote, forty-one years ago, not one single monatomic gas was known, yet he ventured to consider this possible theory, and in the mean time six such gases have been discovered, or on the average one for every seven years that have elapsed, one need not be without hope of future discovery along similar lines; nor will the effort here put forth be considered premature except by those who are without hope of winning truth.

It is difficult to account for the fact that few other writers have touched upon the monatomic theory since LANE's paper was prepared in 1869. This theory is the only one admitting of definite mathematical treatment, for it makes k exactly $1\frac{2}{3}$, and thus defines the internal laws of the Sun with all the rigor of mathematical analysis. Among the other possible values of k it is difficult to assign preference

to any one value over another; and as the smaller values are known from experiments to be inadmissible, owing to dissociation at comparatively low temperatures, $1\frac{2}{3}$ is the only value which can be logically adopted for the conditions existing in the Sun.

Since the results deduced upon this theory accord with LORD KELVIN's prediction regarding the central density, by fixing its value in the region heretofore considered inherently probable on general grounds, between the limits set by LAPLACE's law on the one hand and by the gaseous theory with $k = 1.41$ on the other (cf. *A.N.*, 3992), while at the same time such a distribution of density incontestably follows from the dissociative effects of the enormous temperatures known to exist in the Sun's globe, it seems that the monatomic theory should commend itself to investigators as representing the Sun's actual constitution.

Concluding Remarks on the Dissociation of the Elements.

Since this chapter was finished June 3, 1905, the doctrine of the dissociation of bodies and the transmutation of the elements has been strengthened by the penetrating presidential address of PROFESSOR G. H. DARWIN to the British Association for the Advancement of Science at Cape Town, Aug. 15, and by important experiments at Cambridge, reported by MR. R. J. STRUTT in a letter to the editor of *Nature*, of Aug. 17, 1905. MR. STRUTT points out that it is now proved experimentally that Radium is evolved from Uranium. As SIR WM. RAMSAY found Helium to be evolved from Radium, we seem to have at least three elements showing the property of transmutation, Uranium passing into Radium, and Radium eventually evolving Helium. PROFESSOR DARWIN's address at Cape Town dealt with the general problem of the stability of matter, and the significance of THOMSON's remarkable electronic researches at Cambridge (cf. *Nature*, Aug. 17, 1905).

During the past few years the subject of the transmutation of the elements has engaged the attention of many enthusiastic investigators; and whilst some definite results have been achieved, the progress on the whole has been slow. It was for a time felt that the theory of electrons might lead to the ultimate overthrow of the theory of atoms which was put forth by DALTON in 1805 and now lies at the basis of Chemistry; but more recently it seems to be admitted that it has confirmed and strengthened the atomic theory rather than weakened it. This is the conclusion announced by PROFESSOR E. RUTHERFORD, of Manchester, in his Presidental address to the Mathematical and Physical Section of the British Association at Winnipeg (cf. *Nature*, Aug. 26, 1909).

In his recent presidential address to the Chemical Society of London, Sir Wm. Ramsay expounds the view that the generic difference between elements is due to their gain or loss of electrons; not of such supplementary electrons as convert an element into an ion, but of electrons more closely associated with the atom. The gain or loss of electrons causes the atom to be converted into an atom of some other element. It is held that change from one elemental form of matter into another would be accompanied by unusually large gain or loss of energy; for it is known that the "degradation" of Radium is accompanied by the loss of an enormous amount of energy from a very small mass.

According to Sir Wm. Ramsay, the numbers representing the atomic weights can be represented on the hypothesis that the addition or subtraction of definite groups of electrons is the cause of their divergence from a perfectly regular series. This celebrated investigator has continued his studies on the action of Radium emanation on solutions of copper sulphate and nitrate in glass vessels. They showed traces of Lithium, whereas without the emanation they did not. Silver nitrate gave no corresponding result. A solution of 270gm. of pure Thorium nitrate was left in a flask for three years to test the formation of Helium from Thorium. No satisfactory proof of Helium could be obtained, he said, but several c. cm. of CO_2 were produced, which cannot be accounted for unless evolved from Thorium. With the aid of Radium emanation CO_2 was obtained from Thorium nitrate, Zirconium nitrate and hydrosilico fluoric acid.

Sir Wm. Ramsay concludes that Carbon is therefore most likely a degradation product of Thorium, Zirconium and Silicon. Similar experiments with Lead and Bismuth gave negative or doubtful results. These latest researches on the origin of Carbon are of great interest, because it has long been suspected by Chemists that Carbon was not an ultimate element (cf. *Journal of Chemical Society*, Vol. 95, pp. 624-637, April, 1909; and *Science Abstracts*, Section A, Physics, No. 141, Sept., 1909).

CHAPTER XVIII.

Researches on the Rigidity of the Heavenly Bodies.*

§ 213. *Introductory Remarks on the Rigidity of the Earth.*

Many years have elapsed since Lord Kelvin introduced and Professor Sir G. H. Darwin extended a method for determining the rigidity of the Earth from a comparison of the theoretical and observed heights of the oceanic tides of long period. In a paper "On the Rigidity of the Earth" published in the *Philosophical Transactions of the Royal Society* for May, 1863, Lord Kelvin pointed out that if the matter of the Earth's interior yielded readily to the tidal forces arising from the attraction of the Sun and Moon, the crust itself would respond to these forces in much the same way as the waters of the sea; and the corresponding movements of the crust would mask or largely reduce the height of the oceanic tides calculated for a rigid earth. By actual analysis of long series of tidal observations Kelvin and Darwin subsequently found the observed fortnightly tide to have very nearly its full theoretical height, and hence concluded that our globe possesses a very high effective rigidity (cf. Thomson's and Tait's *Natural Philosophy*, Vol. I, Part II, §§ 832, 847).

When Professor Küstner discovered from observations taken at Berlin in 1890-91 that the terrestrial latitude is really variable, and it was afterwards found, chiefly by Chandler's discussion of various series of observations, that the movement of the pole in the body of the Earth has a period of some 427 days instead of the 305 days long ago inferred from Euler's theory of the rotation of a rigid spheroid, Newcomb pointed out that this observed prolongation of the theoretical Eulerian period indicated some yielding of the matter of the globe, and would afford a new method of evaluating the Earth's rigidity. In his well known paper "On the Dynamics of the Earth's Rotation," he showed that the results already attained essentially confirmed Darwin's conclusion that the rigidity of the Earth

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* Reprinted with slight changes from *Astronomische Nachrichten*, Nos. 4104 and 4152.

is comparable to that of steel (cf. *Monthly Notices of the Roy. Astron. Soc.*, Vol. LII, p. 336, March, 1892).

The essential point in NEWCOMB's explanation is that when the pole changes its position the distribution of centrifugal force shifts with respect to the solid Earth, which is put into a state of stress and must yield to the forces acting upon it, like any other elastic body — the periodic variations of the Earth's figure operating to lengthen the period of the free nutation, by an amount depending on the average rigidity of the whole Earth.

ALBRECHT's continued study of the variations more recently noted at the different international latitude observatories confirms this observational result, and the subject has also been examined theoretically by DARWIN, HOUGH, LARMOR and others, and the validity of the method suggested by NEWCOMB is generally recognized.

In his well known paper "On the Rotation of an Elastic Spheroid" (*Phil. Trans., Roy. Soc.*, A., 1896), HOUGH considered chiefly the case of an incompressible homogeneous spheroid, yet by strictly rigorous reasoning he was enabled to show that the rigidity of the Earth in all probability slightly exceeds that of steel.

In a remarkable paper "On the Period of the Earth's Free Eulerian Precession," read to the Cambridge Philosophical Society, May 25, 1896, PROFESSOR LARMOR showed how to estimate the effect of elastic yielding of a rotating solid on the period and character of the free precession of its axis of rotation, and again confirmed this conclusion from another point of view.

The observed prolongation of the Eulerian period of the variation of latitude is thus fully explained by theory and has strengthened the earlier conclusions of KELVIN and DARWIN drawn from the study of the phenomena of the long period tides of the sea.* But while this method probably admits of somewhat greater accuracy and rigor than the tidal method has yet been proved to be capable of, and is important in confirming conclusions from an independent point of view, it nevertheless grew out of results already fairly well established.

It follows therefore that in the development of our knowledge of the physical state of the Earth, the method of determining its rigidity depending on the long period tides of the ocean has already been of great service to the progress of Science; and the remarkable discovery of KELVIN and DARWIN that under the tidal strains to which it is subjected, the Earth as a whole probably yields less than a corresponding globe of steel, seems destined to be remembered in the remotest ages.

* The new and important theorem on the yielding of the Earth to disturbing forces, developed by PROFESSOR A. E. H. LOVE since this chapter was written in 1905-6, together with the important "Horizontal Pendulum Observations" of DR. O. HECKER at Potsdam, establishing the existence of Bodily Tides due to the action of the Sun and Moon, are treated in § 225.

But striking as this individual result is, the method upon which it depends, like that more recently based on the observed prolongation of the Eulerian period of the variation of latitude, seems to be essentially restricted to the Earth on which we live, and thus unfortunately admits of no extension to the other planets or to the myriads of fixed stars observed in the immensity of space. Indeed it appears that no attempt has heretofore been made to deduce the effective rigidity of the Earth or other heavenly bodies from theoretical considerations based upon the law of universal gravitation. Nevertheless in *A.N.*, 3992, and *A.N.*, 4053, it was pointed out that large masses experience great internal condensation under the enormous pressures to which their matter is subjected, and hence have nuclei of great effective rigidity.

The suggestion there put forth has naturally developed in the course of subsequent investigations to which it led. While considering these problems on Sept. 29, 1905, the writer was not a little surprised to find that a theoretical deduction of the effective rigidity of a body of known mass and internal constitution is always possible, and that the determination admits of the greatest confidence and mathematical rigor.

Indeed, if we assume recognized laws of internal density, which may be supposed to represent the physical constitution, and thus enable us to deduce the pressure to which the matter is subjected at various depths within the mass, it appears that we may calculate not only the rigidity of the Earth, but also that of the Sun, planets and satellites, or other known bodies observed among the fixed stars.

Accordingly many of the principal results embodied in this paper were worked out early in October, 1905, and on October 20, communicated to the editor of the *Astr. Nachr.* for publication. During the writer's absence at his country seat, Blue Ridge, on Loutre, near Montgomery City, Missouri, early in November, some further problems suggested themselves for consideration, and methods were then outlined for solving them; but as the calculations required were of considerable length and difficulty, an opportunity for effecting this extension of the investigation did not present itself till the end of January, 1906, when it was decided to delay publication of the earlier results until all the work could be incorporated in one paper,* which should also include a few corrections to some of the tabular results given in *A.N.*, 4053.

One point in the deduction of the rigidity of the heavenly bodies given in *A.N.*,

* The work done in October included the determination of the mean rigidity of the layers, but the calculations for finding the mean rigidity of all the matter were made in January and February, 1906. Some of the principal arguments against the theory of convection currents were the outgrowth of the later work, which emphasized the untenability of this theory in an impressive way.

4104, seems to have presented some difficulty to a few readers,* and it is advisable therefore to indicate anew the essential steps. In ordinary solids, such as the metals, the property of rigidity is produced by the action of molecular forces which resist deformation. On the other hand the matter within a planet like the Earth is really gaseous but above the critical temperature, and therefore in confinement made to behave as an elastic solid wholly by virtue of pressure which brings the molecules within distances at which they again become effective in spite of the high temperature. Thus in cold solids the property of rigidity is due simply to molecular forces which prevent deformation, while for gaseous matter in confinement under such pressure that it acquires the property of an elastic solid the property of effective rigidity is due wholly to the pressure. In the paper above cited I have therefore taken the rigidity to be directly proportional to the pressure, and ignored all other influences such as temperature, because by hypothesis the density is assumed to follow LAPLACE's law, or the monatomic law in the case of purely gaseous masses, and the temperature is supposed to be conformable to the laws of density.

This hypothesis seems legitimate, and almost certainly as accurate as LAPLACE's law and the monatomic law, upon which the calculated pressures depend. Moreover the validity of the hypothesis that the rigidity is proportional to the pressure appears to be confirmed by the close agreement of the numerical value of the Earth's rigidity found in this way with those found by the recognized empirical processes depending on the tides and the polar motion. By the gravitational method it was found that the rigidity should lie between 750,000 and 1,000,000 atmospheres. Here the effect of the Earth's crust in stiffening the external layers of the globe is neglected, because we do not know the amount of the resulting increase (of the rigidity depending on pressure) due to the crust of solid rock analogous to granite, though we have estimated this increase to be a quantity of the order of 100,000 or 200,000 atmospheres. It seems absolutely certain that the rigidity of the encrusted Earth exceeds 800,000 atmospheres, and highly probable that it surpasses 850,000, the rigidity of steel being 808,000. Is this theoretical confirmation of the results reached by LORD KELVIN and SIR GEORGE DARWIN by empirical methods purely accidental, or is it the outcome of a valid underlying method?

It must also be borne in mind that a variation in the assumed law of density would modify the mean rigidity calculated by this method, and as we do not know the law of the Earth's density with great accuracy, the agreement in the rigidity values is as good as can be expected. An agreement so remarkable as this can with difficulty be ascribed to chance; and while it will no doubt appear differently

* cf. *A.N.*, 4152.

to different minds, it seems to me to indicate that the underlying hypothesis approximates very closely, if it does not represent rigorously, a true law of nature. Under the circumstances it has as much validity as the laws relating to the internal density of the heavenly bodies, and has the great advantage that it is general for all the planets and not restricted to the Earth, as are the empirical methods. The gravitational method, has indeed some minor defects; it cannot, for example, take account of the effect of a solid crust, and in the case of gaseous bodies devoid of crust the outer layers are under too small pressure to behave as solid, and we should stop the integration for the pressure before reaching the surface. But as we do not know at what depth the integration should cease, I have taken the mean pressure of the whole mass as giving its most characteristic property.

The pressure has to be considerable to give a confined fluid element an effective rigidity equal to that of a portion of the solid crust. In the outer layers of encrusted planets the rigidity of an element is always greater than that corresponding to the pressure, and hence in calculating the rigidity of the layers of a planet by the gravitational method, we obtain a value which is too small. In like manner the rigidity of *all the matter* of a planet gives a value which is too large. But the truth, for all masses of considerable size, will, no doubt, lie somewhere between these extreme limits, as was pointed out in *A.N.*, 4104. Notwithstanding these limitations I feel satisfied that those who study the method carefully will find it extremely useful, and in fact our only guide in determining the rigidity of the heavenly bodies generally.

Considerable differences of opinion will naturally exist among investigators as to the effects of the extreme physical conditions prevailing in the interior of the heavenly bodies, and no hope is entertained that a full realization or understanding of these stupendous forces is possible with the limited knowledge of the ultimate properties of matter avaliable at the present time. But even if others should dissent from hypotheses now assumed as highly probable, and thus prefer other processes widely different from those here adopted, yet the present results are perhaps sufficiently remarkable to be of interest to the astronomer and natural philosopher.

§ 214. *Definition of the Modulus of Rigidity, with Table of the Moduluses of Rigidity for Various Substances.*

In his celebrated article on "Elasticity," in the *Encyclopedia Britannica*, ninth edition, Vol. VII, p. 805, LORD KELVIN defines the modulus of rigidity thus:

"The modulus of rigidity of an isotropic substance is the amount of normal traction or pressure per unit of area divided by twice the amount of elongation in

the direction of the traction or contraction in the direction of the pressure, when a piece of the substance is subjected to a stress producing uniform distortion." In § 77, page 815 of the same article, he gives a table of Moduluses of Rigidity expressed in grammes weight per square centimetre, the data being quoted from WERTHEIM (*Annales de Chimie*, 1848) and EVERETT (*Illustrations of the Centimetre-Gramme-Second System of Units*). On page 817 KELVIN gives additional results quoted from his paper "On the Elasticity and Viscosity of Metals," read before the Royal Society and published in the *Proceedings* for May 18, 1865.

By reducing the numbers in these tables to kilograms, and dividing by 1.0333, we obtain the corresponding moduluses of rigidity expressed in atmospheres, so that the results become independent of the area of the section of the substance. Thus we have the moduluses of rigidity given in Table A, based mainly on the experiments of LORD KELVIN.

TABLE A. MODULUSES OF RIGIDITY FOR VARIOUS SUBSTANCES EXPRESSED IN ATMOSPHERES.

Substance	Rigidity in Atmospheres	Adopted Max. Value of the Rigidity in Atmosph.	Authority
American Nickel Steel....	1000000	1000000	Average value of the rigidity concluded from var. tests
Steel, hard..............	822610	825000	TAIT, "Properties of Matter," 4th ed., p. 215
Steel	808000	808000	HOUGH, "Rotation of an Elast. Spheroid," *Phil. Trans. Roy. Soc.*, A, 1896, p. 337
Steel	807123	....	EVERETT, cited by KELVIN
Soft Iron	765896	766000	KELVIN, "Elasticity and Viscosity of Metals," *Proc. Roy. Soc.*, 1865
Wrought Iron	759702	760000	EVERETT, cited by KELVIN
Soft Iron (after stretching)	731637	732000	KELVIN
Platinum	602197	602000	KELVIN
Cast Iron	532275	532000	TAIT
Copper	524533	525000	EVERETT, cited by KELVIN
"	457760	....	KELVIN
"	441304	....	EVERETT, cited by KELVIN
"	380722	....	KELVIN
Brass..................	397077	400000	KELVIN
"	360980	....	EVERETT, cited by KELVIN
"	338721	....	WERTHEIM, cited by KELVIN
"	338817	....	KELVIN
Zinc	348011	348000	KELVIN
Gold	271944	272000	KELVIN
Silver..................	261298	261000	KELVIN
Aluminum	233233	233000	KELVIN
Glass..................	235170	235000	EVERETT, cited by KELVIN
"	232265	....	EVERETT, cited by KELVIN
"	145166	....	WERTHEIM, cited by KELVIN

By this table it appears that the rigidity of hard steel made by the original Bessemer process probably does not exceed 825,000 atmospheres.* It is recognized,

* In his classic paper "On the Rotation of an Elastic Spheroid," *Phil. Trans. Roy. Soc.*, A, 1896, p. 337, MR. S. S. HOUGH uses the smaller value 8.19×10^{11}, equivalent to 808,000 atmospheres.

however, that the modern process developed in the manufacture of armor plate gives steel of much greater toughness and rigidity than that produced when LORD KELVIN's table was prepared nearly thirty years ago. Some authorities place the increase of rigidity in modern American nickel steel as high as 60 per cent., but the smaller value of about 45 per cent., is said to be the general rule adopted by experienced workers. As much depends upon the purity of the material, and the exact percentage of the alloy employed, the increase of rigidity has often been placed no higher than 30 per cent. If we used 45 per cent. for the increase of stiffness due to the alloy, we should have a rigidity of about 1,200,000 atmospheres.

With the smaller value of 30 per cent. the rigidity would be 1,072,500 atmospheres. Under the circumstances there is no doubt that standard nickel steel used for armor plate may easily have a rigidity of at least 1,000,000 atmospheres; and accordingly we adopt this convenient round number in dealing with the rigidities of the heavenly bodies.

§ 215. *Theoretical Evaluation of the Earth's Rigidity, Based on the Internal Pressures Resulting from the Effects of Gravity and* LAPLACE's *Law of Density.*

Since the modulus of rigidity is the amount of pressure per unit of area (or simply pressure, when atmospheres are the units employed) divided by twice the amount of the contraction in the direction of the pressure, it becomes evident that if we could find the mean pressure throughout the body of the Earth, we should have at once a measure of the average rigidity. For the great body of the Earth's matter is above the critical temperature of every known substance, and therefore essentially gaseous, except as condensed by pressure. Under the enormous forces acting throughout the Earth's interior, which ensure the most complete interpenetrability of all the elements, we may always assume that the matter in any layer is isotropic, and the power of resistance developed by secular condensation everywhere directly proportional to the pressure, as in a compressible fluid. Accordingly, in finding the rigidity, we need to consider nothing but the pressure operating on the imprisoned fluid from the surface to the centre of the Earth. Now the pressure throughout the Earth's interior is known with essential accuracy from LAPLACE's law of density, and the corresponding method of calculation developed in *A.N.*, 3992. The pressure is given by the equation,

$$p = \frac{3r}{2(\sigma_1 g)q^2}[(\sigma g)^2 - (\delta g)^2], \tag{493}$$

where r is the Earth's mean radius, g mean gravity, q the constant for LAPLACE's

law, 2.52896 radians = 144° 53′ 55″.2, σ the density at any point, and δ the density at the surface.

To render this expression available for integration throughout the sphere occupied by the Earth's mass, we must put for σ^2 its value $\sigma^2 = \sigma_0^2 \frac{\sin^2(qx)}{q^2 x^2}$, and for δ^2 its value $\delta^2 = \sigma_0^2 \frac{\sin^2 q}{q^2}$, corresponding to the surface, where $x = 1$. Thus we get

$$p = \frac{3(\sigma_0 g)^2 r}{2(\sigma_1 g) q^2}\left[\frac{\sin^2(qx)}{q^2 x^2} - \frac{\sin^2 q}{q^2}\right]. \tag{494}$$

For the total pressure throughout a sphere of radius $\rho = r\,x$, r being the external radius, and $x = \frac{\rho}{r} =$ the fraction of the radius, we have

$$P = \int_0^x p\, 4\pi r^2 x^2 r\, dx = \frac{3(\sigma_0 g)^2 r\, 4\pi r^3}{2(\sigma_1 g) q^4}\left(\int_0^x \frac{\sin^2(qx)}{x^2} x^2 dx - \sin^2 q \int_0^x x^2 dx\right), \tag{495}$$

which by integration gives

$$P = \frac{3(\sigma_0 g)^2 r\, 4\pi r^3}{2(\sigma_1 g) q^4}\left(\frac{qx - \sin(qx)\cos(qx)}{2q} - \sin^2 q \frac{x^3}{3}\right). \tag{496}$$

As our integration is to include the whole sphere of the Earth, we put $x = 1$, and then we have

$$P = \frac{3(\sigma_0 g)^2 r\, 4\pi r^3}{2(\sigma_1 g) q^4}\left(\frac{q - \sin q \cos q}{2q} - \frac{\sin^2 q}{3}\right). \tag{497}$$

The total volume of the Earth is $\frac{4}{3}\pi r^3$, and hence the average pressure per unit of area on all concentric spherical surfaces is

$$R = \frac{P}{\frac{4}{3}\pi r^3} = \frac{9(\sigma_0 g)^2 r}{2(\sigma_1 g) q^4}\left(\frac{q - \sin q \cos q}{2q} - \frac{\sin^2 q}{3}\right). \tag{498}$$

For any other planet, as *Venus* or *Mars*, following the same law of density as the Earth, we merely introduce the factor $n = \frac{G}{g}$, where G is the acceleration of gravity on the planet, and g the value of mean gravity on the Earth. Thus we have in general

$$R = n\frac{9(\sigma_0 g)^2 r}{2(\sigma_1 g) q^4}\left(\frac{q - \sin q \cos q}{2q} - \frac{\sin^2 q}{3}\right). \tag{499}$$

The value of q for the different bodies of the solar system are given in *A.N.*, 3992, that for *Mars* being slightly corrected in *A.N.*, 4053, footnote p. 347. If r in either of these formulae is expressed in metres, the mean pressure, or mean rigidity R, comes out in kilograms per square metre. To reduce the results to atmospheres we divide by 10333, the logarithm of which is (4.0142264). As carried out for the principal bodies of the planetary system, with the data for LAPLACE's law given in *A.N.*, 3992, and *A.N.*, 4053, p. 347, the mean pressures or theoretical rigidities are found by calculation to be as shown in Table C.

It appears from this table that, considered as a mass of fluid, of such a nature that the material resists in proportion to the pressure which it sustains, and therefore no doubt yielded originally in simple proportion to the pressure to which it was subjected in the process of secular condensation, the Earth as a whole has a mean theoretical rigidity of 748,843 atmospheres, which is about identical with that of wrought iron. In other words, if the Earth were in a molten condition but followed the same law of density as at present, pressure alone would impart to it a rigidity equal to that of a wrought iron globe of the same size, provided the iron sphere were devoid of mutual gravitation between its parts, as is sensibly the case with the small iron balls which we hold in our hands, or subject to tests in the Laboratory. This remarkable result shows that when the Earth was in molten condition, it did not have the mobility and plasticity which eminent investigators have sometimes ascribed to it. On the contrary ever since the Earth attained approximately its present dimensions, it has been highly rigid. Accordingly, it must have possessed great effective rigidity throughout Geological History.

The result just obtained appears to give a true lower limit of a planet's theoretical mean rigidity, on the hypothesis that the internal density follows LAPLACE's law.

The amount by which the actual rigidity exceeds this lower limit depends on the effects of the external encrustation and internal viscosity due to the intramolecular forces*, which increase rapidly with the density. The additional augmentations of the mean rigidity due to the enclosing crust of solid rock analogous to granite, and to the viscosity of the matter throughout the planet's body, arising from its considerable density, are difficult to estimate accurately, and perhaps we can only infer that together these two causes would increase the mean rigidity of the Earth by a quantity of the order of 100,000 or 200,000 atmospheres.

* On this point LORD KELVIN says: "MAXWELL's admirable kinetic theory of the viscosity of gases points to a full explanation of viscosity, whether of gases, liquids, or solids, in the consideration of configurations and arrangements of relative motions of molecules, permanent in a solid under distorting stress and temporary in fluids or solids while the shape is being changed, in virtue of which elastic force in the quiescent solid and viscous resistance to change of shape in the non-quiescent fluid or solid, are produced." Article "Elasticity," pp. 801-802.

§ 216. *Determination of the Mean Rigidity of the Earth's Matter as Distinguished from that of the Various Layers Composing the Globe.*

There is, however, another related function of definite character which proves to be of very considerable interest, as fixing a probable upper limit to the rigidity of bodies like the Earth. This function is the mean rigidity of the Earth's matter, found by considering not only the pressure but also the density or mass per unit volume of the imprisoned matter in the several layers; so that the new function represents a mean rigidity in which every element is allowed a weight proportional to its mass, which is multiplied by the pressure to which it is subjected. This conception leads to interesting results also when applied to the Sun and major planets, according to the principles of the monatomic theory, as explained hereafter.

The theory of the determination of the mean rigidity of the Earth's matter is as follows:

$$P' = \int_0^x p\,4\pi r^2 x^2 r\,\mathrm{d}x\,\sigma = 4\pi r^3 \sigma_0 \int_0^x p x^2\,\mathrm{d}x \frac{\sin(qx)}{qx}. \tag{500}$$

Substituting for p its value from (494), we have

$$\left.\begin{aligned} P' &= \frac{3(\sigma_0 g)^2 r 4\pi r^3 \sigma_0}{2(\sigma_1 g) q^6}\left(\int_0^x \frac{\sin^3(qx)\,x^2\,\mathrm{d}x}{x^3} - \sin^2 q \int_0^x \frac{x^2 \sin(qx)\,\mathrm{d}x}{x}\right) \\ &= \frac{3(\sigma_0 g)^2 r 4\pi r^3 \sigma_0}{2(\sigma_1 g) q^6}\left(\int_0^x \frac{\sin^3(qx)\,q\,\mathrm{d}x}{qx} - \sin^2 q \int_0^x \frac{x\sin(qx)\,q\,\mathrm{d}x}{q}\right). \end{aligned}\right\} \tag{501}$$

The integral of the last term is $-\sin^2 q\left(\frac{\sin(qx) - qx\cos(qx)}{q^2}\right)$ (cf. *A.N.*, 4053, p. 345).

The value of the first integral for various values of qx is most conveniently found by quadrature, since expansions in series do not converge readily when the arc is large, as happens in the application of LAPLACE's law to many of the planets. The mean rigidity of the Earth's matter is thus found by the formula

$$R' = \frac{P'}{\frac{4}{3}\pi\sigma_1 r^3} = \frac{9(\sigma_0 g)^2 r\sigma_0}{2(\sigma_1 g) q^6 \sigma_1}\left(\int_0^x \frac{\sin^3(qx)}{qx} q\,\mathrm{d}x - \frac{\sin^2 q}{q^2}\left[\sin qx - qx\cos(qx)\right]\right). \tag{502}$$

On putting $qx = 144° 53' 55''.2$, as in *A.N.*, 3992, the value of the integral is easily found by quadrature to be 0.9592502, and when the rest of the formula is reduced to numbers we find

$$R' = 1{,}028{,}702 \text{ atmospheres.}$$

Thus on the hypothesis that the internal density follows LAPLACE's law it appears that the average rigidity of all the matter of our globe slightly surpasses that of nickel steel.

The actual rigidity of the Earth almost certainly lies between the two limits thus established, namely $R = 748{,}843$, based on the rigidity of the layers deduced from the pressure to which they are subjected, and $R' = 1{,}028{,}702$, derived from the product of the mass of each layer by the pressure acting upon it.

When one considers the effects of the enclosing crust and the viscosity of the whole Earth, which must be assumed to increase towards the centre, owing to the increasing density and rising temperature of the imprisoned matter, it seems not improbable that the actual effective rigidity of our globe may be nearer the upper limit than the lower, and probably we shall not be far wrong in concluding that it is approximately equal to that of nickel steel.

Leaving aside the consideration of the effects of the solidified crust, it is evident from the nature of the forces at work that most of the yielding of our globe, due to the periodic action of small forces, is in the outer layers; and in general the yielding in any concentric layer may be taken to be inversely as the pressure to which the imprisoned matter is subjected. *It is remarkable that the curve of pressure as we descend in the Earth becomes therefore also the curve of effective rigidity for the matter of which the Earth is composed.* Thus the rigidity of the matter at the Earth's centre probably is at least three times that of nickel steel used in armor plate; as we approach the surface the effective rigidity constantly exceeds that of nickel steel until we come within less than 0.4 of the radius from the surface, where the pressure is less than 1,000,000 atmospheres.

To imagine a mechanical substitute for the Earth's constitution, without the introduction of pressure, suppose an alloy of adamant to give the material at the centre of such a globe, of the same size but devoid of gravitation, a hardness three times that of armor plate. The outer layers as we approach the surface must then be supposed softer, and softer, until it is like armor plate at a little over 0.6 from the centre, and finally a very stiff fluid near the surface. In addition to this arrangement of its effective internal rigidity the actual Earth is enclosed in a spheroidal shell of solid rock analogous to granite. One can easily see that tidal forces applied to all the particles of such an artifical armored sphere would produce but very slight deformation, because of the enormous effective rigidity of the nucleus.

The principal uncertainty in this result arises from the admissible variations in the assumed Laplacian distribution of density within the Earth. Both RADAU

and DARWIN (cf. *Monthly Notices, Roy. Astron. Soc.*, Dec., 1899) have pointed out that considerable variations in the internal distribution of density are possible without invalidating the well known argument drawn from the phenomenon of the Precession of the Equinoxes; yet on physical grounds it seems clear that pressure is the principal cause of the increase of density towards the Earth's centre. And since this does not vary greatly for moderate changes in the law of density, the principle of continuity shows that the actual law of density within the Earth cannot depart very widely from that of LAPLACE. The above value of the theoretical rigidity of the Earth may therefore be taken as essentially accurate, and I think no doubt can remain that the rigidity of our Earth as a whole considerably exceeds that of steel. The original conclusions of KELVIN and DARWIN are therefore confirmed by the present dynamical considerations based upon the theory of universal gravitation.

It is perhaps worth pointing out that as a molten Earth, in which the density follows LAPLACE's law, would have a mean rigidity of its layers equal to that of wrought iron, the hypothetical liquid interior would be much less easily deformed by tidal forces than has been generally supposed; so that reaction upon the enclosing crust probably would not be very conspicuous. The amount of this reaction would depend essentially upon the difference between the rigidity of nickel steel and of wrought iron, which is about $\frac{1}{4}$ of the rigidity of the whole Earth as now constituted. Even if one supposed the interior of the Earth to be liquid, the pressure to which it is subjected is so great that the tidal surgings of the nucleus, tending to deform the crust, would be comparatively ineffective; and if the crust of solid rock like granite be moderately thick, it is doubtful if the yielding would be sufficient to reduce sensibly the theoretical height of the fortnightly tides of the oceans. Accordingly it appears probable that the argument drawn from the tides against the fluidity of the Earth's nucleus may in reality be somewhat less conclusive than the most eminent mathematicians have supposed. But from the accordance between the value of the Earth's rigidity obtained from the theory of gravity with those found by DARWIN from observations of the fortnightly tides, and by HOUGH from the prolongation of the Eulerian period for the variation of latitude, it seems impossible to escape the conclusion that the rigidity of our globe as now encrusted probably approaches that of nickel steel.

It is scarcely necessary to add that the traditional theory long held by Geologists that the Earth's interior is a mass of mobile liquid in which currents still persist (cf. FISHER's *Physics of the Earth's Crust*, second edition, pp. 246, 305, *et seq.*) when viewed from a gravitational standpoint is therefore found to be inadmissible. The great effective rigidity or viscosity of the matter within the Earth makes any supposed motion of the imprisoned fluid quite inconceivable.

This view of the Earth's physical constitution, deduced from the theory of gravity, according to rigorous dynamical principles, is in complete harmony with the results of modern Seismology. We have just shown that the elasticity and effective rigidity of the matter as we descend within the Earth increase directly as the pressure. Now the study of the propagation of waves due to a variety of earthquakes appears to show that they are transmitted through the body of the Earth in right lines, and with decidedly greater velocity than through the solid crust curved around the surface of the globe. Since the elasticity depends upon the pressure, and like the density, increases downward according to a definite law, this more rapid propagation through the interior ought to take place. And the observed velocities of waves transmitted through the Earth's body in various directions and at different depths may some day afford a new method for extending our knowledge of the laws of density and pressure within the globe. If, for example n velocities be observed from well determined disturbances propagated in right lines through matter assumed to have known density and elasticity, the conformity of the observed results with those predicted from LAPLACE's law, will throw light upon the accuracy of that law at various depths, and afford n equations of condition for correcting the underlying hypothesis respecting the distribution of density and pressure*.

§ 217. *Applications of the Foregoing Methods to the Principal Bodies of the Planetary System, on the Hypothesis that Their Internal Densities Follow* LAPLACE'S *Law as Developed in A.N.*, 3992.

The formulae to be employed are (499) and (502); the latter, however, being generalized by the introduction of the factor $n = \frac{G}{g}$, as in (499), so that

$$K' = n\frac{9(\sigma_0 g)^2 r\sigma_0}{2(\sigma_1 g)q^6\sigma_1}\left\{\int_0^x \frac{\sin^3(qx)\,q\,dx}{qx} - \frac{\sin^2 q}{q^2}\left[\sin(qx) - qx\cos(qx)\right]\right\}. \qquad (503)$$

* This paragraph was written in October, 1905, and we leave it unchanged, but call attention to the following interesting remarks by PROFESSOR LARMOR in the paper of 1896, previously cited, received here in January, 1906: "It has been estimated by MR. HOUGH (*loc. cit. Phil. Trans.*, 1896) that, were the Earth a homogeneous solid with the actual surface ellipticity, the Chandler period for the free precession would require that it should have the rigidity of steel. The order of magnitude of this result will not be entirely altered by actual heterogeneity; so that, if we can assume the correctness of this period, we may conclude that, as regards the slight stress here involved, the Earth is an elastic solid of about the same order of rigidity as steel. This is in fair accord with the observations of seismologists (cf. PROF. MILNE, *Brit. Assoc. Report*, 1896, "On Earthquake Phenomena"), who finds that earthquake shocks are propagated to distant parts of the Earth not only by the ordinary surface waves, but also by minute tremors which enormously outrun the earthquake proper, arriving in a time that would correspond to propagation in a straght line across the interior of the Earth, provided it possessed an average rigidity about $\frac{3}{4}$ that of steel."

In the following table we give the results of these calculations as applied to the various bodies of our system. For convenience of others who might wish to vary the constants employed, we give the concluded values of the integral $\int_0^x \frac{\sin^3(qx)}{qx}\, q\,dx$ for the several planets and satellites, and also the tabular values of qx. It will be seen that R' is appreciably larger than R only in the case of the larger bodies, and that the difference becomes greater with the increase of the mass. Thus the difference between R' and R is less and less conspicuous in passing from the Sun to the major planets and the Earth and *Venus*, becoming practically insignificant in the smaller terrestrial planets and satellites, which are of nearly homogeneous density throughout.

The results here obtained by LAPLACE's law for the Sun and major planets are no doubt only rough approximations to the truth, but these values are included in order to afford a comparison with the results hereafter deduced from the monatomic theory, which is supposed to approximate the true conditions of the larger masses of our system much more accurately than LAPLACE's law.

TABLE B. VALUES OF THE DIFFERENTIAL ELEMENT $\frac{\sin^3(qx)}{qx}$ AT EACH 5° FROM 0° TO 180°, WITH THE CORRESPONDING INTEGRALS FOR THE VALUES OF qx APPROPRIATE TO THE PLANETS.

Argument qx in Degrees	Values of the Element $\frac{\sin^3(qx)}{qx}$	Argument qx in Degrees	Values of the Element $\frac{\sin^3(qx)}{qx}$	Argument qx in Degrees	Values of the Element $\frac{\sin^3(qx)}{qx}$
°		°		°	
0	0.000	65	0.65620	130	0.19813
5	0.013087	70	0.67918	135	0.15005
10	0.029589	75	0.68848	140	0.10869
15	0.066225	80	0.68405	145	0.074564
20	0.114617	85	0.66640	150	0.047747
25	0.172993	90	0.63662	155	0.027902
30	0.238732	95	0.59625	160	0.014327
35	0.308907	100	0.54724	165	0.0060204
40	0.380422	105	0.49177	170	0.0017648
45	0.45016	110	0.43220	175	0.00037393
50	0.51513	115	0.37090	180	0.0000000
55	0.57260	120	0.31012		
60	0.62024	125	0.25195		

Body	Adopted Value of qx		Value of the Integral	Body	Adopted Value of qx		Value of the Integral
	in Arc	Logarithm in Radians	$\int_0^x \frac{\sin^3(qx)}{qx}\, q\,dx$		in Arc	Logarithm in Radians	$\int_0^x \frac{\sin^3(qx)}{qx}\, q\,dx$
	° ′ ″				° ′ ″		
Mercury	49 28 30	9.9362645	0.173226	*Jup.* Sat. IV	50 7 50	9.9419785	0.178969
Venus	138 32 2.4	0.3834225	0.947947	*Titan*	57 25 12	0.0009423	0.250335
Earth	144 53 55.2	0.4029418	0.959250	*Jupiter*	179 59 41.132	0.4971372	0.9710787
Mars	76 33 26.3	0.1258642	0.469483	*Saturn*	179 59 24.125	0.4971258	"
Moon	39 34 48	9.8393544	0.096796	*Uranus*	179 59 48.475	0.4971422	"
Jup. Sat. I	35 46 0	9.7953576	0.073687	*Neptune*	179 59 49.475	0.4971425	"
" " II	32 2 12	9.7475238	0.055083	*Sun*	179 59 41.839	0.4971367	"
" " III	54 20 50	9.9762553	0.218861				

TABLE C. MEAN RIGIDITY OF THE LAYERS.

Body	R in Atmospheres, according to LAPLACE'S Law	R in Atmospheres, according to the Monatomic Theory
Mercury	25146	. . . .
Venus	590022	. . . .
Earth	748843	. . . .
Mars	102245	. . . .
Moon	11632	. . . .
Jup. Sat. I	14245	. . . .
" " II	15632	. . . .
" " III	18013	. . . .
" " IV	2189	. . . .
Titan	11397	. . . .
Jupiter	6078020	18691010
Saturn	1156730	3560470
Uranus	1501537	4624053
Neptune	1894276	5833133
Sun	658236000	2027861500

TABLE D. MEAN RIGIDITY OF THE MATTER.

Body	R' in Atmospheres, according to LAPLACE'S Law	R' in Atmospheres, according to the Monatomic Theory
Mercury	27859	. . . .
Venus	723148	. . . .
Earth	1028702	. . . .
Mars	108820	. . . .
Moon	23385	. . . .
Jup. Sat. I	21205	. . . .
" " II	30173	. . . .
" " III	23409	. . . .
" " IV	2288	. . . .
Titan	12306	. . . .
Jupiter	12369086	56237501
Saturn	2352839	10712580
Uranus	3053571	13916028
Neptune	3853718	17550730
Sun	1338728600	6101430000

§ 218. *Determination of the Mean Rigidity of the Sun's Globe Considered as a Sphere of Monatomic Gas Made Up of Successive Concentric Layers, Each of Uniform Quality Throughout, but with the Density Varying from the Centre to the Surface.*

Let the surface gravity of the Sun be denoted by G, and suppose the value of gravity at the surface of any concentric spherical shell of radius x to be G'. Then by the method developed in the chapter on the "Physical Constitution of the Heavenly Bodies" (§207, p. 474), Equation [τ] (460), it is shown that

$$\frac{G'}{G} = \frac{3\sigma_0}{x'x^2\sigma_1}\int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2\,\mathrm{d}x = \frac{3\sigma_0}{x'x^2\sigma_1}\mu\,. \tag{504}$$

For a gaseous sphere it is easily shown, Chap. XVI, § 196, p. 435, Eq. (424), that the differential equation for the pressure is

$$\mathrm{d}p = -\sigma G'\,\mathrm{d}x\,. \tag{505}$$

In our present work the minus sign may be omitted, as it simply denotes that the pressure is directed towards the origin of co-ordinates at the Sun's centre, where $x = 0$.

Substituting for G' its value from (504), namely $G' = \frac{3\sigma_0 G\mu}{x'x^2\sigma_1}$ and integrating, we have

$$p = \int_0^x \mathrm{d}p = \int_0^x \frac{3\sigma_0 G\sigma\mu\,\mathrm{d}x}{x'x^2\sigma_1} = \frac{3\sigma_0^2 G}{x'\sigma_1}\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\frac{\mathrm{d}x}{x^2}\mu = \frac{3\sigma_0^2 G}{x'\sigma_1}\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\frac{\mathrm{d}x}{x^2}\int_0^x \left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x\,. \tag{506}$$

The last of these integrals is shown in the former paper to be a certain series called μ, there designated as Equation [β] (443). Using this value in Equation (506) we have

$$\left.\begin{aligned} p &= \frac{3\sigma_0^2 G}{x'\sigma_1}\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\frac{dx}{x^2}\left[\frac{x^3}{3} - \frac{x^5}{20} + \frac{x^7}{240} - \frac{x^9}{3888} + \ldots\right] \\ &= n\frac{3(\sigma_0 g)^2}{x'(\sigma_1 g)}\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3} - \frac{x^3}{20} + \frac{x^5}{240} - \frac{x^7}{3888} + \ldots\right]dx \end{aligned}\right\} \tag{507}$$

where $n = \frac{G}{g}\cdot$

This equation gives the pressure at any point of the Sun's radius, on the hypothesis that the whole mass is composed of gases reduced to the state of single atoms.

The total pressure on the successive concentric sphere surfaces throughout the Sun's globe is given by the integral

$$P = \int_0^{x'} p\,4\pi x^2 dx = n\frac{3(\sigma_0 g)^2 4\pi}{x'(\sigma_1 g)}\int_0^{x'} dx\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3} - \frac{x^3}{20} + \frac{x^5}{240} - \frac{x^7}{3888} + \ldots\right]x^2 dx\,, \tag{508}$$

which must include the whole mass from the centre to the surface. The average or mean pressure per unit of area on all the successive concentric spherical surfaces by which the Sun's mass may be divided is found by dividing this expression by the volume of the sphere,

$$R = \frac{P}{\frac{4}{3}\pi x'^3} = n\frac{9(\sigma_0 g)^2}{x'^4(\sigma_1 g)}\int_0^{x'} dx\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x^3}{3} - \frac{x^5}{20} + \frac{x^7}{240} - \frac{x^9}{3888} + \ldots\right]dx\,. \tag{509}$$

The series in the bracket under the integral signs is called μ, and hence we get

$$R = n\frac{9(\sigma_0 g)^2}{x'^4(\sigma_1 g)}\int_0^{x'} dx\int_0^x \left(\frac{\sigma}{\sigma_0}\right)\mu\, dx\,. \tag{510}$$

The two series $\left(\frac{\sigma}{\sigma_0}\right)$ and μ are fully developed in Chap. XVII, § 202, pp. 458–459, and the adopted logarithms of the coefficients given in Equations [ζ] (447) and [δ] (445). To find R from (510) we have to calculate the coefficients of a square made up of 625 elements, and then perform a double integration, and numerical

evaluation of the terms for the value of $x = x' = 3.653962$, corresponding to the Sun's surface.

The integration thus takes the form

$$\int_0^{x'} dx \int_0^{x} \left(\frac{\sigma}{\sigma_0}\right) \mu \, dx = \begin{Bmatrix} a_1 x'^5 + a_2 x'^7 + a_3 x'^9 & + \ldots + a_i x'^{2i+3} \\ b_1 x'^7 + b_2 x'^9 + b_3 x'^{11} & + \ldots + b_i x'^{2i+5} \\ c_1 x'^9 + c_2 x'^{11} + c_3 x'^{13} & + \ldots + c_i x'^{2i+7} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ y_1 x'^{2i+3} + y_2 x'^{2i+5} + y_3 x'^{2i+7} & + \ldots + y_i x'^{4i+3} \end{Bmatrix} i \gtrequal 1 \tag{511}$$

Substituting this result in (510), and dividing out the factor x'^4, the final integral becomes

$$R = n \frac{9(\sigma_0 g)^2 x'}{(\sigma_1 g)} \begin{Bmatrix} a_1 \quad + a_2 x'^2 + a_3 x'^4 & + \ldots + a_i x'^{2i-2} \\ b_1 x'^2 + b_2 x'^4 + b_3 x'^6 & + \ldots + b_i x'^{2i} \\ c_1 x'^4 + c_2 x'^6 + c_3 x'^8 & + \ldots + c_i x'^{2i+2} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ y_1 x'^{2i-2} + y_2 x'^{2i} + y_3 x'^{2i+2} & + \ldots + y_i x'^{4i-2} \end{Bmatrix} i \gtrequal 1 \tag{512}$$

This gives

$$R = n \frac{9(\sigma_0 g)^2 x' S}{(\sigma_1 g)}, \quad S = 0.00237603\,, \tag{513}$$

where S is put for the sum of the series in the square, and $n = \frac{G}{g}$, denotes the ratio of the acceleration of gravity at the Sun's surface to terrestrial mean gravity. Using the numerical values adopted in *A.N.*, 3992, we find $\log n = 1.4450675$. The numerical evaluation of S, obtained by putting

$$x = x' = 3.653962\,,$$

showed that a little over 300 terms become sensible in the eighth place of decimals. The work has been carefully checked, and the final value found to be

$$S = 0.00237603, \quad \log S = 7.3758519 - 10.$$

Returning now to Equation (513), putting $(\sigma_0 g) = 8417.5$, $(\sigma_1 g) = 1402.6$, as shown in § 196, p. 435, Equation (426), and

$$x' = 696098000 \text{ meters},$$

and dividing the result by 10333, $\log = 4.0142264$, to reduce it to atmospheres, we have

$$R = n \frac{9(\sigma_0 g)^2 x' S}{(\sigma_1 g)\, 10333} = 2027861500 \text{ atmospheres}, \tag{514}$$

which is the mean rigidity of the layers of the Sun's globe calculated on the basis of the monatomic theory.

It thus appears that the Sun's mean rigidity is more than 2000 times that of nickel steel! This means that the Sun's globe considered with respect to all the layers of which it is composed is distorted by the tidal forces due to another body less than 1 : 2000th part as much as the layers of a corresponding sphere of armor plate would be on the hypothesis that the steel has its usual stiffness and is attracted by the neighboring mass, but the particles are without mutual gravitation among themselves. In other words, if we imagine a sphere of nickel steel, solid from the centre to the surface, of the same size as the Sun, but as completely without mutual gravitation between its particles, as those of a small sphere which we can hold in our hands are practically, such an armored sphere under the Sun's attraction would suffer 2000 times more deformation of its layers than that suffered by the average of the layers of our actual Sun.

It may be remarked that formula (513) is immediately applicable to any gaseous mass of monatomic character, which has acquired a settled state, whether in the stellar or approaching the nebular condition, S having been calculated once for all; and thus is likely to be very useful in cosmical investigations covering a wide range of conditions. If the Sun be supposed expanded to fill the orbit of the Earth, for example, and the temperature of the mass be adequate to maintain its figure from collapsing under its own gravitation, it will be found that the mean rigidity varies inversely as the fourth power of the radius, and thus becomes about $1 : (215)^4$, or 1 : 2136750000 of its present value; so that the average rigidity of the layers of the solar nebula at that stage would hardly equal one atmosphere.* The layers of a nebulous mass would therefore present very slight resistance to tidal distortion, while that corresponding to the stellar stage would be multiplied by the factor $\left(\frac{r_0}{r}\right)^4$, and become very large when r, the radius of the star, had shrunk to small dimensions.

When applied to the major planets formula (514) gives the following results:

	R		R
Jupiter	18691010 atmospheres	*Uranus*	4624053 atmospheres
Saturn	3560470 "	*Neptune*	5833133 "

The enormous mean rigidity of the layers of the larger planets is worthy of remark. In this great average pressure acting throughout their masses lies the explanation of their perfect symmetry, regularity and stability of figure under all conditions; and we may be assured that supposed irregularities of figure occasionally reported by observers are due to some kind of delusions, either optical or atmospheric.

* Under these conditions *true rigidity disappears, because the pressure is not great enough to give the gaseous nebula the property of a solid.*

§ 219. *Rigorous Calculation of the Pressures in the Interior of the Sun, with Corrections of Certain Tabular Results Given in A.N.*, 4053.

The pressure on any layer within the Sun's globe depends upon the law of density of the superincumbent matter, the accumulated weight of which is sustained by the reaction of the enclosed nucleus. In the theory of the Sun developed in *A.N.*, 4053, the writer did not feel at liberty to make any assumption as to the internal temperature except that it conformed essentially to the principle of convective equilibrium; and accordingly the temperature and all other parts of the theory were derived from the density, as rigorously determined from the principles of the monatomic theory. But in considering the numerous questions arising in this work, an important oversight was made in the calculation of the pressures within the Sun's globe. The error thus committed has assumed in the writer's eyes the more formidable proportions because it involved a certain violation of the principles of the theory of gravity. For instead of calculating the pressures in the several layers according to the method stated above, the formula for LAPLACE's law, previously employed in *A.N.*, 3992, was again used, with the new densities resulting from the monatomic theory, without examining the implied path of integration for the density. The result was that as the integration for the Laplacian formula presupposes a curve of density similar to that given in *A.N.*, 3992, while that for the monatomic theory is sensibly different, the density in the latter case being less in outer layers and greater in the central parts of the Sun's globe, the concluded pressures came out considerably too small for the outer layers and slightly too large for the region of the Sun's nucleus.* But while the defect thus arising is considerable, it does not greatly affect the general results for the pressures, nor change any of the other conclusions drawn in the *A.N.*, 4053. In fact the results deduced here are decidedly more favorable to the argument for the high effective rigidity of the Sun's outer layers than those given in the former paper, and therefore the present investigation is of great interest to the astronomer.

To determine the pressure throughout the Sun's globe we shall resume equation (507), which involves the series $\frac{\mu}{x^2}$. The logarithms of the coefficients of this series are correctly given in *A.N.*, 4053, Equation [δ] (447). On re-examining nearly all the original calculations of the former paper, however, the author has found that a slight error of copying vitiated the first figures of the printed divisor of the seventh

* The table of solar pressures in *A.N.*, 3992, pp. 139-140, calculated from the same formula, for the curves corresponding to $k = 1.4$, is also vitiated in the same way.

term of the series itself, as given in Equation [β], (443) which should have an 8 instead of the 11 there published.* This did not affect any of the calculations, but attention is called to it here, so that if any one should ever repeat the tedious calculations, the printed error of copy will not prove an embarrassment in the verification of the series by successive expansions and integrations. Indeed these calculations are of such great length that it appears worth while to reprint here the series with the factors of the denominators of the coefficients, which may be useful to other investigators, should they attempt the integrations and expansions required for the determination of the coefficients.

$$\left.\begin{aligned}\mu = \frac{x^3}{3} - \frac{x^5}{2^2\cdot 5} + \frac{x^7}{2^4\cdot 3\cdot 5} - \frac{x^9}{2^4\cdot 3^5} + \frac{19x^{11}}{2^6\cdot 3^4\cdot 5^2\cdot 11} - \frac{2719x^{13}}{2^{10}\cdot 3^5\cdot 5^3\cdot 11\cdot 13} \\ + \frac{20621x^{15}}{2^{10}\cdot 3^5\cdot 5^3\cdot 11\cdot 12\cdot 13\cdot 15} - \frac{193328x^{17}}{2^{10}\cdot 3^5\cdot 5^3\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 17} \\ + \frac{39667364x^{19}}{2^{11}\cdot 3^7\cdot 5^3\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 16\cdot 17\cdot 19} \\ - \frac{8078124341x^{21}}{2^{11}\cdot 3^8\cdot 5^4\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 16\cdot 17\cdot 18\cdot 19\cdot 21} + \cdots\end{aligned}\right\} \tag{515}$$

Using the logarithms of the coefficients given in Equations [δ] and [ζ], and integrating the expression,

$$\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3} - \frac{x^3}{20} + \frac{x^5}{240} - \frac{x^7}{3888} + \cdots\right]dx\,, \tag{516}$$

in the form of a square made up of 625 elements, of which nearly 400 are sensible in the eighth place of decimals, when $x = x' = 3.653962$, we finally obtain for the integral the value

$$I = 0.0394230\,. \tag{517}$$

And the expression for the pressures at the Sun's centre becomes

$$\varpi_0 = n\,\frac{3(\sigma_0 g)^2 x' I}{(\sigma_1 g)\,10333} = 11{,}215{,}403{,}000 \text{ atmospheres.} \tag{518}$$

To find the pressure at any part of the radius, the Sun being considered a sphere of monatomic gas in convective equilibrium, it is sufficient to use the expression

$$\varpi = \varpi_0\left(\frac{\sigma}{\sigma_0}\right)^{1\frac{2}{3}}. \tag{519}$$

As the value of $\left(\frac{\sigma}{\sigma_0}\right)$ for each tenth of the radius and also each hundredth of the outer tenth are correctly given in *A.N.*, 4053, a simple calculation gives the

* All these corrections are of course duly entered in Chapter XVII, of the present volume of these *Researches*, but it is thought best to let the discussion stand essentially as in *A.N.* 4104.

pressures indicated in the following table, which exhibits the results of the monatomic theory as applied to the Sun and major planets in a form which is rigorously exact.

The distribution of temperature in two monatomic spheres is defined by the relation

$$T' = T\left(\frac{p'}{p}\right)\left(\frac{\sigma}{\sigma'}\right), \tag{520}$$

and hence the new values of the pressures in the Sun do not affect the tabulated internal temperatures of *Jupiter* and *Saturn* given in *A.N.*, 4053. Accordingly we add merely the temperatures for *Uranus* and *Neptune*, which were found to be somewhat affected by an undetected error in the former paper. While it is not probable that the monatomic theory is more than approximately true for bodies so dense as the outer planets, yet it seems desirable to remove every known inconsistency from the theory finally adopted.

TABLE E. PRESSURES IN THE SUN AND MAJOR PLANETS, CALCULATED RIGOROUSLY, ACCORDING TO THE MONATOMIC THEORY.

R Radius	Logarithm of $\left(\frac{\sigma}{\sigma_0}\right)^{1\frac{2}{3}}$	*Sun* ϖ in Atmospheres	*Jupiter* ϖ in Atmospher.	*Saturn* ϖ in Atmospher.	*Uranus* ϖ in Atmosph.	*Uranus* T'	*Neptune* ϖ in Atmosph.	*Neptune* T'
						C °		C °
1.00	....	....	....	....	....	.....	....	.....
0.99	4.6876531 − 10	54635	503	96	125	543	157	625
0.98	5.4506015 − 10	316531	2918	556	722	1096	910	1263
0.97	5.9018620 − 10	894699	8246	1572	2040	1661	2574	1913
0.96	6.2253176 − 10	1884224	17367	3308	4296	2237	5420	2577
0.95	6.4786910 − 10	3376805	31124	5929	7700	2825	9713	3255
0.94	6.6877495 − 10	5464675	50368	9595	12461	3425	15719	3946
0.93	6.8661574 − 10	8240856	75957	14469	18791	4037	23705	4651
0.92	7.0222149 − 10	11804019	108799	20725	26916	4661	33954	5370
0.91	7.1610551 − 10	16250623	149784	28532	37056	5297	46745	6103
0.90	7.2863735 − 10	21636565	199886	38077	49451	5945	62381	6849
0.80	8.1446101 − 10	156467430	1442153	275356	356787	13105	450078	15098
0.70	8.6794610 − 10	536137160	4941631	941337	1222533	21448	1542196	24710
0.60	9.0702821 − 10	1318551200	12153220	2315083	3006640	30741	3792806	35334
0.50	9.3716965 − 10	2639437700	24327970	4634265	6018605	40576	7592330	46748
0.40	9.6047276 − 10	4513802000	41604170	7925235	10292650	50291	12983930	57940
0.30	9.7800711 − 10	6759055500	62298900	11867400	15412410	59105	19442380	68095
0.20	9.9029024 − 10	8968448000	82663100	15746600	20450400	66185	25797700	76251
0.10	9.9758125 − 10	10607851000	97773660	18625030	24188660	70782	30513420	81548
0.00	0.0000000	11215403000	103373500	19691755	25574040	72343	32261040	83193

The net results of this re-examination of the pressures in the Sun and major planets is therefore a considerable increase in the tabular pressures near the surfaces of these bodies and a sensible decrease in the pressures at their centres. And the effect is to augment the average rigidity of a body following the monatomic law when the integration for the pressure is extended to the whole mass.

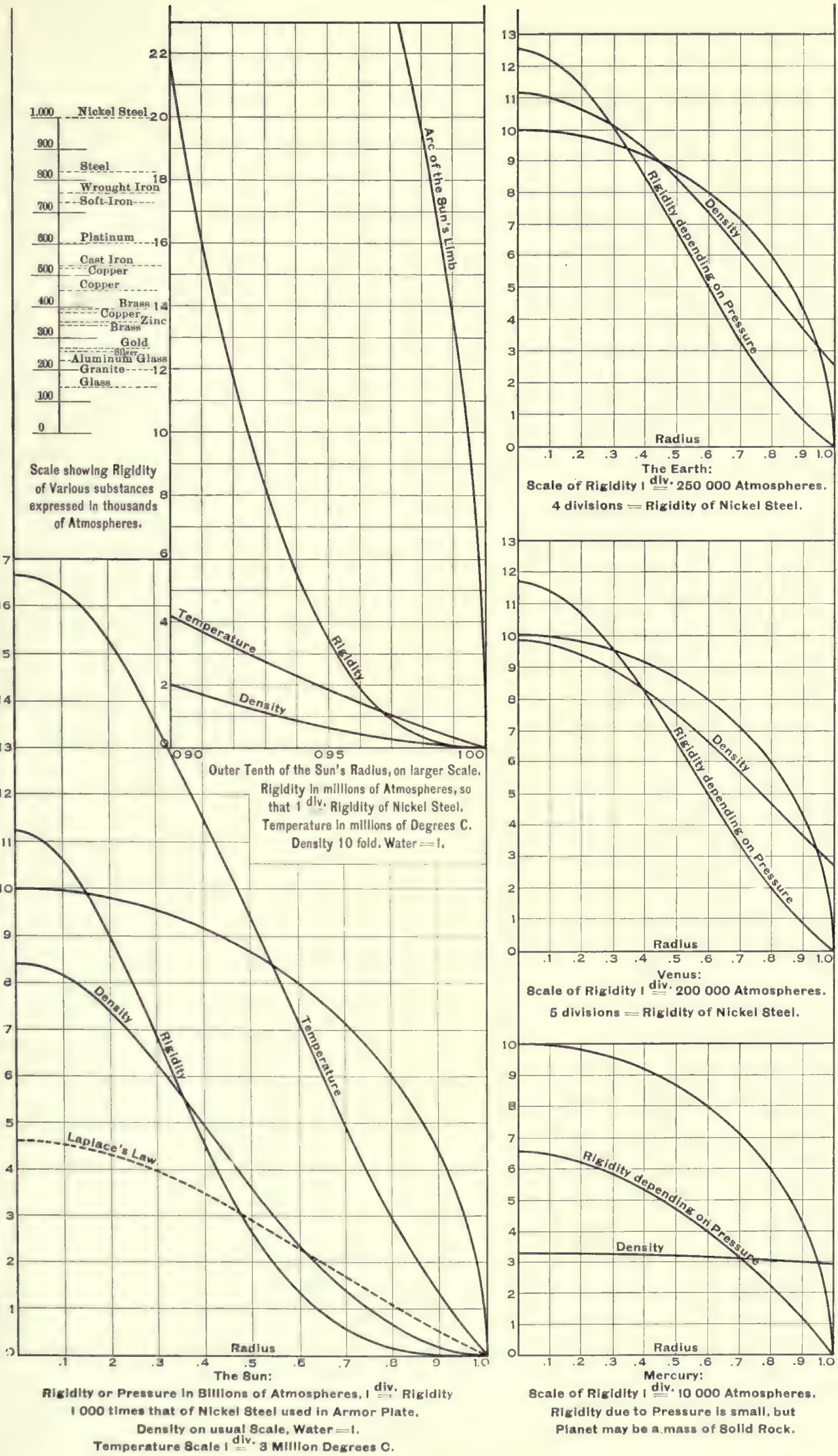

PLATE XXVI. PHYSICAL CONSTITUTION OF THE SUN AND OF THE PLANETS MERCURY, VENUS, AND THE EARTH.

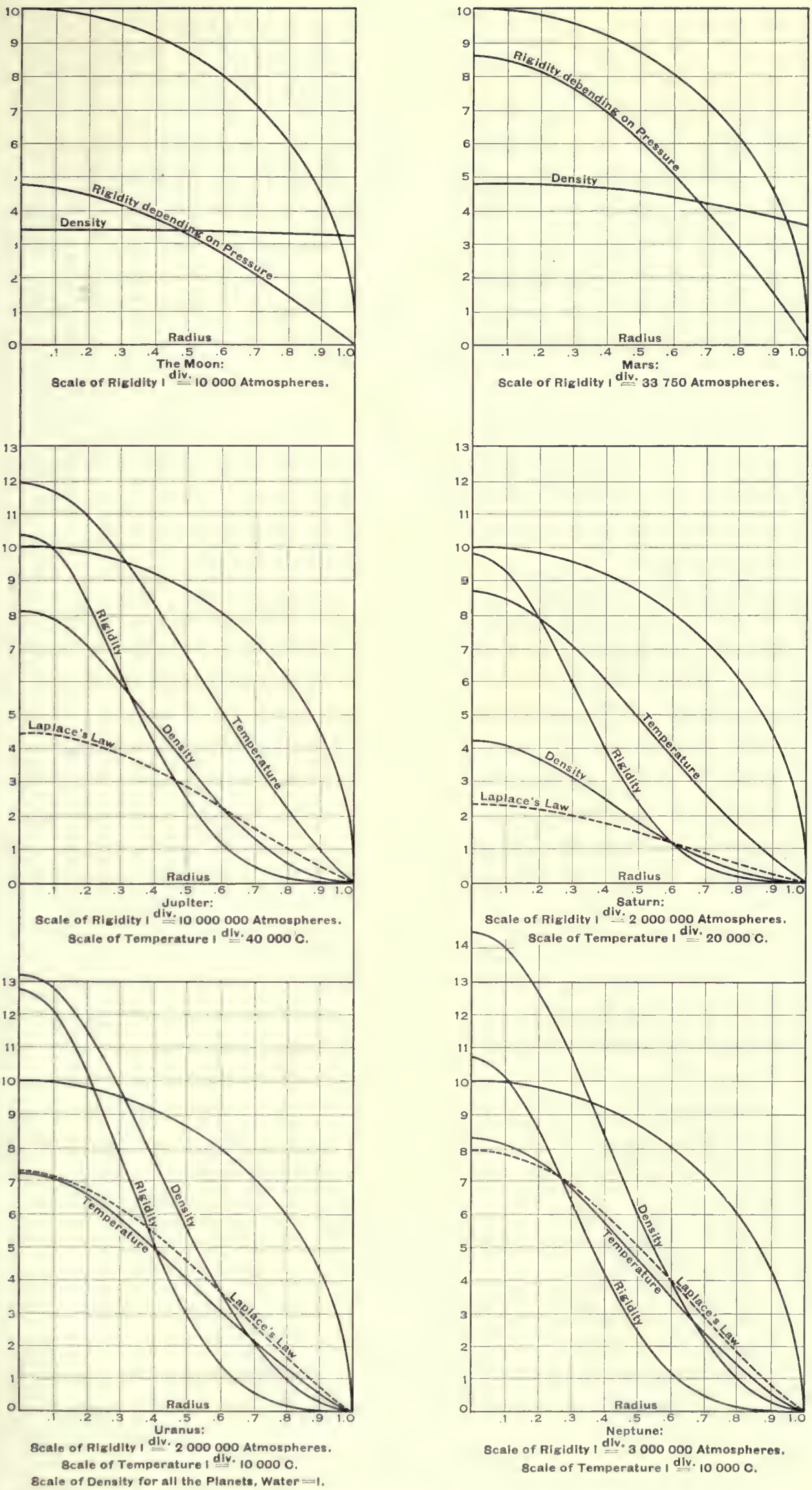

PLATE XXVII. PHYSICAL CONSTITUTION OF THE SUPERIOR PLANETS AND OF THE MOON.

Instead of a pressure of 8,000,000 atmospheres at a depth of one-tenth of the Sun's radius, the exact pressure at that depth resulting from the monatomic theory is 21,636,565 atmospheres, about 2.7 times the former value. Thus with an effective rigidity at that depth of nearly 22 times that of armor plate, it is quite inconceivable that the mass can ever be appreciably disturbed by external explosions such as are witnessed in the ejection of prominences. Indeed any theory which postulates convection currents in matter of this rigidity labors under difficulties altogether too great to be overcome; and thus the view that the Sun's heat and light are supplied in the main by direct radiation without any appreciable stirring of the rigid mass of flaming fluid acquires additional confirmation.

§ 220. *Theory of the Mean Rigidity of the Sun's Matter, as Distinguished from that of the Successive Concentric Spherical Layers Composing his Globe.*

In this investigation we consider not only the pressures acting upon the several layers of flaming fluid composing the body of the sun, but also the density or amount of matter per unit volume in each layer, so that in the final result any layer has a weight in proportion to its mass. It is therefore strictly a weighted mean of the rigidities of the several layers of the fluid composing the Sun's globe. We have already seen that the total element of pressure exerted upon any spherical shell distant x from the Sun's centre, and whose surface is therefore $4\pi\, x^2$, is

$$dP = 4\,\pi n \frac{3\,(\sigma_0 g)^2 x^2}{x'\,(\sigma_1 g)} \int_0^x \left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3} - \frac{x^3}{20} + \frac{x^5}{240} - \frac{x^7}{3888} + \;.\;.\;.\;\right] dx\,. \qquad (521)$$

The density of the matter in this layer of thickness dx is $\sigma = \sigma_0\left(\frac{\sigma}{\sigma_0}\right)$, and hence the element of pressure multiplied by the mass of the shell gives the following relation:

$$dP' = 4\,\pi n \frac{3\,(\sigma_0 g)^2 x^2 dx\, \sigma_0 \left(\frac{\sigma}{\sigma_0}\right)}{x'\,(\sigma_1 g)} \int_0^x \left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3} - \frac{x^3}{20} + \frac{x^5}{240} - \frac{x^7}{3888} + \;.\;.\;.\;\right] dx\,. \qquad (522)$$

By integrating this expression for the whole volume of the Sun, so as to include the product of the pressure in each layer of the Sun's globe by the corresponding mass of the layers, and dividing the resulting integral by $\frac{4}{3}\pi\sigma_1 x'^3$, which is the Sun's mass, we obtain the theoretical mean rigidity of the Sun's matter, on the hypothesis that the internal density conforms to the monatomic theory.

Accordingly

$$R' = \frac{P'}{\frac{4}{3}\pi\sigma_1 x'^3} = n\frac{9(\sigma_0 g)^2\sigma_0}{(\sigma_1 g)x'^4\sigma_1}\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x\int_0^{x}\left(\frac{\sigma}{\sigma_0}\right)\left[\frac{x}{3}-\frac{x^3}{20}+\frac{x^5}{240}-\frac{x^7}{3888}+\ .\ .\ .\right]\mathrm{d}x\,. \tag{523}$$

The series in the bracket under the last integral may be derived from the series called μ by dividing out the factor x^2, and hence the final form of the expression for the mean rigidity of the Sun's matter becomes

$$R' = n\frac{9(\sigma_0 g)^2\sigma_0}{(\sigma_1 g)x'^4\sigma_1}\int_0^{x'}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x\int_0^{x}\left(\frac{\sigma}{\sigma_0}\right)\frac{\mathrm{d}x}{x^2}\int_0^{x}\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x\,. \tag{524}$$

The integral of this expression involves nothing but x, which gives the distance of the matter from the Sun's centre, and $\left(\frac{\sigma}{\sigma_0}\right)$, which defines the internal distribution of density according to the principles of the monatomic theory as originally suggested by LANE and more fully developed in the paper on the "Physical Constitution of the Heavenly Bodies" (*A.N.*, 4053). Full account is taken of the gravitational forces acting on every atom of the Sun's mass; and when the integration is once completed, the resulting expression will give the mean rigidity of the Sun's matter with all the rigor of mathematical analysis.

§ 221. *Evaluation of the Triple Integral, with Application of the Resulting Formula to the Sun and Planets.*

We shall now show how to evaluate the above triple integral, and shall eventually apply the resulting formula to the Sun and major planets on the hypothesis that they are essentially globes of gaseous matter in the monatomic state. The last integral is the series called μ, made up of 25 terms, and as an equal number of terms is included under each of the other integrals, through the recurrence of the series $\left(\frac{\sigma}{\sigma_0}\right)$, the final value of the triple integral involves the summation of all that are sensible of some 15,000 terms. From the nature of the series $\left(\frac{\sigma}{\sigma_0}\right)$, however, it is easy to see that only a fractional part, which proves to be a little over one-third, of these elements will become sensible in units of the eighth place of decimals; and we are thus required to calculate only about 6,000 terms. But as the final numerical value of the triple integral, after the factor x'^5 is divided out, is very small, the work must be abbreviated as much as possible, and checked with the greatest care.

Denoting the coefficients of the μ series by $a_1, a_2, a_3, \ldots a_i$, and those of the $\left(\frac{\sigma}{\sigma_0}\right)$ series by $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i$, it is easy to see that the elements under the second integral of Equation (524) by multiplication give

$$\left(\frac{\sigma}{\sigma_0}\right)\frac{1}{x^2}\int_0^x\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x = \left\{\begin{array}{l} \alpha_1 a_1 x + \alpha_1 a_2 x^3 + \alpha_1 a_3 x^5 \quad + \ldots + \alpha_1 a_i x^{2i-1} \\ \alpha_2 a_1 x^3 + \alpha_2 a_2 x^5 + \alpha_2 a_3 x^7 \quad + \ldots + \alpha_2 a_i x^{2i+1} \\ \alpha_3 a_1 x^5 + \alpha_3 a_2 x^7 + \alpha_3 a_3 x^9 \quad + \ldots + \alpha_3 a_i x^{2i+3} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_i a_1 x^{2i-1} + \alpha_i a_2 x^{2i+1} + \alpha_i a_3 x^{2i+3} + \ldots + \alpha_i a_i x^{4i-1} \end{array}\right\} i \gtrless 1 \qquad (525)$$

This may be written

$$\left.\begin{array}{l} \left(\frac{\sigma}{\sigma_0}\right)\frac{\mu}{x^2} = \alpha_1 a_1 x + (\alpha_1 a_2 + \alpha_2 a_1)x^3 + (\alpha_1 a_3 + \alpha_2 a_2 + \alpha_3 a_1)x^5 + (\alpha_1 a_4 + \alpha_2 a_3 + \alpha_3 a_2 + \alpha_4 a_1)x^7 \\ \quad + \ldots + (\alpha_1 a_i + \alpha_2 a_{i-1} + \alpha_3 a_{i-2} + \alpha_4 a_{i-3} + \ldots + \alpha_i a_1)x^{2i-1}. \end{array}\right\} \qquad (526)$$

Integrating this last expression the two integrals of the Equation (524) become

$$\left.\begin{array}{l} \int_0^x\left(\frac{\sigma}{\sigma_0}\right)\frac{\mathrm{d}x}{x^2}\int_0^x\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x = \alpha_1 a_1\frac{x^2}{2} + (\alpha_1 a_2 + \alpha_2 a_1)\frac{x^4}{4} + (\alpha_1 a_3 + \alpha_2 a_2 + \alpha_3 a_1)\frac{x^6}{6} \\ \quad + (\alpha_1 a_4 + \alpha_2 a_3 + \alpha_3 a_2 + \alpha_4 a_1)\frac{x^8}{8} + \ldots + (\alpha_1 a_i + \alpha_2 a_{i-1} + \alpha_3 a_{i-2} + \ldots + \alpha_i a_1)\frac{x^{2i}}{2i}. \end{array}\right\} \qquad (527)$$

The coefficients of this series have been found by the actual calculation of 338 terms to be as follows, the brackets indicating logarithms:

$$\begin{aligned} \int_0^x\left(\frac{\sigma}{\sigma_0}\right)\frac{\mathrm{d}x}{x^2}\int_0^x\left(\frac{\sigma}{\sigma_0}\right)x^2\,\mathrm{d}x = \; & [9.2218487]_{-10}x^2 - [8.5228787]_{-10}x^4 + [7.6432701]_{-10}x^6 - [6.6445383]_{-10}x^8 \\ & + [5.5608210]_{-10}x^{10} - [4.4131494]_{-10}x^{12} + [3.2104464]_{-10}x^{14} - [1.9777828]_{-10}x^{16} \\ & + [0.7070485]_{-10}x^{18} - [9.4087219]_{-20}x^{20} + [8.0876769]_{-20}x^{22} - [6.7448748]_{-20}x^{24} \\ & + [5.3850897]_{-20}x^{26} - [4.0095923]_{-20}x^{28} + [2.6200107]_{-20}x^{30} - [1.2176798]_{-20}x^{32} \\ & + [9.8036980]_{-30}x^{34} - [8.3789913]_{-30}x^{36} + [6.9443442]_{-30}x^{38} - [5.5004313]_{-30}x^{40} \\ & + [4.0478336]_{-30}x^{42} - [2.5870619]_{-30}x^{44} + [1.1185636]_{-30}x^{46} - [9.6427375]_{-40}x^{48} \\ & + [8.1599383]_{-40}x^{50} - [6.6705280]_{-40}x^{52} + \ldots \end{aligned} \qquad (528)$$

By this process of condensation 338 terms are reduced to a series of 26 terms; and when the product is again taken by the series $\left(\frac{\sigma}{\sigma_0}\right)$, the resulting square of 676 terms represent 8,788 of the original elements of the triple integral.

Of the 676 terms it was found that 519 are sensible in units of the eighth place of decimals, thus making a total of 5,932 of the original 15,000 elements actually taken into account in the final value of the triple integral.

The work of summing these 519 terms was carefully checked, and after x'^5 was divided out, and the terms of the square numerically evaluated for $x = x' = 3.653962$, the final value of the triple integral was found to be

$$\left.\begin{aligned} &\int_0^{x'} \left(\frac{\sigma}{\sigma_0}\right) x^2 \, \mathrm{d}x \int_0^{x} \left(\frac{\sigma}{\sigma_0}\right) \frac{\mathrm{d}x}{x^2} \int_0^{x} \left(\frac{\sigma}{\sigma_0}\right) x^2 \, \mathrm{d}x = x'^5 S' \\ &\text{where} \qquad S' = 0.0011915, \quad \log S' = 7.0760940 - 10\,. \end{aligned}\right\} \tag{529}$$

Hence the mean rigidity of the Sun's matter becomes

$$R' = n \frac{9\,(\sigma_0 g)^2 \sigma_0 x' S'}{(\sigma_1 g)\,\sigma_1} = 6{,}101{,}430{,}000 \text{ atmospheres.} \tag{530}$$

The resulting mean rigidity of the Sun's matter, 6101.43 times that of nickel steel, is sufficiently remarkable to excite the astonishment of astronomers who may have been accustomed to viewing the Sun's globe as a soft mass of gas. Yet it is not too much to say that while few results in the whole domain of Physical Science fill one with greater wonder than this, probably none have been deduced by more rigorous processes than those here employed; which are based on the law of density as defined by the principles of the monatomic theory, and determined with great mathematical accuracy in *A.N.*, 4053. In a mass of which the matter has such a great average rigidity as that of our Sun, it is impossible to believe that convective processes, even if established, could continue any length of time, and accordingly this result seems effectively to disprove the general theory of solar convective currents as ordinarily held by astronomers.

As in the case of formula (514) for the mean rigidity of the layers, so also here, formula (530) gives the average rigidity of the matter of any star of known mass and dimensions, and is likely to be extremely useful in various cosmical investigations. Thus when the solar nebula extended to the orbit of the Earth, the average rigidity of the matter was less than that of our present Sun in the ratio of $6.(215)^4$ to unity, or 1 : 12820500000 of its present value. The criteria established by formula (514) and (530) ought to enable us to understand the mean state of condensing masses at any stage of their history, and thus afford safe guides for the interpretation of observed phenomena in the case of some of the stars and nebulae, when accurate data become available for the use of the investigator.

When the above formula (530) is applied to the major planets the results are as follows:

Planet	Mean Rigidity of the Matter R'
Jupiter	56237501 atmospheres
Saturn	10712580 "
Uranus	13916028 "
Neptune	17550730 "

In concluding this investigation we may observe that the value of $S' = 0.0011915$, calculated in Equation (529) is almost exactly one-half that of S as found independently by numerical calculation in Equation (513), which gives $\frac{1}{2} S = 0.00118802$. As the forms of these two equations otherwise are identical, except for the factor $\frac{\sigma_0}{\sigma_1} = 6$, occurring in (529), the result is that for the Sun considered as a sphere of monatomic gas $R' = 3 R$; and we are led to the interesting law that the mean rigidity of all the matter of such a globe is always exactly three times that of the layers in which it is arranged. We shall not stop here to inquire into the cause of this remarkable result, as depending on the nature of the preceding double and triple integrals, beyond remarking that the close agreement in the above values of S' and $\frac{1}{2}S$, shows the extreme accuracy of the numerical work. If self-luminous bodies of large size are globes of monatomic gas, as astronomers have abundant reasons to suppose, this simple physical law will no doubt be applicable generally to myriads of fixed stars constituting the sidereal universe.

§ 222. *On the Gravitational Tenacity of the Heavenly Bodies.*

The tenacity, or breaking force, required to pull the two hemispheres of a planet apart against gravity affords another useful estimate of the strength of the various bodies of our system. This is calculated on the hypothesis that the mass is fluid, and that forces can be applied in such a way as to produce division along a diametral plane. To determine this tenacity of a planet we have only to find the mean gravitational pressure on a diametral section, and compare the result to the tenacity of the corresponding materials as found by terrestrial experiments. The following table gives the tenacity of various substances expressed in atmospheres, and also the calculated tenacity of the several bodies of our system. In view of the preceding discussion of the rigidities, the method of calculation employed in finding the tenacity of the planets scarcely requires explanation. In any infinitely narrow zone of the diametral section of a body like the Earth, of width $r\,\mathrm{d}x$, the pressure is $p\,2\,\pi\,r\,x\,r\,\mathrm{d}x$, and for the whole diametral section, the general expression becomes

$$P = 2\pi r^2 \int_0^x px\,dx\,, \tag{531}$$

which admits of integration for the several laws of pressure corresponding to the different laws of density given in the foregoing equations. For our present purposes, however, it has been thought sufficient to use the numerical data on the internal pressures already published in *A.N.*, 3992, and those calculated in Table E of the present paper, and simple quadrature of the form corresponding to the weighted mean,

$$\Pi = \frac{\sum_{i=1}^{i=i} P_i}{\pi r^2} = p_1 x_1^2 + p_2(x_2^2 - x_1^2) + p_3(x_3^2 - x_2^2) + \ . \ . \ . + p_i(x_i^2 - x^2_{i-1})\,, \tag{532}$$

where $p_1, p_2, p_3, \ldots p_i$ are the mean pressures for the successive shells into which the sphere is divided, and $x_1, x_2, x_3, \ldots x_i,$ the respective fractions of the radius. As the pressures are already given for each tenth of the radius, the expressions $p_2\,(x_2^2 - x_1^2)$, $p_3\,(x_3^2 - x_2^2)$, . . . , &c., are simple products by differences of these squares, and thus easily evaluated with all necessary accuracy.

The great tenacity of the larger heavenly bodies affords little indications of the probability of the disruption of a planet by an explosion due to any combination of elements such as is likely to occur in actual nature. The hypothesis proposed by Olbers for the explanation of the Asteroids, and long since abandoned by astronomers, was really too improbable for serious consideration.

The gravitational tenacity of a planet becomes small for a small mass. But to reduce it to equality with the adhesive force of an equal globe of sandstone, for example, the diameter of a mass of the same density as the Earth could not exceed 100 kilometres; and in the case of a substance having the same density (5.5) but a tenacity equal to that of steel the diameter would be approximately 1280 kilometres.

The strength of a large mass of stone, as a mountain of granite, or a small satellite, like those of *Mars*, depends almost wholly on cohesion, and may have any resisting power appropriate to the given substance. The bodily strength of the planets, and the Sun, and stars, on the other hand, depend almost entirely on the mutual gravitation of their parts, the cohesion becoming relatively insensible, especially when the masses are made up of compressed gases at enormously high temperatures. From these considerations it seems difficult to imagine conditions which could endanger the stability of a planet or star when it has once attained any considerable size. Probably nothing short of a violent bodily collision with

another hard dense mass would produce a decided disturbance of the planet's figure.

TABLE F. THE TENACITY OR BREAKING STRENGTH OF VARIOUS SUBSTANCES, BASED ON LORD KELVIN'S TABLES IN THE ARTICLE "ELASTICITY," *Encycl. Brit.*, NINTH EDITION, AND LATER AUTHORITIES.

Substance	σ Density	τ Tenacity in Atmosph.
American Nickel Steel pianoforte wire	7.760	30000
British Steel pianoforte wire	7.727	22859
American Nickel Steel, drawn	7.740	18707
Steel Wire, drawn	7.718	9591
Steel Cast, drawn	7.717	8110
American Aluminum Bronze, rolled (10% Al., 90% Cu.)	8.30	6803
American Bronze Manganese, rolled	8.85	6803
Iron Wire, Common	7.553	6300
Copper Wire, drawn	8.933	3968
Platinum Wire, fine	21.166	3387
Brass Wire	8.383	3319
Silver, drawn	10.369	2865
Gold, drawn	18.514	2748
Palladium	11.35	2632
Bronze	8.660	2439
Zinc, Common, drawn	7.008	1529
Ivory	1.917	1188
Tin, Cast	7.404	403
Glass	3.329	391
Lead	11.215	213
Marble, Italian white, Statuary	2.837	70
Sandstone, Connecticut Red	2.50	40
" Scotch (according to TAIT)	2.41	34
" Missouri, hard	2.40	30
" Missouri, soft	2.30	10

TABLE G. CALCULATED TENACITY OF BODIES OF THE SOLAR SYSTEM.

Body of the Solar System	τ Average Tenacity in Atmospheres Due to the Attraction of Gravitation	N_1 Number of Times Greater than the Adhesive Force of an Equal Globe of Solid Red Sandstone, $\tau = 40$, the Stone Being Supposed Devoid of Gravitation Between Its Parts	N_2 Number of Times Greater than the Adhesive Force of an Equal Globe of Solid Steel with Tenacity of the Strongest Pianoforte Wire, Taken at 30,000 Atmospheres, the Steel Globe Being Supposed Devoid of Gravitation for Itself
Mercury	31358	784.9	1.0453
Venus	832490	20812.2	27.7497
Earth	1068188	26704.7	35.6063
Mars	131030	3274.8	4.3677
Moon	16208	405.2	0.5403
Jup. Sat. I	12696	317.4	0.4232
" " II	13336	333.4	0.4445
" " III	20911	522.8	0.6970
" " IV	1924	48.1	0.0641
Titan	9790	244.7	0.3263
Jupiter	17664880	441622	588.8
Saturn	3370610	84265	112.3
Uranus	4357660	108940	145.2
Neptune	5524800	138120	184.1
Sun	1919738000	47993450	63991

§ 223. *General Considerations on the Theory of the Sun and Stars.*

Many important problems are suggested by the theory of the physical constitution of the heavenly bodies, but it is obvious that only a few of them can be touched upon here. In the first place it should be remarked that some of the views set forth in *A.N.*, 4053, have been more or less anticipated by PROFESSOR SCHUSTER of Manchester, who also adopted the monatomic theory in a thoughtful address of wide range delivered to the Royal Philosophical Society of Glasgow, November 6, 1901, and published in the *Proceedings* of that learned Society. At the time of composing the paper in *A.N.*, 4053, the writer was not aware of this address. The views set forth by PROFESSOR SCHUSTER do not in general differ widely from those held by the writer, except on some particular points. For a complete exposition of his views on the numerous problems under review the reader must be referred to his paper "On the Evolution of the Solar Stars." He says: "As it is exceedingly likely that at the temperature of the Sun and stars, all molecules are monatomic like mercury vapor, we may base our calculations on that assumption;" and he calculates some tables for the internal density, pressure and temperature of the Sun, based on the general principles of convective equilibrium as developed by LANE, RITTER and LORD KELVIN. PROFESSOR SCHUSTER assumes the constancy of the atomic weights of the elements as found by chemical experiments, and makes the internal temperature depend upon the kind of atoms present, inclining to the view that the centre of the Sun is composed mainly of vapor of iron in the monatomic state. He adds: "It is a curious fact, which is not perhaps without significance, that the central density of the Sun, as calculated on the assumption of its being a perfect gas, is only very little in excess of the density, which, according to the most careful recent estimates is to be ascribed to the central portion of the Earth, and again that that estimate is very little in excess of the density of solid iron." He thus seems inclined to favor an iron nucleus for the Sun and Earth.

The writer has elsewhere (*A.N.*, 3992, 4053) assigned reasons for distrusting this popular hypothesis; and at present it will perhaps be sufficient to call attention to the great rigidity of all these cosmical masses long before the existing stage is reached in the general process of their secular condensation. Such great rigidity of the matter would very effectively prevent a separation of the elements according to their atomic weights, and it is difficult if not impossible to see how distinctively metallic nuclei could form in any of these masses.

In *A.N.*, 4053, no assumption was made as to the nature of the atoms at different depths in the Sun's mass, nor were their atomic weights taken to be

constant and invariable under the fierce conditions of temperature and electric excitation* prevailing in the Sun's interior.

Attention has been called to the fact that one is not at liberty to impose upon the atoms of the nucleus temperatures resulting from the law of density combined with the Sun's observed effective temperature, regardless of the weight of the atoms. But to those who consider the observed effective temperature well established, and the atomic weights within as probably dependent upon the temperature and the conditions of electric excitation, and therefore not necessarily constant under all the extreme conditions prevailing in the Sun's globe (a view also much emphasized by SIR NORMAN LOCKYER), the procedure in *A.N.*, 4053, will appear sound and unobjectionable. Which of these alternative methods is the correct one time alone can decide; but in the present state of our knowledge nothing seems more certain than that dissociation will eventually become a well established fact.

Some writers on cosmical physics appear to consider that the liquid state of matter is often met with in the heavenly bodies, and that when masses once reach that condition, they are henceforth practically incompressible, cease to contract, cool down and soon become non-luminous. From the considerations adduced in Chapter XVII, §212, pp. 487–492, regarding the secular accumulation of heat within the stars during the gaseous stage, it is impossible to entertain this view seriously.

On account of the enormous temperatures thus developed it seems probable that liquids play a very small part in the physics of stars which are still self-luminous, such as those composing the visible universe; and therefore we need not dwell upon the problems suggested. But one very common remark of Physicists, that all liquids are practically incompressible, must not be allowed to pass unnoticed. As measured with imperfect apparatus under the small forces available to the experimenter in the Laboratory this is no doubt approximately a fact of observation; but even if liquids existed universally in nature it would not hold true under the enormous forces operating in the interior of the heavenly bodies. Indeed it seems certain that all matter, whether solid, liquid or gaseous, under such stupendous forces would yield like a sponge; because it is generally admitted that the intervals separating the particles are always perfectly enormous in comparison with their dimensions.

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* In speaking of the dynamics of the electron in his address on "Mathematical Physics" at the St. Louis Congress of 1904 POINCARÉ says: "It is the motion of electrons that produces the lines of the spectrum; this is proved by the phenomenon of ZEEMANN; what vibrates in an incandescent body is affected by a magnet, and is hence electrified. This is a very important point; but no one has gone into the question any further." Translation by PROF. J. W. YOUNG, *Bulletin Am. Math. Soc.*, February, 1906, p. 258.

In his address on "Mathematical Physics" delivered at the St. Louis Congress of Arts and Sciences in 1904, POINCARÉ says:*

"The astronomical universe consists of masses, undoubtedly of great magnitude, but separated by such immense distances that they appear to us as material points; these points attract each other in the inverse ratio of the squares of their distances, and this attraction is the only force which affects their motion. But if our sense were keen enough to show us all the details of the bodies which the physicist studies, the spectacle thus disclosed would hardly differ from the one which the astronomer contemplates. There, too, we should see material points separated by intervals which are enormous in comparison with their dimensions, and describing orbits according to regular laws. Like the stars proper, they attract each other or repel, and this attraction or repulsion, which is along the line joining them, depends only on distance."

In his recent Presidential Address to the British Association for the Advancement of Science at Cape Town, SIR GEORGE DARWIN discusses the theory of electrons, and continues: †

"I have not as yet made any attempt to represent the excessive minuteness of the corpuscles, of whose existence we are now so confident; but, as an introduction to what I have to speak of next, it is necessary to do so. To obtain any adequate conception of their size we must betake ourselves to a scheme of threefold magnification. LORD KELVIN has shown that if a drop of water were magnified to the size of the Earth the molecules of water would be of a size intermediate between that of a cricket ball and of a marble. Now each molecule contains three atoms, two being of hydrogen and one of oxygen. The molecular system probably presents some sort of analogy with that of a triple star; the three atoms, replacing the stars, revolving about one another in some sort of dance which cannot be exactly described. I doubt whether it is possible to say how large a part of the space occupied by the whole molecule is occupied by the atoms; but perhaps the atoms bear to the molecule some such relationship as the molecule to the drop of water referred to. Finally, the corpuscles may stand to the atom in a similar scale of magnitude. Accordingly, a threefold magnification would be needed to bring these ultimate parts of the atom within the range of our ordinary scales of measurements." . . .

"The community of atoms in water has been compared with a triple star, but there are others known to the chemist in which the atoms are to be counted by fifties and hundreds, so that they resemble constellations."

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* cf. *Bulletin Am. Math. Soc.*, February, 1906, p. 241; authorized translation by PROFESSOR J. W. YOUNG.

† cf. *Science*, August 25, 1905, pp. 233-34.

The passages here cited from two of the most eminent authorities afford us conclusive proof that all matter is compressible to an almost indefinite extent by sufficiently great forces; and accordingly it appears that under the stupendous forces operating in the interior of the heavenly bodies nothing is more certain than the total failure of all the experimental laws of physics relating to the incompressibility and the impenetrability of matter.

Accordingly we conclude that while the usual compressibility of matter is greatly obstructed by molecular forces as the density increases, and decided discontinuity intervenes at the solid and liquid stage, when only small forces are applied; yet in view of the smallness of the particles in comparison with the spaces which intervene between them, no doubt increasing pressure always augments the condensation, in spite of the powerful molecular repulsion, which MAXWELL showed in some cases to vary inversely as the fifth power of the distance. The limit of condensation is usually fixed by the very high temperatures prevailing in the larger heavenly bodies. In the same way the pressure in cosmical masses is generally sufficient to ensure complete interpenetrability of all the elements, especially at the enormous temperatures operating in the interior of the Sun and stars.

§ 224. *The Effects of Great Rigidity Upon the Circulation of the Matter of Cosmical Globes.*

It is well known that while the observation and photography of the Sun have been greatly extended of late years, very little progress has been made in the theory of the Sun, and the extension of this subject is therefore of high interest to all contemporary investigators.*

To photograph the outside surface of the Sun, in the hope of deducing the under-lying laws of the internal movement, is a good deal like photographing the wave-swept surface of the sea, far from the land, with a view of finding the laws of the tides and other currents of the ocean. For the surface of the Sun is in a state of incessant change, and the instantaneous appearance of the photosphere does not indicate the nature of the movement beneath, any more than the billows of the open sea will disclose the laws of the ocean currents.

It may be assumed that there is some continuity in the phenomena witnessed on the Sun's surface, but the order of events is difficult to establish, because the changes noted cannot be interpreted without an approximate understanding of

* The first part of this Section is based on a paper entitled "Results of Recent Researches on the Physical Constitution of the Sun," prepared at the request of PROFESSOR A. RICCO of Catania, and published in the *Memorie della Società degli Spettroscopisti italiani*, Vol. XXXVII, *anno*, 1908.

the underlying physical conditions, which heretofore have been very little studied, owing to the fact that there has been no adequate method for attacking the problem. Having already developed the principles of the monatomic theory, we may again recur to the interpretation of the observed phenomena.

In *A.N.*, 3992, we have developed the theory of the internal density and pressure within the principal bodies of the solar system, on the hypothesis that the density conforms to LAPLACE's celebrated law for the density of the Earth, which naturally is somewhat modified in its application to other masses of different physical constitution. While this gives a better approximation to the truth than the simple hypothesis of uniform density, and proves to be satisfactory for the encrusted planets; yet the result is found to be unsatisfactory for the Sun and major planets, which are masses at high temperature, and certainly of small density near the surface, and therefore of correspondingly greater density in their central parts than is indicated by the Laplacian law.

On several grounds it may be held that the monatomic theory, according to which the heavenly bodies are gaseous masses reduced by their enormous temperature to the state of single atoms, gives the closest known approximation to the true condition of the Sun and main body of the fixed stars. If any departure from this simple theory takes place, it will be chiefly in the outer layers, where the physical conditions change rapidly, and entire continuity under the conflict of antagonistic forces can scarcely be expected. In the neighborhood of the photosphere and chromosphere, the fluid is subjected to great velocity of movement, while the force of gravity is such as to produce an abrupt change in the pressure; on the other hand the fall of temperature is exceedingly rapid as the upper limits of the photosphere is approached, and the radiation pressure of the Sun's light is so intense as to largely if not entirely overcome the force of gravity. Under the circumstances the discontinuous physical conditions existing at the Sun's surface necessarily operate to modify the full validity of ordinary physical laws. But while this takes place in the surface layers of the Sun, it can hardly be true in any of the deeper portions of that globe, where there is no sudden transition, as at the boundary of the sphere.

In regard to the significance of these results, it suffices to point out that above the critical temperature gaseous matter passes into the solid state under the effects of pressure alone, without going through the intermediate liquid state ordinarily observed at common temperatures.

Hence even with the enormous temperature of the Sun, the whole interior of that immense globe is under such pressure that the matter in confinement behaves as a solid. The nucleus of the Sun is therefore a globe of enormous effective

rigidity. At what depth the matter acquires the property of a solid is not certainly known, but it is impossible to believe that the layer which behaves as a gas can have a depth equal to one-tenth of the radius; for the pressure at that depth is about twenty-two million atmospheres. Matter sustaining such pressure must be effectively solid; and if the rigidity is proportional to the pressure, as may be inferred from physical laws which are observationally verified in the case of the Earth, the Sun's matter at a depth of 0.10 of the radius must be about twenty-two times more rigid than nickel steel used in armor plate.

Accordingly only the outermost layers behave as gaseous matter, while the whole nucleus is rendered effectively solid and highly rigid by pressure. And as convection currents could not be maintained in a mass reduced to the state of a rigid solid, it follows that convection currents can exist only in the outermost layers of the Sun's globe.

Heretofore it has been very generally assumed by astronomers that convection currents extend down to the centre, and thus produce a steady transfer of heat from the interior outward, so as to maintain the enormous surface luminosity of the Sun's disc. But it is obvious that researches for the interpretation of phenomena noticed at the Sun's surface, to be of any effect, should be guided by the general principles here laid down. Before we can make progress in the treatment of this problem we must base our reduction and discussion of the observed phenomena upon the monatomic theory, or something better, and proceed by successive approximations to form a tenable theory of the radiation and fluid movements of the Sun's surface, which will explain observed phenomena in accordance with admissible underlying conditions.

The first need of solar physics to-day is therefore the theoretical study of the radiation of a globe such as we have shown the Sun to be. An effort might be made to trace the observed movements to their appropriate depths. This would give us the true theory of spots, periods, solar drift and kindred phenomena observed on the Sun's disc, and afford the observer a clue to the periodicity and distribution of the eruptions noticed in the photosphere.

The results established in this chapter appear to bring to light in a clear and definite way the forces of resistance which must necessarily array themselves against any system of currents imagined to exist in the depths of cosmical masses. Great pressure always produces great effective rigidity of the matter, and any motion of matter under such strain would be accompanied by the most enormous molecular friction. It follows therefore that in general when the pressure is very great, enormous resistance would be exerted against currents already existing, and would tend to bring them rapidly to rest; so that the continuation of the motion

would presuppose the constant exertion of correspondingly great forces to overcome the friction due to the pressure. As the pressure increases rapidly with the depth in all large masses of moderate density, it seems almost inconceivable that rapidly moving currents could descend to any appreciable depth, and hence all currents observed externally might be expected to be shallow, and to die out rapidly as they descend into the mass.

These considerations strongly support the views set forth in *A.N.*, 4053, that the Sun's supply of light and heat is maintained by direct radiation from beneath rather than by a system of convection currents descending to great depths against the frictional resistance of the surrounding matter. Not only is the resistance to such supposed motion very great, but it is shown that the gases in the outer parts of the Sun's globe are sufficiently rare to admit of a very powerful direct radiation without bodily motion of the flaming fluid, which is also rendered perfectly transparent by the enormously high temperatures. Moreover it is difficult to conceive of a regular system of double tubes in the continuous body of the Sun, with the hot fluid ascending in one set and the materials cooled by radiation descending in the other. Indeed it seems almost impossible to imagine such a system of currents at work against the resistance inevitably arising from the contrary motions of such neighboring streams. Neither can one see how such a mutually obstructing and antagonistic series of currents could be maintained if once established. Obviously they would all very soon die out, and in view of the proved effective rigidity of the matter it is impossible to see how they could ever be re-established by any natural process.

When one reflects that direct radiation does away with this highly complex hypothetical system of movement, while at the same time it affords an adequate supply of energy from below, the conclusion seems irresistible that the general conception of convection currents sinking to great depths in large masses of high average rigidity is a mistaken one; and that the simpler process of direct radiation is the general law of nature. Convection currents might indeed exist to some extent in the very shallowest of the photospheric layers of the Sun and stars, and also in nebulae of small average rigidity, but surely not in the interior of well developed stars having an average effective rigidity thousands of times greater than that of armor plate.

The rapid outflow of energy in the case of a typical star would seem to be mainly from the surface layers, though a smaller amount of energy comes up more slowly from greater depths; and no doubt some process of radiation permits the vibrations to be transmitted or exchanged through the intervening medium without bodily motion of matter effectively thousands of times

more rigid than nickel steel, though at a temperature of many millions of degrees.

In the paper "On the Evolution of Solar Stars" previously cited, but unknown to the writer at the time of preparing the views set forth in *A.N.*, 4053, it appears that PROFESSOR SCHUSTER also has given some consideration to the effects of direct radiation, though on the whole he seems to adhere to the system of convection currents. He says: "It may not be unnecessary to say a few words as to the evidence we possess that the convection currents which play so important a part in the theoretical investigation actually exist in the Sun. The surface radiates an amount of heat into space of which we can form a very fair estimate by measuring the quantity which reaches the Earth. The number so obtained is 1,340 million per square metre of the solar surface, the unit of heat being the amount necessary to raise one gramme of water through one degree. We obtain an idea of what that number means if we imagine the Sun to be surrounded by a shell of ice, the heat supplied by radiation could melt in each minute a layer of ice fifty-eight feet thick. Or expressing it with LORD KELVIN in terms of power, we may say that the solar surface does work by radiation equivalent to 131,000 horse-power for each square metre of his surface. The heat thus lost by radiation must be supplied from the inside of the Sun, otherwise the solar surface would cool down in a fraction of a second to a temperature at which it would cease to be luminous. If the heat is carried from the inside to the outside by convection alone, the velocity of the currents of vapor must be very great. Taking the pressure of the vapor near the surface to be one atmosphere, we may say that all the heat contained in a layer having a thickness of 370 metres is lost by radiation in each second of time, and this number does not depend on the nature of the vapor or on its temperature. A layer of that thickness would have to be replaced by convection in every second if the temperature of the surface is to be maintained. From this I calculate that if the difference in pressure between the descending and ascending currents is one atmosphere, the velocity of the convection currents must be 616 metres per second, or about 1,000 miles per hour. These up and down draughts of vapor must take place with the calculated velocity, unless an appreciable portion of the heat is supplied from the inside in some other way, as for instance by radiation. It is difficult to form an estimate as to how far radiation can help to keep up the temperature of the surface. I have made some calculations on this point which, though they have yielded interesting results, cannot at present be expressed in definite numbers. It is sufficient for the present argument to maintain, that even if radiation takes a prominent part in the determination of the distribution of temperature, we cannot escape the conclusion that

convective currents must bring about a continuous interchange of matter between the inside and outside of the Sun. This theoretical conclusion is amply confirmed by observation."

Again SCHUSTER adds: "If convection currents could be completely stopped, the heavier gases would sink to lower levels, and the outer layer of a star would be made up of hydrogen and the lighter metallic vapors. It is owing to convection that a mixing takes place, and the stronger the convection the more complete the mixing."

A particular line of argument will naturally appeal with different force to different investigators, and some may still feel that the subject of cosmical circulation is open to further research; but to most minds the argument drawn from the theory of the rigidity of the Sun's matter will no doubt seem absolutely conclusive against the general doctrine of convection currents as applied to well developed stars.

Conclusions Regarding the Constitution of the Sun and Stars.

(1) The Sun viewed as a globe of monatomic gas, has a mean rigidity of all its layers of more than 2027 times, and a mean rigidity of all its matter of 6101 times that of nickel steel. The variation of the rigidity in the different layers is sufficiently exhibited by the accompanying table of pressures, or by the curve of rigidity drawn in the plate. These results enable one to see that the principal tidal movements in a star like the Sun would be chiefly of a superficial character, and that great bodily distortion would occur only in bodies of small mass, or in large masses still greatly expanded, and thus approaching the nebular condition.

(2) The mean rigidities of the layers of *Jupiter*, *Saturn*, *Uranus* and *Neptune* turn out to be respectively 18.691010, 3.56047, 4.624053, and 5.833133, and the mean rigidities of their matter, measured by the same unit, 56.237501, 10.712580, 13.916028, 17.550730 times that of nickel steel.

(3) Of the four inner planets, the Earth unquestionably has decidedly the highest rigidity. It certainly exceeds that of Bessemer steel, and probably is but little inferior to that of the nickel steel used in armor plate and sometimes encountered in iron meteorites observed to fall from the heavens.

(4) *Venus* is found to have a rigidity certainly greater than that of Platinum; and if that planet (like the Earth) is surrounded by a considerable crust of solid rock analogous to granite, it may closely approach that of wrought iron.

(5) In smaller masses the effect of the surrounding crust of solidified rock due to secular cooling rapidly augments, and while it is difficult to make an exact estimate of this modifying cause, it probably is safe to infer that *Mars* has a rigidity

at least equal to that of glass, and more probably about equal to that of gold, which corresponds closely to the pressure upon the planet's nucleus (285,000 atmospheres). The exact value will depend on the thickness and stiffness of the crust, which in this case appears likely to be considerable, owing to the advanced stage of the planet's secular cooling, and the ruddy color indicating iron ores in the rocks which make up the surface.

(6) The rigidities of *Mercury* and the Moon appear likely to be about equal to that of the softer grades of glass. If, however, the surface rock is very stiff, the mean rigidity may possibly approach that of silver. This estimate includes both the effects of pressure and of the thick, solid crust by which these masses are surrounded.

(7) The other satellites no doubt have still lower rigidities. In general they probably are comparable in stiffness to the various kinds of glass; though a mass so rare as the fourth satellite of *Jupiter* may have a rigidity which is even smaller.

(8) It will be seen from the foregoing tabular moduluses of rigidity, quoted mainly from LORD KELVIN's experiments, that aluminum, silver and gold have successively slightly higher rigidities than glass. Accordingly, owing to the small amount of its primordial heat, and the present advanced stage of its development, it seems not unlikely that the planet *Mars* may approach the rigidity of gold, and *Mercury* that of silver; while the stiffer satellites may compare in rigidity with corresponding spheres of aluminum. But it seems extremely improbable that any of the smaller masses of our system, composed chiefly of amorphous rock, which has never been subjected to great pressure, have rigidities corresponding to the harder metals, such as zinc, brass, copper, or iron.

(9) The remarkable result obtained in *A.N.*, 4053, that in stars which are composed of gaseous matter reduced to the state of single atoms, the central density is always exactly six times the mean density, should be recalled in connection with the even more remarkable result reached in § 221 of the present chapter, namely, that in monatomic stars the mean rigidity of the matter is always exactly three times that of the layers. The occurrence of such simple relations in a theory which seems likely to have wide application to the physical universe may well raise the question in the minds of philosophers whether exact numerical relations are not somewhat more frequent among the ultimate phenomena of nature than has been generally supposed. In his recent address on "Mathematical Physics" at St. Louis, POINCARÉ has treated of the relations of numbers and other mathematical expressions to the order of nature, and pointed out some lines along which discoveries of the deeper secrets of the universe may be expected. Those who accept NEWTON's view that the ultimate laws of nature are essentially simple,

will see in the results here reached an argument which will tell strongly in favor of the monatomic theory as correctly representing the phenomena of the heavens; but others who reject these views will no doubt hold that these laws are only occasionally applicable to existing physical bodies. It would seem, however, that such simple relations must augment in some degree the probability of a theory which on other grounds appears rational and inherently probable under the extreme conditions prevailing in the stars.

(10) As respects the concluded rigidity of the Earth different investigators will naturally attach different degrees of importance to the values resulting from the several methods of determination. Thus in his well known paper on the "Rotation of an Elastic Spheroid," p. 320, Mr. S. S. Hough has shown that Newcomb's reasoning in respect to the polar motion is not strictly rigorous, though his general result is approximately confirmed; and whilst by strictly rigorous process Hough reaches the conclusion that the rigidity of the Earth somewhat exceeds that of steel, he adds that prior to the use of the method depending on the observed* prolongation of the Eulerian period of the variation of latitude "the only knowledge we have of the amount of the Earth's rigidity arises from the very vague indications furnished by Tidal Theory." Judging, however, by the complete confirmation of the inferences first drawn from the Theory of the Tides this seems too low an estimate of the importance of that method.

Whatever defects may exist in the method developed in the present paper, it does not lack in simplicity and definiteness as respects the outside limits of the Earth's rigidity. And it depends so directly on the familiar principle of pressure that the Geologist will no doubt grasp the point of view of the Astronomer more readily than in the case of the other methods depending on the tides and the polar motion.

In a recent review of the problem of the rigidity of the Earth as found by various methods, but especially the yielding indicated by the prolongation of the Eulerian period of the variation of latitude, Professor Sir G. H. Darwin concurs in the result reached by Mr. S. S. Hough and remarks:

"It would seem that the average stiffness of the whole Earth must be such that it yields a little less than if it were made of steel. But the amount by which the surface yields remains unknown, because we are unable to say what portion

* From the theoretical rigidities given in this chapter it would be possible for a mathematician to predict with considerable accuracy the period of the variation of latitude on some of the other planets, thus reversing the procedure followed historically in the case of the Earth. In the case of *Mars*, for example, we should first compute the appropriate Eulerian period for a rigid spheroid of its known mass and dimensions, and then calculate the prolongation of that period resulting from a mean rigidity equal to that of Gold. In the case of the major planets and the Sun the problem would involve the movement of the surface material relatively to the nucleus, which could be regarded as fixed, on account of the great rigidity of the interior portions of those masses.

of the aggregate change is superficial and what is deep seated." (cf. *Tides and Kindred Phenomena of the Solar System*, pp. 254-5).

Does not the method for investigating the rigidity of the heavenly bodies developed in the present paper satisfactorily answer this question? It is obvious that there will naturally be some uncertainty respecting the exact law of density within the Earth, but probably all will agree that it increases in an essentially continuous manner from the surface to the centre, and that the central density must therefore approximate that of lead. Moderate variations of the density as we go downward produce only moderate effects on the calculated pressure, effective rigidity, and viscosity at various depths; and thus it would seem that the yielding of the Earth's mass must everywhere follow closely, if not absolutely, the law of the effective rigidity depending on the pressure. This method therefore will doubtless enable us to conclude the approximate distribution of the yielding throughout the different strata of the terrestrial spheroid, and may prove useful in various investigations of the Physics of the Earth.

It only remains to add that the investigation of the physical properties of cosmical globes suggested in *A.N.*, 3992, seems to be capable of more complete realization than the author could have anticipated when that paper was prepared in 1904. The method of investigation developed in the present paper appears to be a very general one; and perhaps it will be possible for investigators to approximate the laws of density with sufficient accuracy to afford a clear conception of the actual state of the planets and the Sun, and stars of corresponding mass in like stages of development. It seems probable that all the large masses may be taken as essentially gaseous, made up chiefly of elements in the monatomic state; and therefore the surface density, viscosity and rigidity will be infinitely small. At greater depths the density, pressure, elasticity and viscosity increase rapidly, according to laws carefully worked out in the paper on the physical constitution of the heavenly bodies and made more exact in Table E of the present chapter; and when the depth is sufficient to give a pressure comparable to the rigidity of armor plate, taken at 1,000,000 atmospheres, it would seem that all bodily circulation of currents must essentially cease, though the atomic movements remain intense on account of the high temperature. It thus appears that the elasticity, rigidity, and viscosity increase rapidly with the development of density and pressure. Consequently Bodily Tides developed in immense globes flaming fluid, like the Sun and stars, would present a great variety of phenomena, depending chiefly on the changing viscosity of the successive layers near the surface, while at great depths the viscosity could be taken to be practically infinite. The principal tidal effects would thus be confined to the surface layers,

where the viscosity is comparatively small; but some modification of this simple result would no doubt be produced by the lesser and often antagonistic tidal movements in the lower layers of great viscosity. In a star approaching the nebular condition the rigidity would be small throughout, on account of the small pressure due to the rarity and great expansion of the mass; and consequently when the tide-raising forces are adequate, the tidal distortion would be very large.

These considerations, when applied to systems supposed to approximate the conditions of the stars, may aid us in explaining observed phenomena, and afford a better understanding than has yet been attained of the physical conditions pervading the sidereal universe. Accordingly the problem of the rigidity of the heavenly bodies, first investigated by LORD KELVIN and SIR G. H. DARWIN for the case of the Earth, by means of the tidal phenomena of our seas, thus seems to be one of wide interest; and evidently admits of general treatment from the point of view of gaseous masses subjected to the action of universal gravitation.

When we have some simple dynamical principles deduced from the theory of gravitation in connection with the mechanical theory of gases, which are capable of throwing light upon the physical state of the matter within the heavenly bodies, especially as respects density, pressure, rigidity, elasticity, and viscosity, perhaps some solid progress can be made in explaining the phenomena presented by the stellar universe. Under the circumstances it will not appear wholly inappropriate that the beginning of this line of thought should be traced back to LORD KELVIN'S ingenious use of the Tides, the general theory of which promises to play a considerable part in the ultimate explanation of phenomena observed in the sidereal heavens.

§ 225. LOVE'S *Recent Investigation of the Yielding of the Earth to Disturbing Forces.*

The *Proceedings* of the Royal Society for January 14, 1909, contains an important paper by PROFESSOR A. E. H. LOVE, of Oxford, on the "Yielding of the Earth to Disturbing Forces," in which the problem of the rigidity is treated from a general and simple point of view. The following account of this new work is very brief, but it may be sufficient to give some insight into the method employed, and thus add to the completeness of this chapter, most of which was written in 1905-6. When the form of the Earth yields under the stress of disturbing forces, the yielding can be most appropriately specified by two numbers h and k. The first of these quantities indicates the actual yielding of the surface and the second the alteration of the gravitational potential of the Earth arising from the action of the disturbing forces producing the yielding of the surface. Observations of

the fortnightly tides of the Indian Ocean led LORD KELVIN to the approximate value

$$\left.\begin{aligned} 1+k-h &= \tfrac{2}{3}\,, \\ \text{or} \qquad h-k &= \tfrac{1}{3}\,. \end{aligned}\right\} \tag{533}$$

This value has been confirmed by W. SCHWEYDAR by an analysis of more extensive series of observations of the fortnightly tides, and by observations made with the horizontal pendulum (cf. *Beiträge Zur Geophysik*, Vol. 9, 1907, p. 41). The very exact observations of DR. O. HECKER taken at Potsdam by means of a horizontal pendulum mounted on a sand foundation twenty-five metres below the surface of the ground show that the equation (533) is remarkably exact (cf. *Beobachtungen an Horizontal pendeln, &c., Veröffentlichung d. k. Preuss. geodätischen Institutes*, No. 22, Berlin, 1907). He succeeded in measuring deflections of gravity as small as $0''.001$, the total range of variation being about $0''.016$. In his early tidal researches KELVIN tried to evaluate h and k separately by making certain hypotheses, and he thus reached the values

$$h = \frac{5}{6} \;\; ; \;\; k = \frac{1}{2}\cdot \tag{534}$$

Now by PROFESSOR LOVE's theorem we may find k directly from the expression

$$k = A \times \frac{427-306}{427}, \tag{535}$$

where A is a numerical coefficient differing but little from unity, and depending upon the oblateness of the Earth's meridians, its angular velocity of rotation, and gravity at the surface. LOVE takes the diminution of gravity due to rotation as $\frac{a\omega^2}{g} = 1{:}289$, and the mean oblateness $\epsilon = 1{:}\,297$, and then the equation

$$1 - \frac{\tau_0}{\tau} = \alpha = k\frac{a\omega^2}{2g} \Big/ \left(\epsilon - \frac{a\omega^2}{2g}\right) \tag{536}$$

determines k. In this formula a is the Earth's mean radius, ω the angular velocity of rotation, g the value of gravity at the surface, while τ_0 and τ denote the periods 306 and 427 days, respectively. By means of this equation LOVE finds

$$k = \frac{4}{15}, \text{ nearly,} \tag{537}$$

And since by (533) $h-k = \frac{1}{3}$ nearly, it follows that

$$h = \frac{3}{5}, \text{ nearly.} \tag{538}$$

Hence he announces the following theorem: "The inequality produced in the potential of the Earth, near its surface, by the action of the Sun and Moon, is about four-fifteenths of the tide-generating potential, and the inequality produced in the surface of the Earth is about three-fifths of the true equilibrium height of the tide. The results hold for each of the partial tides answering to the several periodic terms of the tide-generating potential."

If the matter of the Earth be taken to be homogeneous and incompressible, LOVE finds $k = \frac{3}{5}h$. When the rigidity, μ, is taken to be uniform throughout the globe, the theory of the deformation of an elastic sphere gives

$$h = \frac{5}{2}\left(1 + \frac{19}{2}\frac{\mu}{g\sigma a}\right)^{-1} \tag{539}$$

But by observations of the tides and of the variation of the vertical it is found that $h—k = \frac{1}{3}$, and if $k = \frac{3}{5}h$ this gives $h = \frac{5}{6}$. Using this value in Equation (539), we find $\mu = \frac{4}{19}g\,\sigma a$. And since in C. G. S. units $g\sigma a = 3.5 \times 10^{12}$ nearly, we find $\mu = 7.6 \times 10^{11}$ nearly. This is very nearly the rigidity of steel and agrees with LORD KELVIN's estimate of the average rigidity of the Earth.

If we put $h = \frac{3}{5}$, the Equation (539) gives $\mu = \frac{1}{3}\,g\,\sigma\,a$ or about 1.2×10^{12}, which is considerably larger than the value just found, and agrees more nearly with the rigidity deduced from the rate of transmission of the second phase of earthquake waves to great distances. PROFESSOR LOVE does not, however, attach much importance to this agreement, because by varying the assumptions he obtains a variety of results (cf. *Proc. Roy. Soc.*, Nov. 28, 1908, pp. 81-82). He shows that the value of the rigidity deduced depends upon certain hypotheses respecting the internal density, compressibility, etc., which cannot be accurately determined at present. All the methods point to the validity of LORD KELVIN's general conclusion that the Earth has approximately the rigidity of steel, some estimates being higher and others lower.

On certain assumptions it is shown also that the nucleus is more than twenty times more rigid than the outer shell imagined by WIECHERT to have a depth of 0.22 of the radius. LOVE concurs in the conclusion that the nucleus is more rigid than ordinary material at the surface, but that the shell may be less so; yet not small enough to permit us to imagine a layer of fluid under the crust, except it be isolated and restricted to small areas. This accords with the conclusions drawn from the author's study of earthquake phenomena, in which it is shown that no movement ever takes place within the globe, save just beneath the crust, as when mountains are being formed by the transfer of lava from beneath the sea

and its injection under the land; and even then it takes the throes of a world-shaking earthquake to produce the movement.

In regard to the internal state of the Earth, LOVE remarks: "It appears that increase of density towards the centre compensates to some extent for defect of rigidity, and that increase of rigidity towards the centre can compensate for a considerable defect of rigidity in the superficial portions. The effect of compressibility is not known, but it seems improbable that the yielding of a compressible sphere with an assigned rigidity should be less than that of an incompressible one. The result that the rigidity of WIECHERT's shell may be less than that of most surface rocks led SCHWEYDAR to adopt WIECHERT's suggestion that there may exist a plastic sheet between the nucleus and the shell. I think it may be regarded as certain that there is not within a depth of 1400 km. a continuous layer of molten matter, separating the inner portions of the Earth's body from the outer portions, and behaving as a fluid in respect of forces of the type of tide-generating forces. In order that the astronomical motions may be performed as we know they are, and that the surface may not yield to such forces more than we know it does, the portions of the Earth which are outside such a sheet, if it exists, must be much more rigid than we can reasonably conceive them to be. No amount of rigidity of the nucleus would enable us to satisfy the conditions."

After an analysis of the problem, LOVE concludes as follows: "It appears, therefore, that, even if the solid nucleus were absolutely rigid, and the enclosing shell were 1400 km. thick, the presence of a layer of fluid separating the nucleus from the enclosing shell would increase very much the yielding of the surface. To prevent the surface from yielding more than it actually does, the rigidity of the enclosing shell would have to be nearly five times that of steel. If the enclosing shell were thinner, a still higher rigidity would be needed. For example, if it were 64 km. thick, and of density half the mean density, or about 2.8, the requisite rigidity of the enclosing shell, the nucleus being absolutely rigid and the fluid layer thin, would be about 50×10^{12} dynes per square centimetre, or about sixty-six times the rigidity of steel. These numbers seem to me to be decisive against the hypothesis of the fluid layer. This conclusion does not negative the possible existence of areas of continental dimensions beneath which there may be molten matter. It means that such areas must be isolated; the molten matter beneath them cannot form a continuous sheet separating a central body from an enclosing crust. The conclusion does not negative the possible existence of a layer of comparatively small rigidity; but, if there is such a layer, it must be rigid enough to prevent a finite slipping of the enclosing crust over the central body."

The following considerations on the rigidity of the Earth, with two slight

changes, are quoted from the author's "Further Researches on the Physics of the Earth," etc. (*Proc. Am. Philos. Soc.*, 1908, pp. 191-193):

We may therefore take the outer layers of our globe to have a rigidity about equal to that of glass,* and assume that at a depth of 0.1 of the radius it becomes nearly 2.5 times as great as it is at the surface.

Whether it becomes at a depth of twenty miles less than it is at the surface we cannot tell, but such a decrease is not impossible, perhaps not improbable; because at this depth the molten rock moves in earthquakes, and yet in confinement it must have a very sensible rigidity, though probably not more than half that of granite.

Accordingly, it looks as if the rigidity at the surface is about equal to that of glass, at a depth of 20 miles about one-half that at the surface, and at the depth of 40 miles nearly the same, but increasing below that depth, and at 160 miles again equal to that at the surface, and at a depth of 400 miles considerably larger yet, or about 1.4 times that of glass. Increasing below this depth according to the pressure, it becomes at the center over three times that of nickel steel used in armor plate. The rigidity of steel is attained at a little over 0.3 of the depth to the center of the Earth. If this be the distribution of rigidity in the Earth, the curve of rigidity is as follows:

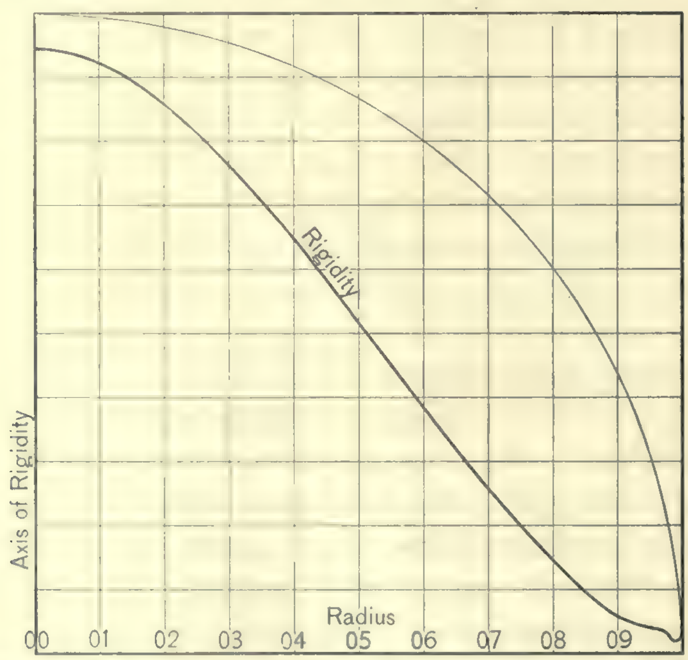

FIG. 38. RIGIDITY OF THE EARTH, SHOWING THE PLASTIC LAYER JUST BENEATH THE CRUST.

* Since this was written the rigidities of many kinds of Granite and Marble have been found by F. D. ADAMS and E. G. COKER (*Publications of Carnegie Institution*, No. 46) to lie between 2×10^{11} and 3×10^{11} C. G. S. Units, or say, between 200,000 and 300,000 atmospheres. These results seem to show that Marble and Granite have about the rigidity of glass.

This postulated fall in the rigidity just beneath the crust is probable for several reasons:

1. The temperature increases quite rapidly as we go downward, while the pressure increases proportionately more slowly, so that a depth would be reached at which the matter would become a plastic if not a viscous fluid.

2. The eruption of volcanoes and lava-flows on a vaster scale show that a molten layer underlies the crust, and occasionally is forced to the surface.

3. This underlying molten rock moves in world-shaking earthquakes, and frequently is expelled from beneath the sea under the land to form mountain ranges along the coast.

4. We may prove this expulsion of lava by the observed seismic sea waves, which indicate a sinking of the sea bottom, and by the simultaneous uplift of mountains and coasts.

From these considerations it follows that the Earth is most nearly liquid just beneath the crust, and has the greatest rigidity at the center. As the plastic or quasi-viscous layer beneath the crust is thin, and possessed of considerable rigidity, *it too remains quiescent except when set in motion by the dreadful paroxysms of an earthquake.*

In tidal and other observations the Earth therefore behaves as a solid, and the rigidity of the Earth inferred by KELVIN and DARWIN is confirmed. Yet a layer of plastic matter or quasi-viscous fluid exists just beneath the crust, and when disturbed by earthquakes gives rise to the development of ridges in the crust called mountains, chiefly by the expulsion of lava from under the sea.

CHAPTER XIX.

Theory of the Nebulae.

§ 226. *Historical Sketch of the Earliest Discoveries of Nebulae.*

The word *Nebula* is the Latin equivalent of νεφέλη, which is used by Greek writers, from Aristotle to Ptolemy, to denote a cloud or cloud-like object. Thus in the catalogue of the fixed stars given in the *Almagest*, Ptolemy describes five objects as cloudy stars (νεφελοειδής), because they presented a somewhat blurred appearance, due to the proximity of other stars which made a small cluster and therefore could not be distinctly separated by the naked eye. Accordingly it is not remarkable that the *nebulous stars* noted by the ancients were few in number and confined to objects so grouped together that they presented a hazy aspect.

After the invention of the telescope by Galileo in 1610, they were all readily resolved into separate and distinct stars, but found to be situated in such proximity as to cause the light to be somewhat blurred when viewed by the unaided vision.

Two years later, in 1612, Simon Marius discovered the Great Nebula of *Andromeda* and found it to be an object of totally different character from those previously known. To the naked eye, indeed, it appears nebulous, but this does not seem to have been noted by the earlier observers. On examining it with the telescope, Marius was surprised to find that the same nebulous aspect was preserved, and that, unlike the objects previously classed as nebulous, owing to the compression of individual stars, it did not seem to be of sidereal constitution. He justly compared the light of this great nebula to that of a candle shining at night through a transparent horn, which is a very appropriate description of the general character of this remarkable object.

The earliest true nebulae were all discovered by accident, and for a long time the list of them was very small. The next object of this kind to be noticed was the great nebula of *Orion*, discovered by Huyghens in 1656. He was astonished

at the unusual aspect of this object, and could not believe it to be of sidereal nature. And when he noticed the intense blackness of the heavens in the region about this great nebula, the luminosity of the object suggested to his mind the idea that it might be caused by an aperture in the sky through which glimpses of the empyrean or luminous region beyond might be obtained. It is not probable that he seriously entertained this view, but he records it as an impression which came to him; and a similar impression has been made on the minds of many modern observers.

Among the other early observers who labored to distinguish the nebulae from the starry aggregations of the heavens, none are so deserving of mention as HALLEY, LACAILLE, and MESSIER.

In 1714 HALLEY presented to the Royal Society an account of the six nebulae then known to astronomers: namely, the great nebula of *Andromeda*, discovered by SIMON MARIUS in 1612; the nebula of *Orion*, discovered by HUYGHENS in 1656; and the great nebula in *Sagittarius*, discovered by ABRAHAM IHLE in 1665; the cluster *Omega Centauri*, discovered by HALLEY at St. Helena, in 1677; the star cloud in *Antinous*, discovered by KIRCH in 1681; finally, the great cluster in *Hercules*, discovered by HALLEY in 1714.

It will be noticed that some of these objects are truly nebulous, but that most of them are of stellar constitution. HALLEY supposed the light of the truly nebulous objects to come from an extraordinary distance, and he imagined it to be occasioned by a lucid medium diffused throughout the ether, and shining by its own light.

The veteran astronomer HEVELIUS had already noted sixteen nebulae, in addition to those mentioned by HALLEY; and in 1755 LACAILLE communicated to the Paris Academy of Sciences a catalogue of forty-two nebulae in the southern hemisphere, recently observed at the Cape of Good Hope. He classified them according to their apparent resolvability. The members of the first group contained no indications of sidereal structure, but were compared to the milky light noticed in certain small patches of the Milky Way; these of the second consisted wholly of congeries of stars, so close together as to be separately invisible to the naked eye, but resolvable in a telescope; those of the third group consisted of stars surrounded by real nebulosity, and therefore related to the first class.

MESSIER's first catalogue of forty-five nebulae and clusters of stars was published in the *Memoirs* of the Paris Academy of Sciences for 1771; but subsequently reprinted in the *Connaissance des Temps*, and in 1783 and 1784 increased to 103 objects. He pronounced no opinion on the nature of these nebulous objects, but commended them to the attention of astronomers in the hope of detecting changes of form or structure.

This brief account of the study of the nebulae prior to the time of HERSCHEL enables us to appreciate the gradual abandonment of the Aristotelian theory that the nebulae are made of diffuse luminous matter suggested by the hazy light noticed in the Milky Way, and thus closely related to the Galactic Circle.

§ 227. *The First General Exploration of the Nebulae Made by* SIR WM. HERSCHEL.

It will be seen from this brief sketch that up to the time of SIR WM. HERSCHEL, the total number of nebulae known did not exceed 150. This unrivaled man was the first to attempt a systematic exploration of the heavens with powerful instruments, and he laid the foundations of all our knowledge of the sidereal universe.

In 1786 he presented to the Royal Society a catalogue of 1,000 new nebulae and clusters of stars; and three years later, in 1789, a second catalogue of 1,000 additional nebulae; while in 1802 he added a third catalogue of 500 more of these objects. He divided the nebulae and related bodies into some twelve different groups, arranged according to his idea of the contents of the sidereal universe. It is thus evident that SIR WM. HERSCHEL completely revolutionized our knowledge of the nebulae.

SIR JOHN HERSCHEL afterwards extended and perfected the nebular surveys begun by his father; so that when he published his General Catalogue in the *Philosophical Transactions* for 1864, he was able to give the places of 5,079 nebulae and clusters of stars, of which only about 450 had been located by other observers. Accordingly our catalogues of the nebulae date from the epoch of the HERSCHELS, but they have since been considerably extended by others.

The principal modern work on the subject is DR. J. L. E. DREYER'S *New General Catalogue of Nebulae and Clusters of Stars*, published in the *Memoirs of the Royal Astronomical Society*, Vol. XLIX, Part I, 1888. This work gives a very good history of the discoveries in the twenty-four years subsequent to the publication of SIR JOHN HERSCHEL'S *General Catalogue*. In the twenty-two years since the *New General Catalogue* was published, many additional nebulae have been discovered, both by visual and photographic means, but a general catalogue is not yet available.* Indeed, photography has so greatly augmented the number of small, faint nebulae, that the future catalogues will have to be of enormous extent. But of the new objects, only a few are striking, so that most of the remarkable nebulae are found in the existing catalogues, and our earliest records of them date from the explorations of SIR WM. HERSCHEL.

* In the *Memoirs of the Royal Astronomical Society* for 1908, DR. DREYER has, however, published a supplement to the *New General Catalogue* of 1888.

§ 228. *Spiral Nebulae Not Continuous Masses of Fluid in Equilibrium Under Hydrostatic Pressure, but Discontinuous Vortices Circulating and Condensing Into Systems.*

We have seen that for a long time after the epoch of LAPLACE and HERSCHEL the impression prevailed that the nebulae were continuous masses of fluid in equilibrium under hydrostatic pressure. This assumption underlies LAPLACE'S formulation of the nebular hypothesis, and has held its ground almost up to the present time. The continued dominance of these ideas is due to the great authority of LAPLACE, and of such followers as SIR JOHN HERSCHEL, NEWCOMB and DARWIN, none of whom departed very radically from the teachings of the founder of the nebular hypothesis.

It is true that a note of dissent was occasionally heard, as when sagacious thinkers began to meditate over the vast extent of the nebulae, their great transparency, and their enormous distance from the Earth. This seemed to make it difficult to believe that the nebulae are continuous masses of fluid in equilibrium under conditions of hydrostatic pressure. For, with such extreme rarity, hydrostatic pressure could not be exerted unless the mass was at enormous temperature; and a very high temperature could not be maintained in a mass of small density and great transparency, since the heat would be almost instantly radiated away into space. Some doubt was therefore thrown on the validity of the current ideas of the nebulae. The dissenters, however, were embarrassed by the fact that LAPLACE'S theory of detachment alone enabled them to account for the roundness of the orbits of the planets and satellites, and their mutual relationships, all of which were supposed to follow from the condensation of rings of vapor.

And whilst the problem of the condensation of a ring into a single large body was felt to be difficult, it was not yet believed that it could not take place, except by PROFESSOR NEWCOMB (*Popular Astronomy*, page 504), who held that it would produce a group of small bodies like the asteroids; nor was it realized that any other agency, such as a nebular resisting medium, could have produced the observed roundness of the orbits. Therefore, whilst it was not at all evident how a ring could condense into a planet or satellite, it was not doubted that it had occurred in the early history of our solar system. Accordingly, if rings had been abandoned and had afterwards condensed into planets and satellites, it necessarily followed that hydrostatic pressure had been exerted from the centre outward in their detachment. Under the circumstances, the dominance of Laplacian conceptions during the whole of the past century is easily understood.

From these considerations we see how the nebulae continued to be viewed as fluid masses in equilibrium, and not as discontinuous cosmical clouds, often with vortices formed of streams circulating without the exertion of hydrostatic pressure between the coils. After what we have already proved in this work, we see clearly that the nebulae are too transparent and too vast in extent to be anything else than discontinuous clouds of cosmical dust, devoid of true fluid properties. The origin of such vortices in the condensation towards a centre has already been explained; and we have found that the principle of hydrostatic pressure finds little or no application among the rarest nebulae. This is easily proved by numerical calculation in the case of the solar system, from the data supplied by BABINET'S criterion. The solar system is characterized by very small attendant bodies, the satellites being very small compared to their planets, and the planets very small compared to the Sun; but systems of this type are also common among the stars, though necessarily invisible in our existing telescopes. The facts of the solar system therefore afford a just basis for estimating the tendency among the nebulae as a class.

§ 229. *Many Nebulae Not Yet Settling as a Whole, and Therefore Devoid of the Spiral Form.*

If we study the nebulae as a class we shall perceive among these immense clouds of cosmical dust a very large number of objects which have no regular form. Thus we have globular or planetary nebulae, ring nebulae, elliptical nebulae, and spiral nebulae,— the two last named being very intimately associated — but there still remains a large number of irregular nebulae, which have not assumed any definite shape. Among these irregular nebulae one readily recalls the great nebula of *Orion*, the *Trifid* nebula, the *Crab* nebula, the *Omega* nebula, the *America* nebula, and most of the nebulae of very wide extent discovered by photography during the past twenty years. This diffuse nebulosity is sometimes found to cover whole constellations, and is almost devoid of form or definite boundaries. Obviously such nebulae are much too tenuous and widely extended to be regarded as gaseous masses in equilibrium under the pressure and attraction of their parts, and we must regard them as clouds of cosmical dust produced by the expulsion of particles of matter from the stars by the radiation pressure of their light and by electric forces, thus constituting portions of the universal chaos in which order has not yet been introduced.

When we consider the forms of these irregular nebulae, their extreme tenuity and vast extent, we realize that the forces of attraction exerted by one part of such

a nebula upon another, must be excessively feeble. Different parts of such a mass have different proper motions, and there may thus arise some streaming movement, as where dark lanes are shown on our photographs. Dark bodies traversing such a nebula and clearing out a path might, however, produce a similar appearance. Many of these nebulae thus have a wispy aspect, something like the cirrus clouds of our sky; but the streams thus exhibited in space are due to the movement of the nebulosity itself, not to that of the medium in which the cloud is suspended, as in the case of the cirrus noticed in our atmosphere.

The motion in all such diffuse nebulae under the feeble forces acting on them must necessarily be slow; and therefore it is not surprising that so many nebulae

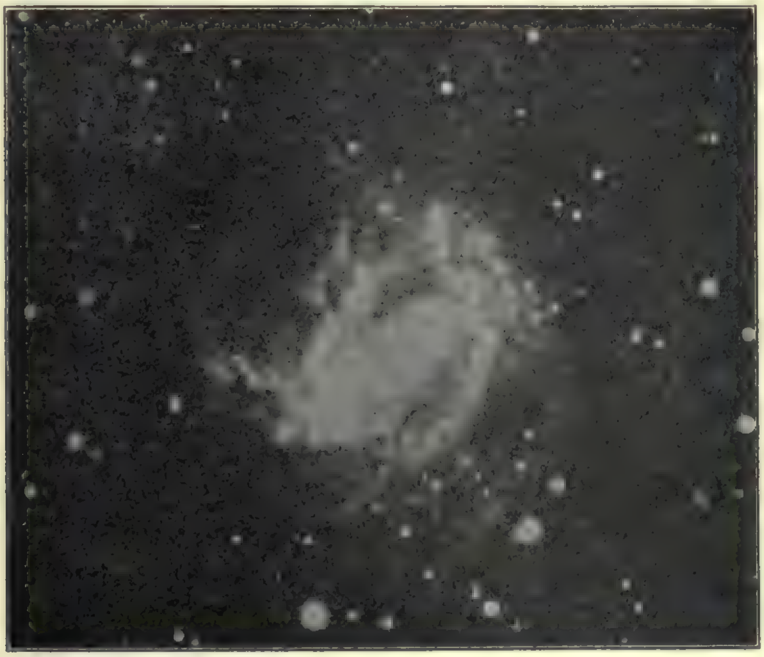

Fig. 39. Harvard Photograph of the Nebula Messier 83, *N.G.C.* 5236, Showing Spiral Movement in an Early Stage of Development.

are "without form and void." On the contrary, it is surprising that regular forms are exhibited by so many nebulae of comparatively large size. We can explain the development of regular forms among the nebulae only by the great length of time during which the Universe has existed. Such immense ages alone suffice to produce out of widely diffused chaos a partial approach to order. *Yet this tendency to regular forms, while nowhere perfectly attained, is at work throughout the immensity of space.*

In time it will give rise to settlement toward centers, and thus produce spiral nebulae and clusters of stars, all endowed with spiral movement. The progress

already made by many nebulae towards order and system in their internal arrangement is a sure indication of what will some day be visible in all nebulae.

The existence of irregular nebulae is therefore to be expected. It is surprising that observations of the Universe should make it possible to exhibit to our minds such tangible proof of progressive development among the nebulae. This result would hardly be possible without the aid of modern photography, which traces the faintest streams and thus discloses to us the nature of the movements going on in the sidereal universe.

§ 230. *The Immense Extent of the Larger Nebulae and Their Ultimate Spiral Movement — Photographs of Nebulae Made at Lick Observatory.*

The amazing extent of the larger nebulae has long been known, but has become more obvious with the progress of astronomical photography during the past twenty years. BARNARD's photographs of the Milky Way, taken at the Lick Observatory from 1888 to 1894, mark a new epoch in bringing out the cloud forms shown to exist among the stars; but along with this interesting revelation came another, *showing that the background of the sky is almost everywhere covered with faint nebulosity. In fact it has proved extremely difficult to find any part of the sky which was perfectly black.* The nebulosity is everywhere spread among the stars, and in the nebulae proper becomes so conspicuous as to attract instant attention. The secret of BARNARD's celebrated photographs of the Milky Way consists in the wide field of a hundred square degrees or more obtained with the portrait lens. DR. MAX WOLF has done almost equally extensive and celebrated photographic work, showing diffused nebulosity spread over vast areas of the sky.

Some of the regions of nebulosity are of about the same extent as the cloud forms of the Milky Way, thus covering whole constellations. It is scarcely necessary to dwell on the absolute extent of such regions of nebulosity. At the very lowest estimate of distance they must cover billions of times the space occupied by the solar system, though always so exceedingly tenuous that the nebulosity often shows merely as the faintest haze on the background of the sky.

If instead of diffuse nebulosity we consider actual nebulae, such as the great nebula of *Orion*, or *Andromeda*, we shall perceive that the space occupied by these masses cannot well be less than a billion times that occupied by the solar system. Such a nebula may eventually develop into a cluster of thousands of stars; and in many cases this seems to be what is already going on. These nebulae are of such immense extent that the separate stars are enabled to have their own autonomous spheres of influence, without much disturbance from without; and therefore

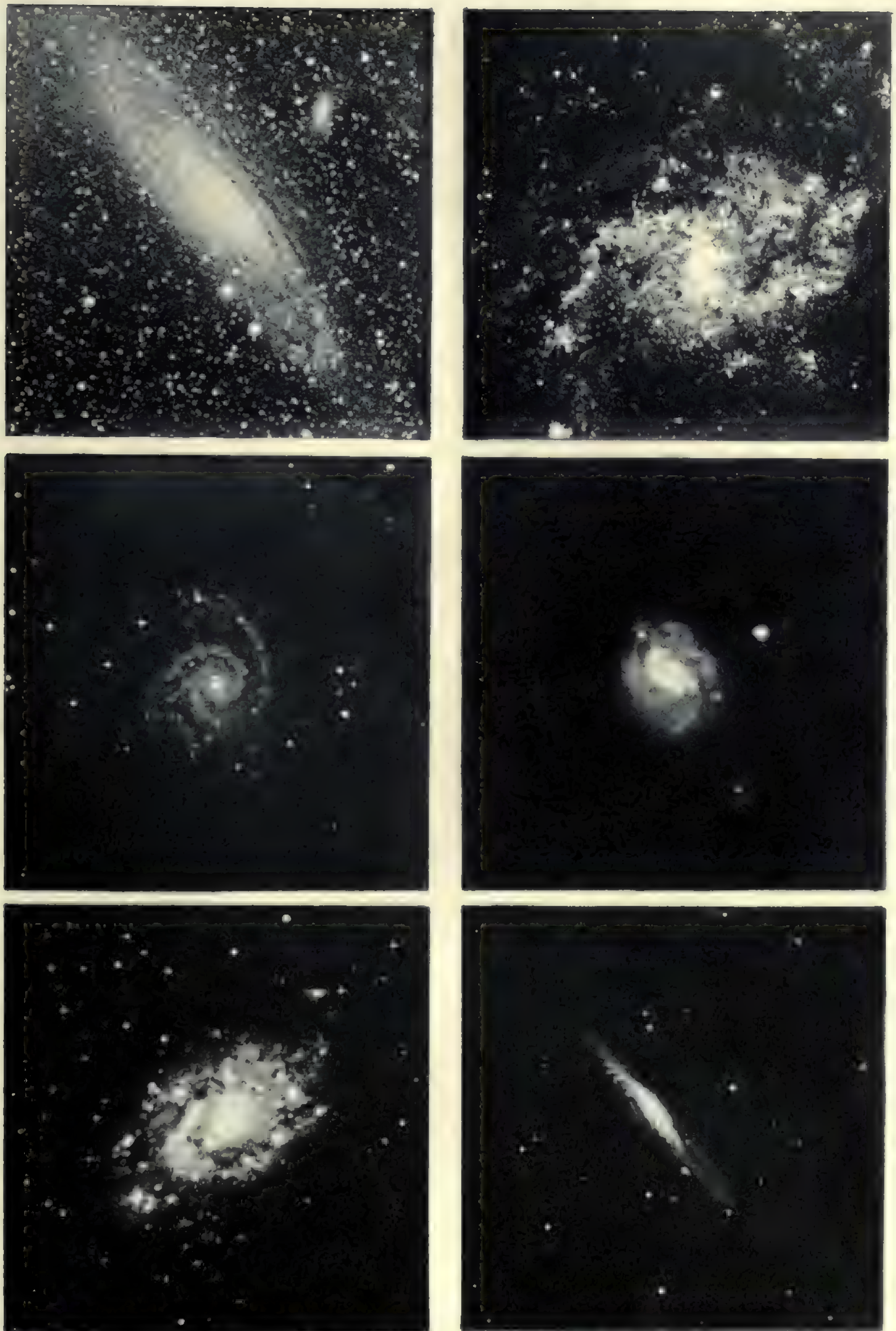

Plate A

SPIRAL NEBULAE PHOTOGRAPHED AT LICK OBSERVATORY:

THE GREAT NEBULA IN ANDROMEDA; M 33, TRIANGULI;

M 74, PISCIUM; M 77, CETI;

H.V 44, CAMELOPARDI; H. I 200, LEONIS MINORIS.

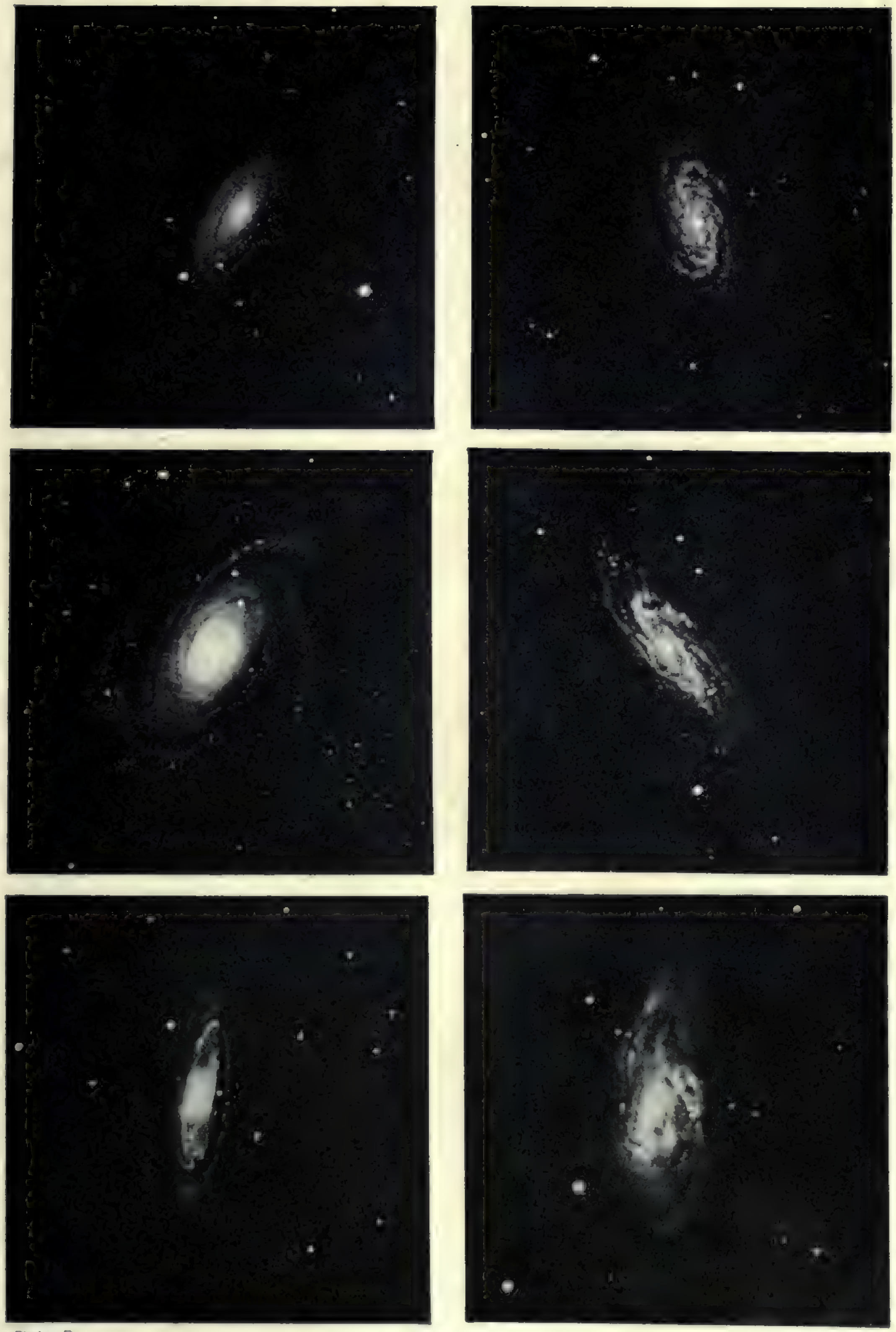

Plate B.

SPIRAL NEBULAE PHOTOGRAPHED AT LICK OBSERVATORY:

H.I.205, URSAE MAJORIS;

M 81, URSAE MAJORIS;

M 65, LEONIS;

H.I 56-57 LEONIS;

H.I 199, URSAE MAJORIS;

M 66, LEONIS.

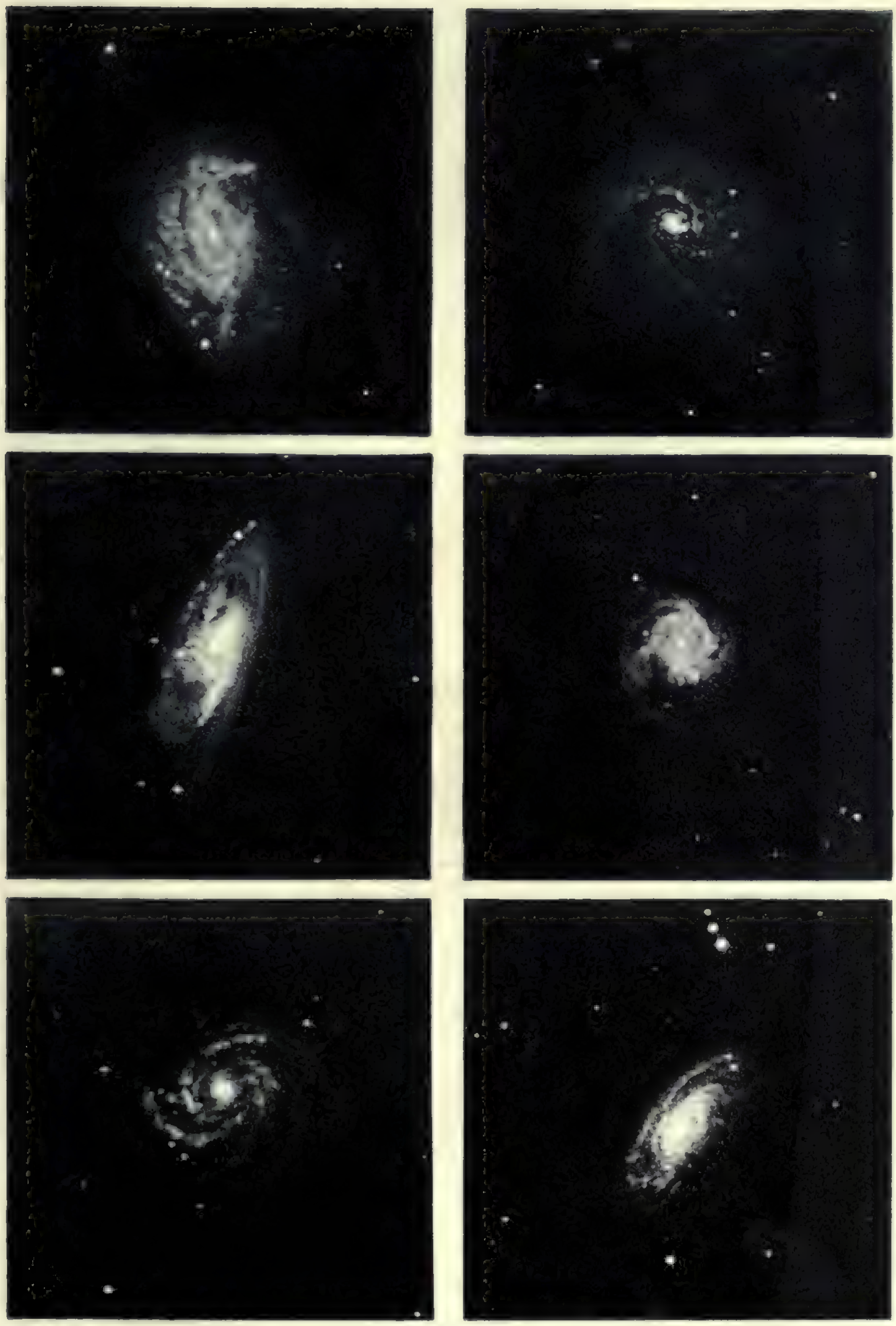

Plate C

SPIRAL NEBULAE PHOTOGRAPHED AT LICK OBSERVATORY.

H. II 730, URSAE MAJORIS; M 99, COMAE BERENICES;

H. V 43, URSAE MAJORIS; M 61, VIRGINIS;

M 100, COMAE BERENICES; M 88, COMAE BERENICES.

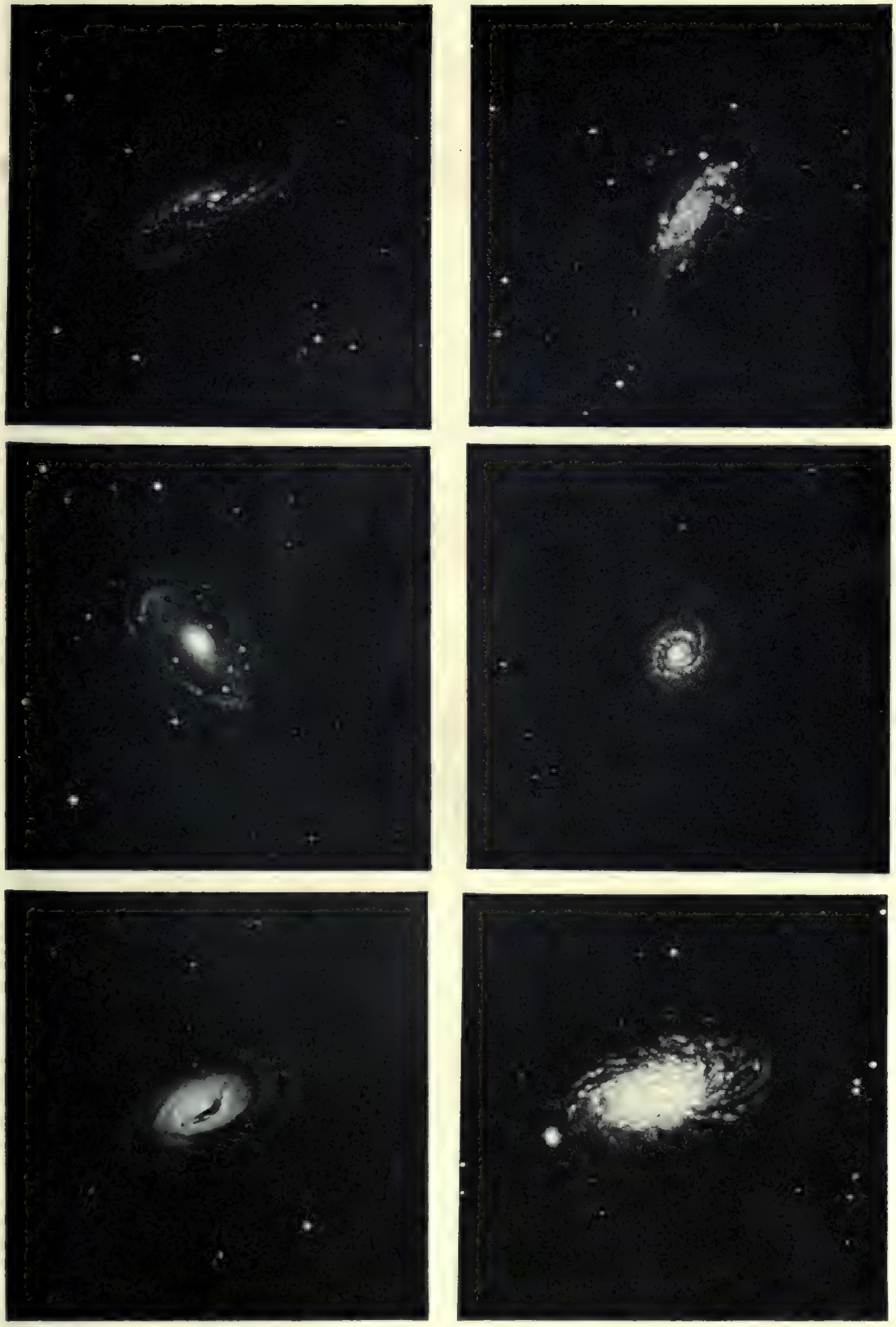

Plate D.

SPIRAL NEBULAE PHOTOGRAPHED AT LICK OBSERVATORY:

H.V 2, VIRGINIS; H.I 92, COMAE BERENICES;

H. I 84, COMAE BERENICES; M 94, CANUM VENATICORUM;

M 64, COMAE BERENICES; M 63, CANUM VENATICORUM.

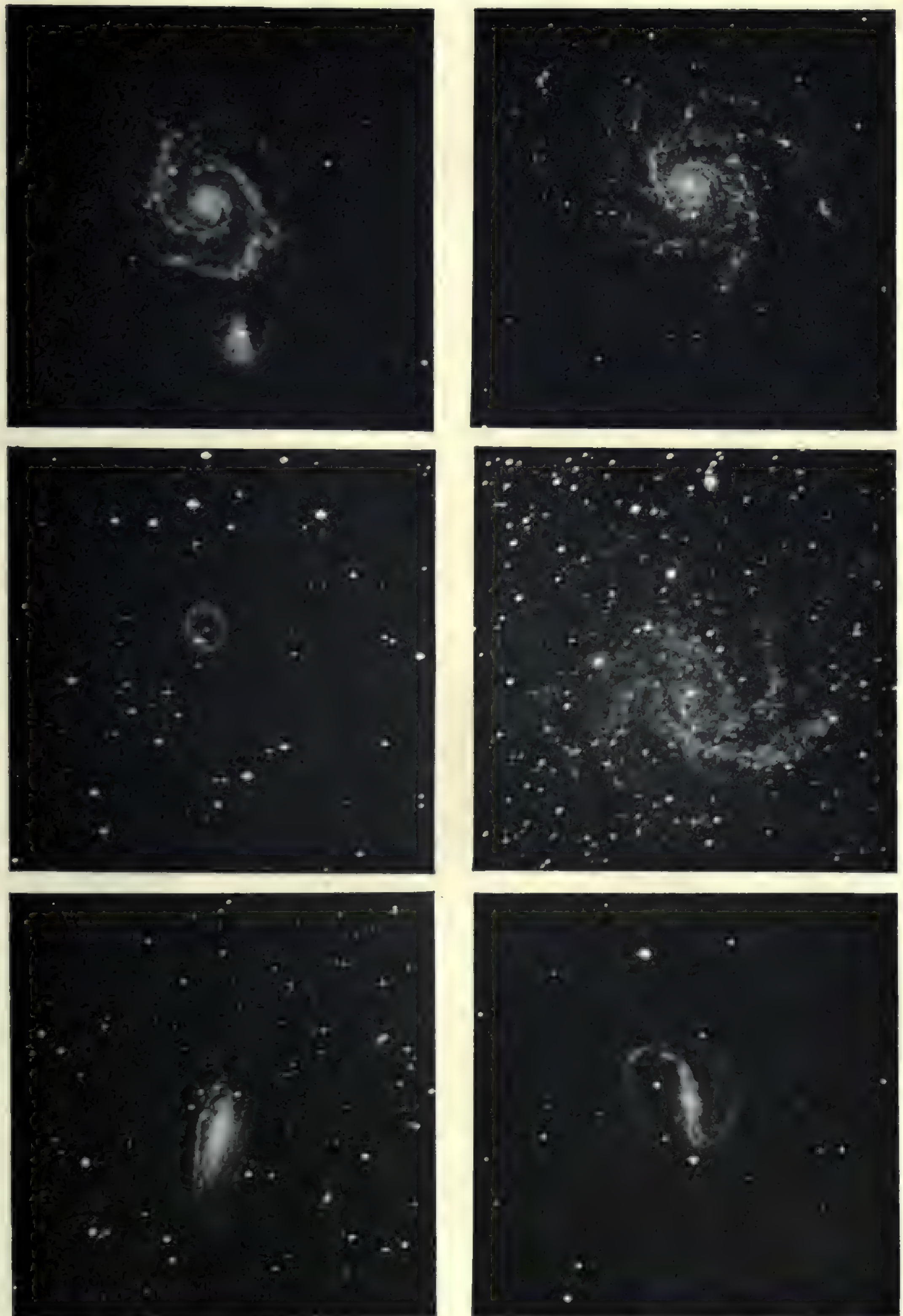

Plate E

SPIRAL NEBULAE PHOTOGRAPHED AT LICK OBSERVATORY:

M 51, CANUM VENATICORUM; M 101, URSAE MAJORIS;

H. IV 13, CYGNI; H. IV 76, CEPHEI;

H. I 53, PEGASI; H. I 55, PEGASI.

the tendency is to produce star clusters. Every such cluster will eventually be found to have a spiral movement, due to the action of the whole mass; but this is not yet disclosed by observation, because our records do not extend over a sufficiently great interval of time. Yet the visible streams of nebulosity shown among the spiral nebulae of ordinary size convey to us a clear impression of what is going on among the nebulae and clusters of the widest extent.

Photographs of Nebulae Made at Lick Observatory.

The beautiful plates of the spiral nebulae here reproduced were all taken at the Lick Observatory. The plate of the Great Nebula in *Andromeda* was taken by PROFESSOR E. E. BARNARD, with the Willard Lens, exposure $1^h\ 15^m$, November 21, 1892. The rest of the spiral nebulae were photographed by KEELER and PERRINE with the Crossley Reflector, and are reproduced from the magnificent plates given in Vol. VIII of the *Publications of the Lick Observatory* (1908). This fine volume is a suitable memorial to the late DIRECTOR J. E. KEELER, who did so much for the development of our knowledge of the nebulae, and it ought to be in the hands of every student of the subject. It is by far the finest work on nebulae ever published, and will long remain the most convenient and authoritative treatise for disclosing to us the nature of the nebulae as recorded by the photographic plate under the light-gathering power of a great reflector. KEELER was a master in the use of the reflector, and the fine pictures obtained under his skillful manipulation and that of PERRINE have given us the observational basis for a sound knowledge of the evolutionary processes at work among the nebulae. These photographs when correctly interpreted in the light of mathematical theory afford a secure dynamical foundation for the Laws of Cosmical Evolution.

§ 231. *Theory of the Formation of Annular Nebulae.*

The nature of the annular nebulae has always been quite obscure, and heretofore no very satisfactory explanation of this singular form has been put forward. The accompanying figure shows the mode of formation of this type of nebula adopted by the present writer. The theory is simply this: Two equal streams of cosmical dust drift past each other at such distances that they pass without collision except at the ends; and the whole then becomes one unbroken girdle of nebulosity revolving under the mutual gravitation of its parts.

The union of two streams is clearly indicated by the blurred ends of the ring nebula in *Lyra*. The overlapping of the streams where they meet gives rise to the

hazy nebulosity at the extremities of the ellipse. The arch of the ellipse on either side is of immense extent, and the mutual attraction of the two streams has merely served to produce curvature in the two original streams.

After the nebula whirls for a long time it will finally pass from an elliptical to a more circular girdle, and the interior of the ring will also become covered with a thin gauze of nebulosity due to wastage from the ring. This condition seems to be partially attained by the ring nebula in *Lyra.*

The directness and simplicity of this explanation is so striking as to leave little or nothing to be desired. It explains all the details of the Ring Nebula in *Lyra,*

Fig. 40. Theory of the Formation of the Ring Nebula in Lyra, by the Coiling Up of Separate Streams of Cosmical Dust, or by the Mere Gravitational Settling of a Single Stream of Nebulosity of Very Unsymmetrical Figure.

the brightest and most remarkable object of the kind in the heavens; *and it also makes such annular forms incidental and occasional products of the same process by which the spiral nebulae originate.* Among the vast number of streams passing one another or simply settling and in general coiling up into spiral nebulae, it would necessarily happen that a few of them would merely be curved and by union of the extremities pass into the annular form. *The annular form is therefore a special type of spiral nebulae, at a particular stage of its existence.*

Later on it may develop into a spiral of the ordinary type; or if it continues to be symmetrical about the centre, the friction and collision of the particles of dust will cause them to fall towards the centre and give not only the "gauze drawn over the hoop" spoken of by SIR JOHN HERSCHEL, in the case of annular nebula in *Lyra*, but finally also a more or less uniform disc of planetary character.

Thus the annular nebulae, the spiral nebulae, and the planetary nebulae are all closely related; but the annular form is very rare, because the mechanical conditions for producing a perfect annulus are seldom realized in actual nature.

The following discussion taken from the late MISS A. M. CLERKE's *Problems in Astrophysics* (1903), will show both the extreme uncertainty heretofore attaching to the theory of the annular nebulae, and the great superiority of the present theory over those heretofore current: "It is certainly by no accident that the striae within this (*Lyra*) nebula coincide in direction with the major axis; nor can the termination of the transverse axis by maxima, of the longitudinal axis by minima of brightness be regarded as casual features. All the details of the edifice, in fact, are arranged with obvious reference to its apparent shape, and this amounts to a demonstration that the elongation is real. The nebula, then, is not simply projected into an oval; it is not a circular formation viewed obliquely. There seems no escape from the conclusion that it is an ellipsoid of revolution — that the bands follow the line of the equator, and originate under conditions prescribed by the rotation of the body; while the partial interruption of luminosity at the ends overrides the holding power of gravity. Everything, indeed, leads us to suppose that this nebula, like the rest of its kind, is actually a hollow spheroid of shining fluid, the marginal brightness resulting from the increased thickness of the luminous shell penetrated by the visual ray. The 'hoop' and the 'gauze' drawn around it are then two aspects of the same thing. Nebulous rings, as such, probably do not exist. They would be subject to perspective effects, no traces of which are to be found in the heavens. Annularity in nebulae may accordingly be considered as a purely optical modification of a different structural plan" (*Problems in Astrophysics*, pp. 485-486).

§ 232. *Increase of Angular Motion Near the Centres of Spiral Nebulae.*

If we consider a nebula of vast extent, all parts of which are more or less dominated by the attraction of a central nucleus, which may be an imperfectly developed star, it will become evident that nebulosity revolving about it at different distances will have different periods. According to KEPLER's law the angular motion will be most rapid towards the centre, as observed among the planets revolving about the Sun and among the satellites revolving about the several

planets. Consequently it follows that the central parts of such a spiral nebula will revolve most rapidly; and the result will be the formation of an indefinite number of adjacent spirals, the outer ones being left behind by the more rapid whirling of the central parts.

This relative shearing of the particles initially in a line is the inevitable consequence of the variation of the central attraction with the distance. It has been considered by PROFESSOR E. J. WILCZYNSKI, now of the University of Chicago, in the *Astronomical Journal*, No. 465, June 30, 1899. In considering the coiling up of a nebula this more rapid winding of the spiral towards the centre must not be overlooked. All nebular vortices have this tendency to spiral arrangement, and the effect is to increase the internal friction and facilitate collision and degradation of energy of the system, so that the progress of central condensation is accelerated.

§233. *What is a Nebula?—A Mass of Gas? Or Does It Include Also Solid Globes Like Our Moon and the Planets?*

This is an important question, to which heretofore we have been unable to return a definite answer; but we shall now endeavor to examine it in the light of the latest knowledge of the development of the solar system. The evidence of the spectroscope shows that many of the nebulae are gaseous, while others behave as if made up partly of solid bodies and give a spectrum which is largely but not wholly continuous, with superposed lines so faint as to be recognized with difficulty by any means yet known to science.

The powers of the spectroscope, however, are slowly augmenting; and within a few years we may expect considerable additional knowledge from the great reflectors now coming into use. It is from an unexpected source that much new light has come in regard to the nature of the nebulae; namely, the history of the solar system, as made out by the capture theory.

We have seen that our system was originally a spiral nebula of vast extent, and that in the course of immeasurable ages it condensed to much smaller dimensions, while the surviving orbits were all worn down to the round form by the secular effects of a resisting medium.

The planets were formed in the nebula, and are as old as the Sun itself. In time those which survive have gradually neared the Sun, and meanwhile captured families of satellites, and at the same time augmented their own masses and modified their obliquities. Now when our system was in the nebular stage it was evidently filled with planetary bodies — such as the Moon, the Earth, *Venus*, *Jupiter*, and the other major planets. In the course of ages most of the smaller planets have been captured and swallowed up by the Sun and larger planets. When

the system was still filled with nebulosity these bodies as seen from a distance appeared immersed in excessively faint nebulosity. At that time, however, the embryo bodies were not as large as the finished planets and satellites are to-day. Of this there is not the slightest doubt.

We conclude, therefore, from the known events of the solar nebula, that other nebulae in space go through a similar order of development. Whilst therefore the light of many nebulae seems to disclose mainly gaseous matter, it is absolutely certain that solid globes like the Moon and our terrestrial planets abound also in most of the nebulae. Bodies somewhat larger than *Jupiter* often shine as small stars in the spiral nebulae and clusters observed in space, but they seldom can be seen individually, so that the nebulosity presents a continuous aspect.

Thus from the study of the solar system we derive considerable additional knowledge of the nature of spiral and other nebulae. As the solar system has developed from a spiral nebula, it is the best possible witness as to the contents of such masses in other portions of space; and since fortunately we live in this system, it becomes possible to infer the contents of a typical spiral nebula with an unexpected degree of confidence.

Accordingly nothing could contribute more to our positive knowledge of the nebulae than a correct theory of the development of the solar system. In the celestial spaces we see cosmical systems in various stages of development, but we know of no other system so far advanced in development as our own. Yet, by combining our data of the system, near at hand, with the data given by those far away, it becomes possible to interpret the appearances of the spiral nebulae as a whole, and to infer the universal tendency among all nebulae whatsoever. They are all settling under the effects of universal gravitation, but it is only in certain cases that the motion has become well ordered and gives visible evidence of the development of cosmical systems.

§ 234. *Researches of* DR. MAX WOLF *on the Ring Nebula in Lyra.*

In *Nature*, of April 8, 1909, an account is given by PROFESSOR BOHUSLAV BRAUNER, of the Bohemian University, Prague, of the recent researches of PROFESSOR MAX WOLF, of Heidelberg, on the Ring Nebula in *Lyra*. By spectrum photography it is shown that the ring nebula consists of four different gases. It will of course be understood that this is consistent with the presence of an infinite number of solid bodies of all sizes from meteorites to moons and planets.

DR. WOLF's investigation indicates that the ring nebula is in rapid rotation, and that this centrifugal movement in the course of ages has operated to cause the gases to be concentrated in four different concentric layers. BRAUNER says:

"On using the image of the ring itself instead of the slit of a spectroscope, photographic images of the rings corresponding to the different spectral lines were obtained on the plates, but the dimensions of the rings were found to be different and to correspond to four gases of which the ring nebula is composed. The smallest ring, *A*, representing the innermost part of the ring, is composed of an unknown gas; the next largest ring, *B*, is composed of hydrogen; the next largest ring, *C*, consists of helium; and the largest ring, *C*, consists of an unknown gas. The question arises — What is the nature of the two unknown gases?

"Bredig found in 1895 that if a mixture of two gases is subjected to centrifugal rotation, the relative concentration of the gas of higher molecular weight (*i.e.* higher density) increases with the radius of rotation. We must, therefore, assume that in the series of our four gases *A*, *B*, *C*, and *D*, the density or molecular weight increases from the smallest value of *A* to the largest value of *D*, and this is, indeed, proved by the fact, found by Wolf, that the gas *B* consists of hydrogen, molecular weight = 2.016, and the gas *C* of helium, molecular weight = 3.96. From this it follows that the gas concentrated in the smallest zone of the ring *A* must have a smaller molecular weight than hydrogen. This gas has not yet been isolated upon our Earth, but its existence and atomic weight were predicted by the great Russian chemist and natural philosopher, Mendeléeff in a popular article published in Russian in 1902, the essential part of which was translated into English in 1904 under the title "An Attempt Towards the Chemical Conception of the Ether.

"Mendeléeff shows that if the elements of the rare or inactive gases, He, Ne, Ar, Kr, and Xe, discovered by Rayleigh, Ramsay, and Travers, are placed in the well-known nought-group, we must expect the existence of elements of the same group possessing smaller atomic weights than helium and hydrogen. Mendeléeff assumes that in the first horizontal series of the system, on the left side of, or before, hydrogen in the nought-group, where we find hitherto an empty place, an element stands possessing an atomic and molecular weight of 0.4, and he adds that this element might be identical with Young's 'coronium.' This part of the periodic arrangement is:

Series	Groups 0	1
1.	? = 0.4 . . .	. . . H = 1.008
2.	He = 4.0 . . .	. . . Li = 7.00

"As there must be a definite ratio between the densities of the four gases *A*, *B*, *C*, and *D* and their radius of rotation corresponding to their maximal molecular concentration, it is not impossible that from the data obtained by Wolf the density

of the lightest gas, *i.e.*, its molecular weight, which must be identical with its atomic weight, might be calculated. As regards the heaviest unknown gas, *D*, if this is not a gas of the helium-argon group, we may be allowed to point out that the existence of a gas possessing a larger atomic weight than hydrogen and a smaller atomic, but a larger molecular, weight than helium is not absolutely excluded."

This account of PROFESSOR MAX WOLF's work is of extreme interest, and has an important bearing on the phenomena which may be found to exist in all spiral nebulae when they come to be carefully investigated, as they doubtless will be some day. Meanwhile we may remark that the separation of the gases does not tell us what other chemical elements are not disclosed. Perhaps more than one interpretation of this stratification of the gases, with the lighter elements towards the centre, may be made; but the following seems the most probable: namely, the gases are liberated in the nebula by the impacts among the smaller solid bodies of which it is so largely composed, and the elements of lowest molecular weight escape and diffuse themselves throughout the whole region of the nebula. The smaller the atomic weights the more rapid the diffusion. And as the nebula forms one gigantic system revolving about its centre of gravity, the diffused gases will naturally drift towards the centre, the lighter elements outstripping the heavier ones. This explanation is simple, and it accounts for all the known phenomena by means of recognized physical laws. The investigations of DR. JOHNSTONE STONEY and others, on the escape of the atmospheres of the bodies of the solar system, have a direct bearing on the problem here discussed. In general the larger the body and the more powerful the force of gravity the lighter the elements which may be retained. In the case of small moons nearly all the elements escape. Even if a system such as the ring nebula in *Lyra* included planets of the size of *Jupiter* or larger, which could retain the gases, yet the small moons would numerically so greatly predominate, that the escape of these elements would become conspicuous in the nebula, from the impacts occurring among the smaller bodies. And the more rapid drift of the lighter elements towards the centre of the annulus would be inevitable.

§ 235. *Researches of* DR. E. A. FATH *on the Spectra of the Spiral Nebulae and of Globular Clusters.*

In *Lick Observatory Bulletin*, No. 149, DR. E. A. FATH, who is now connected with the solar observatory at Mt. Wilson, examines the question as to whether the

spectra of the spiral nebulae are really continuous, as frequently stated in many works on Astronomy, and heretofore generally believed. He found the light of the spiral nebulae so faint that very long exposures would be required to give satisfactory photographs of their spectra. It was estimated that from 450 to 500 hours' exposure would be required to give a spectrum of the *Andromeda* nebula as good as can be obtained with the Lick Mills spectrograph on a bright star such as *Arcturus* in two minutes. Notwithstanding the extreme difficulty of the investigation, considerable light was shed on the constitution of the spiral nebulae. DR. FATH's conclusions from the study of eight spiral nebulae (namely N. G. C. 224, 650-651, 1023, 1068, 3031, 4736, 5194, 7331), and three globular clusters (N. G. C. 6205, 7078, 7089) are as follows:

"The results of this preliminary investigation may be summed up in the statement: No spiral nebula investigated has a truly continuous spectrum. While this may be a step in advance, nevertheless it is wholly inadequate to answer the question as to the real nature of these interesting objects which the work of KEELER brought so prominently before the astronomical world. Their spectra vary from those having principally bright lines such as are found in the gaseous nebulae to those containing only absorption lines of the solar type.

"In trying to interpret these results it must be remembered that the spectrograms obtained record only the spectra of the denser central portions of the nebulae. In the case of the *Andromeda* nebula this amounts to five minutes of arc. Then too the very low dispersion of the spectrograph used undoubtedly masks much that is of fundamental importance. Whether the spiral arms will give the same spectrum as the central portion is a question which will be difficult to answer with present appliances.

"Careful consideration has been given to various hypotheses to account for the character of the spectra photographed. Only one hypothesis seems at all tenable and serious objections can be advanced against this one. It may be termed the 'star-cluster' theory, reached by the following considerations.

"The only known sources of continuous spectra are luminous solids, liquids, very dense gases or possibly masses of gas of great thickness. To produce bright lines or bands we require gases or vapors rendered luminous by heat, electric discharges or chemical change, or substances made to fluoresce by energy supplied by some external agency. For absorption lines or bands to be present the necessary condition is an absorbing medium, usually of a gaseous nature, between a source of continuous radiation and the observer. These are, for the most part, well established experimental facts.

"The primary or fundamental part of the spectra of the spiral nebulae is a continuous background. This is interrupted by absorption lines, and superimposed upon it in some cases are bright lines or bands. The matter producing the lines, bright or dark, must be assumed to lie between the source of continuous radiation and the observer. Hence we conclude this source to be surrounded by a gaseous envelope, of physical condition varying in the different nebulae and corresponding to the various spectra obtained. The only celestial bodies of this type with which we are acquainted are the stars.

"The hypothesis that the central portion of a nebula like the famous one in *Andromeda* is a single star may be rejected at once unless we wish to modify greatly the commonly accepted ideas as to what constitutes a star. Assuming, however, an unresolved star cluster consisting of stars mainly of solar type we would seem to have a sufficient explanation of the spectrograms of the *Andromeda* nebula. But the question arises: Is it reasonable to assume that in a condensed cluster we should have stars of one spectral type strongly predominating?

"In seeking an answer to this question I have found but two references to observations of the spectra of known dense star clusters. These are not sufficiently definite to settle the matter. HUGGINS (*Phil. Trans.*, 156, 389, 1866), in reporting an observation on the great star cluster in *Hercules*, says: 'Spectrum of central blaze continuous. Spectrum ends abruptly in the orange. The light of the brighter part is not uniform; probably it is crossed by bright lines or lines of absorption.' VOGEL (*Astr. Nachr.*, 78, 245, 1871) likewise observed this cluster and writes as follows: 'Der bekannte Sternhaufen im *Hercules* zeigte ein sehr schwaches continuirliches Spectrum, welches sich von ca. 620 bis Milliontel Millimeter verfolgen lies. In demselben waren einige dunkle Streifen zu erkennen, konnte aber wegen zu grosser Lichtschwäche nicht gemessen werden.'

"It was accordingly necessary to investigate some known star clusters. From the CROSSLEY negatives three dense clusters were selected for trial; N. G. C. 6205, 7078, and 7089. The first, the famous cluster in *Hercules*, as stated above, gave evidence of containing stars of different spectral types, while the other two gave F-type spectra only. Photographs show clusters such as these to be made up of two groups of stars (*Lick Observatory Bulletin*, 3, 49, 1904, and *Astrophysical Journal*, 20, 354, 1904), the one of magnitudes 11 to 13 and the other of magnitude about 16. The brighter group was probably the only one giving the spectrograms. If we could view such a cluster from a distance so great that it could not be resolved into its component stars there is no question but that it would continue to give

such spectrograms as have been described. We thus have the question as to whether clusters can be found in which stars of one spectral type predominate answered in the affirmative.

"The 'star cluster' interpretation of the results obtained stands or falls by the question of parallax. It does not seem reasonable to assume stars that should be many times smaller than the Sun. We also require the cluster to be at such a distance that it can not be resolved. Now the only determination of the parallax of a spiral nebula that the writer has been able to find is of the *Andromeda* nebula by BOHLIN (*Astr. Iaktt.* o. Unders. Å Stockholms Obs., 8, No. 4, 66, 1907). He finds this to be $0''.17$. His result rests on two separate determinations, each of which gave positive values for the parallax both in α and δ. The result is therefore entitled to some confidence. If, now, the parallax of the *Andromeda* nebula is of this order of magnitude, the 'star cluster' theory, at least for this particular object, is not very satisfactory. For, assuming this parallax; a surface brilliancy of the component stars equal to that of the Sun; and a ratio of brightness of the central portion of this nebula to that of the Moon of $\frac{1}{180,000}$ (derived from CROSSLEY negatives): a simple computation shows that if the stars are so closely packed as to be irresolvable their dimensions are of the order of the asteroids; while if assumed only as large as *Jupiter* they would appear some seconds of arc apart. This is not found to be the case. The 'star cluster' theory, however, seems to be the only one that can at all adequately explain the spectrum obtained, and another determination of the parallax, preferably by another observer with a different instrument, would be of great interest."

§ 236. *Part Played by Repulsive Forces in Producing the Appearances of the Nebulae.*

The above quotation from DR. FATH's important paper is of considerable length, but it was found difficult to abridge it; and moreover there appeared to be an advantage in having the evidence of others that the spiral nebulae are filled with bodies varying in size between the asteroids and the planet *Jupiter*. The "star cluster" theory of spiral nebulae is thus seen to be supported by spectroscopic evidence, and at the same time by the capture theory of the origin of our solar system, which is based on dynamical principles combined with observed phenomena among the bodies still surviving and circulating as planets and satellites. For we have found in our solar system the most convincing evidence that our primordial nebula was filled with great numbers of moons and planets. The recognition of a similar condition in the spiral nebulae, by which these objects

become allied to star clusters, is therefore of deep interest. We should in all probability conclude that the spiral nebulae have an abundance of moons and planets, but that individually they are seldom continuously self-luminous. We should rather infer that the light of the whole nebula is due to certain transformations of energy, with luminescent effects produced by electric discharges in high vacua, much of the energy being liberated by impacts of various kinds of satellites. The light thus alternately arising among an infinite number of small bodies would render the entire nebula faintly luminous, like a candle shining through thin horn, as was remarked by SIMON MARIUS, in describing the great nebula of *Andromeda*, soon after the invention of the telescope.

The light of a nebula in many respects is analogous to that of a comet, the tail of which is illumined only by Cathodic rays emanating from the Sun. In the case of a nebula naturally there are many centres of radiation, and charges of positive electricity are developed, giving rise to repulsive forces, as well as phosphorescent effects, and Roentgen rays, with the resulting precipitation of ions. The condensation of vapors under the powerful influence of X-rays produces the peculiar molecular structures found in meteorites, and therefore known to exist in all nebulae. And just as the charges borne by the particles of a comet's tail causes the repulsion of minute corpuscles from the Sun, so also this same dispersion takes place from other stars, and even from the most powerful centres of radiation in nebulae which are developing stellar nuclei. *The matter repelled from these centres in a nebula, however, must necessarily be made up of excessively minute particles, and it therefore constitutes but an infinitesimal part of the vast mass which has been gathered together by capture from the surrounding regions of the heavens.* Accordingly it appears that the principal agency operating on the figures of the nebulae is the attractive force of universal gravitation, while the electric and other repulsive forces exert a secondary influence which is sometimes traceable in the forms of the nebulae as observed in the immensity of space. Thus the outer parts of the *Orion* and also of the *Trifid* nebula have been thought to give evidence of repulsive forces, but the centres of repulsion, if such movement really took place, are not definitely indicated, and it is uncertain whether the repulsion proceeded from within the nebulae, or from external sources among the stars, but probably the latter. As the nebulae are formed by the gathering together of diffused particles of cosmical dust, it is difficult to distinguish the gravitational from the dispersive tendencies. The approach to order noticed in the spiral nebulae, however, is decisive in showing that under normal conditions attractive forces are predominant, while repulsive forces exert only secondary influence, which is scarcely recognizable in the vast majority of the nebulae.

§ 237. *How the Moons and Planets in a Nebula Originate.*

We have found that the nebulae are full of solid globes which, for convenience of diction, we call moons; and we have seen that most nebulae are originally of vast extent, but with the lapse of ages undergo secular shrinkage. How, then, do the nebular moons originate? We answer: By the precipitation of ions under the influence of X-rays and the condensation of the resulting cosmical dust to many centres, while the nebula is still so diffused that each centre is undisturbed by the neighboring centres of attraction.

Under these conditions even a very small moon may slowly increase its mass, because its sphere of influence is comparatively large; and the individual moons revolve for immeasureable ages without passing near disturbing bodies. Accordingly, in the early stages of nebular development, accretion goes on at an infinite number of centres. In the later stages the satellites become dense enough to introduce collision as a common occurrence, and this destroys some of the independent moons, while others grow correspondingly in size. In the early stages of nebular condensation the bodies gathered up by the moons are mainly particles of cosmical dust; in the later stages the nebulosity is more exhausted and the impact of solid globes increases, owing to the shrinkage of the nebula as a whole, and the increasing density or number of these globes in a given amount of space.

In the early stages of a nebula the moons are isolated and control a considerable sphere of influence; in the later stages this sphere of influence is more restricted, owing to the action of the other bodies, and especially the Sun. Accordingly, it follows that the process of growth of moons is a very slow one, and occupies billions of years. A large part of this time corresponds to the purely nebular stage, during which small globes are forming, but the nebula is not yet developing into a planetary system. Just how long the nebular stage is, compared to the planetary stage, it is difficult to determine; probably the two stages are not very unequal in length. Moreover, the two stages run together, and are not clearly divided one from the other, so that at present we can only estimate the lengths of the two periods.

On the one hand, it is clear that the nebular stage is of exceedingly long duration, because in the beginning the density is insensible, and the expansion extreme; so that the central attraction is feeble, and the movement of circulation correspondingly slow. On the other hand, the planetary stage must be of very great duration, otherwise such round orbits could not be produced by the secular effects of the resisting medium. The total history of such a system extends over billions of years, and the formation of moons by the slow process of accretion therefore presents no difficulty.

§ 238. *Impacts Such as Formed the Lunar Craters are Frequent Occurrences Within a Growing Planetary System, Because of the Abundance of the Small Bodies in Such a Condensing Nebula; but Collisions in the Stellar Universe Between Independent Stars and Systems are Extremely Rare Events.*

In Chapter XIV, which deals with the phenomena of the Lunar surface, we have found that impacts of satellites have produced the craters and other mountains now observed on the Moon. As these craters are very abundant, the conclusion was drawn that impacts of bodies against the Lunar surface at one time were very common phenomena. This frequency of impact arose from the great number of small bodies formerly pervading our system, most of which have at length disappeared by absorption into the Sun and planets.

The abundance of impacts thus proved to have occurred in the solar system raises the question whether impact may not after all be a comparatively frequent event between individual stars in nature, as BICKERTON, LORD KELVIN, ARRHENIUS, and others have imagined (cf. § § 64, 65, etc.). To answer this question in a satisfactory manner, we may observe that the two cases considered, namely, *collisions within a given system, and between separate systems are quite distinct, in several ways.* In any given planetary system, such as that of which our planets and satellites are the component bodies, collisions are frequent for the following reasons:

(1) The bodies are comparatively close together, and whilst the density of them in the interplanetary spaces is small absolutely, it is very large compared to the density of such masses in interstellar space.

(2) Therefore occasional collisions in the planetary spaces are to be expected. This will necessarily follow if the orbits change slowly under the periodic and secular variations of their elements due to universal gravitation and the secular action of a resisting medium.

(3) For whilst the initial motions might be so arranged that no two of the masses would ever collide, yet, under the influence of perturbations and resistance, these conditions of stability hold only for a limited time. On the other hand, the bodies revolve for indefinite ages, in shifting orbits of which the major axes and eccentricities slowly decrease, while the other elements change continually, and this finally brings about great numbers of collisions.

(4) Accordingly it follows that sooner or later the small bodies will be absorbed by the large ones. The impacts still shown by the indentations on the face of the Moon bear witness to the terrible character of the resulting collisions.

As between separate systems collisions are extremely rare, owing to the following circumstances:

(1) In the stellar universe the stars are enormously distant from one another.

(2) They are moreover endowed with motion which is nearly rectilinear.

(3) Unless the path of a star is so much curved that it is made to close in passing, one approach is all that ever takes place between two stars.

(4) And as these approaches can seldom be close, and bodily impacts of stars are practically impossible, it follows that stellar collision is the rarest of phenomena.

(5) This striking contrast between satellite impact within a planetary system, and collision between separate stars, rests primarily on the great density of the satellites in our primordial system, or in a nebula, and on the infinitely small density of bodies in interstellar space. Moreover, the motions in systems like our own are periodic with slowly changing elements, whilst among the stars the movements are essentially restricted to single passages in nearly rectilinear paths. It is not strange, therefore, that satellite impacts should prevail, while interstellar collisions are nearly if not quite unknown.

§ 239. *The Existence of Planetary Systems About the Fixed Stars, and the Physical Conditions Which Lead to the Development of Life.*

We have seen that in the condensation of a spiral nebula some of the bodies of which it is composed come to revolve in nearly circular orbits about the central Sun and thus constitute planets such as we find in the solar system. No doubt the planets revolving about the fixed stars have usually captured systems of satellites; so that planetary systems such as our own are not special but general phenomena which may be assumed to prevail throughout the universe.

Now, since a nebula condenses into solid globes and some of these planets and satellites come to revolve in systems of great stability and permanency, it becomes interesting to inquire into the physical conditions which might lead to the development of life.

In our solar system life developed on one or more planets, which were originally parts of a nebula at the temperature of space. As the planets have gradually neared the Sun their average surface temperatures have probably increased; but the only really high temperatures they have ever experienced arose from their own internal heat, which was largely cut off as soon as they became encrusted. The same conditions would arise among the fixed stars, and thus the situation in nature generally is similar to that in our solar system.

The problem of the origin of life in the solar system is a difficult one, in the solution of which but little progress has been made; but there is an increasing interest in the subject and a growing probability of its ultimate solution. In his work on "Worlds in the Making," Chapter VIII, ARRHENIUS discusses the process by which the germs of life might be transferred from one planet to another under the effects of electric charges and radiation pressure; and he seems to show that in this way spores of life from the Earth might be diffused throughout the universe. This is a very interesting possibility, and will be welcomed by all philosophic thinkers.

But the question arises whether we are entitled to assume for the Earth any such pre-eminence among the planets of space. Might not life have come from some other planet to this one? In the solar system alone conditions favorable to the development of life exist not only on the Earth, but perhaps also on *Mars* and especially on *Venus,* whilst countless planets attend the other fixed stars.

§ 240. *The Physical Conditions on Venus Nearly Identical with Those on the Earth.*

Of all our planets *Venus* most strikingly resembles the Earth, as to size, situation, duration of year, light and heat, and lastly as respects atmosphere and day and night. For we must in all probability conclude that the Hesperian day is of about the same duration as the Terrestrial. The chances are almost infinity to one that the rotation period is not greatly different from that of the Earth. The same causes have operated to determine the rotations in both cases, and this conclusion that the Hesperian and the Terrestrial day are of about the same length is confirmed by the observed length of the Martian day. The latter is 41 minutes longer than our day, while the Hesperian day appears to be shorter by 35 minutes. In view of the situation of *Mars,* about as much without the Earth's orbit as *Venus* is within, this close agreement is remarkable and can scarcely be due to chance.

Moreover, many observers of markings on *Venus* agree in making the observed rotation of that planet about 23 hours and 21 minutes.

It is well known that the early observations of HUGGINS and VOGEL indicated the presence of water vapor in the atmosphere of *Venus,* which is almost as dense as that of the Earth. With day and night and seasons similar to those on the Earth, the obliquity being probably about 20°, or less, it will be readily seen how extremely probable it is that life exists on the Hesperian globe. The physical conditions are evidently much more suitable to the maintenance of life than are those on *Mars,* which have long been such a subject of discussion. It is probable that

the atmosphere of *Venus* is lighter than that of the Earth, but still dense enough to give an atmospheric pressure of some 27 inches.

The average temperature on *Venus* is also doubtless higher than on the Earth, but not high enough to exclude the formation of ice and snow. Though large polar caps of ice have not been proved to exist, they might be so completely obscured by clouds that we could not detect them even if very prominent. The blunting of the southern horn of the planet shows that the surface is not smooth, but most likely traversed by mountain ranges of great height, as was long ago inferred by SCHROETER. The formation of unusually high clouds, if the terminator irregularities were caused in this way, would hardly be confined to one region at the southern horn, unless the mountains modified the air currents, which probably is the case.

The high albedo of the planet, 0.77, which shows that it reflects almost as much light as new fallen snow, indicates that the Cytherean globe is veiled in clouds; and the vapor producing these clouds can be none other than that of water, as indicated directly by the spectroscopic researches of HUGGINS, VOGEL, TACCHINI and LOWELL.

Altogether it is practically certain that *Venus* is an abode of life, and that the life on the planet is closely similar to what we find on the Earth. The planet is so largely covered with clouds as to greatly increase the difficulty of fixing the rotation period. Whilst this is disappointing, it assures us that the planet has an abundance of water, with continents, mountains, rivers, seas, and oceans, and is therefore refreshing as showing the unquestioned possibility of the highest form of life on another planet of our solar system.

It is well known that *Venus* is the most beautiful of all the planets. This was recognized by the Greeks from the earliest ages; for in an account of the brightness of *Achilles'* spear, HOMER has a significant description:

οἷος δ' ἀστὴρ εἶσι μετ' ἀστράσι νυκτὸς ἀμολγῷ
ἕσπερος, ὃς κάλλιστος ἐν ὀυρανῷ ἵσταται ἀστήρ.

"Just as the star *Hesperus*, which is classed the fairest star in heaven, pursues its course with the stars in the darkness of night," etc. (*Iliad*, XXII, 317–319).

In view of the substantial proof now arising from many circumstances, but more especially from the short period of the planet's rotation, that *Venus* is certainly habitable, and therefore undoubtedly an abode of life, probably of almost as high and varied an order as we have upon the Earth, HOMER's classification of the beautiful planet as the fairest star in heaven seems wonderfully appropriate and even prophetic.

In a paper on the existence of planets about the fixed stars presented to the American Philosophical Society, held at Philadelphia at the General Meeting, in April, 1910, the author has discussed more in detail the development of planets about other suns. It is shown that our planets *survive from the solar nebula, but never were thrown off as formerly supposed;* so that they have developed independently of the Sun, and been added on from without, as evolutionary products of the outer parts of the nebula. By the same process it follows that in the condensation of the nebulae, which have formed the other fixed stars and given them rotations about their axes, *planets will necessarily have survived about them also; so that planetary systems unquestionably exist about the great majority of the fixed stars.* And if worlds of habitable character are so abundant throughout space, of course the natural phenomenon called life is not wanting. Such a narrow view would be inconsistent with the *fundamental doctrine of uniformity,* which was long ago made the basis of Natural Philosophy, as formulated by the illustrious NEWTON, and which has given us the great development of the sciences during the past two centuries.

Considerable numbers of men of science in our time have fallen into such materialistic habits of thought that they are loth to admit the existence of life on other planets. This is, however, a proper subject of scientific inquiry, and liberal minds will meet it in a straightforward and candid manner. In time we shall gain sufficient light to bring out the folly of materialism and enable the earnest seeker after truth to return to the ancient but more comprehensive philosophy of PLATO and ARISTOTLE, who taught the Greeks and all subsequent ages that the ultimate problems of life are worthy of the study of the greatest minds.

CHAPTER XX.

Theory of Double and Multiple Stars.

§ 241. *General Considerations on the Discovery of Spectroscopic Binary Stars.*

The discovery of physically connected double stars with components so close together as to be beyond the resolving power of any telescope, which was made at the Harvard and Potsdam Observatories in the early nineties, together with the great extension of this branch of research in recent years, chiefly at the Lick, Yerkes, Allegheny, and Potsdam Observatories, may be regarded as one of the most remarkable developments in modern astronomy. By the application of the Döppler-Huggins principle in modern spectrographs, an entirely new light has been thrown upon the constitution of the sidereal universe. Systems of a markedly different type from our own are shown to be abundantly scattered throughout space; and although our solar system should not be considered unique among cosmical systems, it may be so regarded among those which at present are known to us by the direct evidence of the telescope or spectroscope.

For at the great distance of the fixed stars we could not see non-luminous bodies so small as our planets. And heretofore we have been so completely in the dark regarding the mode of formation of the solar system that we could not safely infer the existence of planetary systems about the fixed stars. If, however, the Capture Theory be admissible, and our Sun has captured and at length produced an orderly planetary system out of the chaotic nebulosity once whirling about it, it will follow that other stars likewise have developed similar orderly systems by the same process. Our knowledge of other planetary systems is thus deductive, yet being based on known mechanical laws, may be said to be well established, but not by direct observation.

Visual double star Astronomy, although considerably transformed at certain epochs, has been of comparatively steady growth. It began essentially with the explorations of Sir Wm. Herschel in the latter part of the 18th century; and in the early part of the 19th century was greatly extended and placed upon a firm basis by the researches of W. Struve. It was worthily maintained by Sir John Herschel, Otto Struve and Dembowski; and of late years has been vastly extended by

Burnham and the observers whom his incomparable example has inspired. And not only has observation been greatly extended, but also calculation and higher theoretical research on the orbits and perturbations of the stellar systems; until at the present time the number of fairly accurate orbits will be not less than 80. The author of these *Researches* and others are still occupied with the revision of these orbits, and doubtless we may expect more accurate data in the course of a few years.

The most important element of the orbit of a double star is the eccentricity. For the 80 orbits now determinable this will average about the same as that found from the 40 orbits treated by the writer in 1896. Perhaps the average eccentricity will be slightly increased by the latest calculations, but the change from 0.5 will be only a few hundredths; and for our present discussion we may therefore adhere to the value deduced fourteen years ago.

For a long time after the orbits of double stars began to be carefully studied it was customary to think of the periodic time as the most important element, partly because of the convenience of observers, and partly because, when the annual parallax is known, this and the semi-axis major gives the mass of the system.

Sir John Herschel, however, many years ago remarked that the eccentricity is physically speaking the most important of all the elements. This was evident to the author from the time of his earliest detailed study of the elements of binary stars in 1888. Probably the correctness of this view is now generally recognized by astronomers. And fortunately the eccentricity is an element which may be determined with accuracy both in the visual and spectroscopic systems, so that we are now in a position to study their leading characteristics, although the data at hand are not yet so complete as might be desired.

§ 242. *Rapid Progress of Stellar Astronomy During the Last Fourteen Years.*

Until recently no one could have anticipated the results of researches on spectroscopic and visual binaries which we have to-day. It is curious and instructive to reflect over the great changes of opinion which have occurred within the past fourteen years. Even no longer ago than 1896 any one who dared to believe in the existence of a great number of stellar systems was considered an enthusiast, and encountered the usual opposition from conservative men of science who are always in the majority. In some well known cases a few of these individuals declared that certain branches of photography were the only lines of research in which rapid progress was being made; and emphatic objection was made to work in so unpromising a subject as double stars. Who can name a branch of scientific research to-day in which greater and more enduring progress has been made than in visual and spectroscopic double star astronomy? Certainly the change of opinion has been remarkable!

According to estimates based on work done at the Lick Observatory, CAMPBELL found that about one star in five of those examined proved to be a spectroscopic binary; and in certain groups of stars FROST has found the ratio of binaries to run as high as one-third. If, then, the stellar universe be taken as made up of two hundred million stars, it will follow from CAMPBELL'S estimate that there are in our sidereal system some forty million double and multiple stars, all with relatively large masses; and it is impossible to estimate how many smaller bodies may attend the stars which appear to be single. But for reasons set forth in this work, based on the analogous effects of similar causes, it appears that in all probability the other one hundred and sixty million stars have systems of planets and satellites of their own essentially like those of our solar system.

The telescope discloses especially the widely separated, and probably only the luminous, companions among the systems comparatively near us in space; while the spectrograph reveals equally all attendant masses which are large enough to disturb perceptibly their luminous attendants, whatever be their distances in the depths of space. It was this independence of the spectrograph of angular separation, so long as the orbit merely remained telescopically visible, that suggested the method proposed by the author for measuring the distance of the Milky Way (cf. *A.N.*, 3323; and these *Researches*, Vol. I, Chapter I, § 7). Although the application of this method is somewhat restricted, it looks as if it might become available for actual use in the course of the present century.

It is impossible to close this account without recalling the inestimable debt which Astronomical Science owes to SIR WILLIAM HUGGINS, the illustrious founder of spectroscopic astronomy, who was the first to render DÖPPLER'S principle practically useful in the study of the heavens. Not only was he the first to measure the motions of stars in the line of sight, but also one of the first to recognize the fertility of the lines of research here laid down. And if now, after twenty-two years' labor, we may concur in PROFESSOR SCHIAPARELLI'S view, that a secure foundation has been laid for a scientific cosmogony, we must recall that it is made possible largely by the extension of the labors which SIR WILLIAM HUGGINS began in the early sixties. What lover of truth will not rejoice that our illustrious contemporary has lived* to see the ripe fruits of his pioneer labors, when few would consider and still fewer believe in the new methods, which have so powerfully revolutionized every branch of astronomical science?

* We grieve to record that since this was written the venerable SIR WILLIAM HUGGINS has departed this life, in the 87th year of his age, May 12, 1910. By researches extending over more than half a century he placed the Science of Astrophysics on a secure foundation, and will always deserve to be remembered among the greatest and noblest philosophers of all time. It was he who first established the *chemical uniformity of Nature* by showing that the elements are the same wherever a star twinkles. He thus earned the title of the NEWTON of Astrophysics. (cf. the writer's "Tribute to the Memory of SIR WILLIAM HUGGINS," in *Popular Astronomy*, for August, 1910.)

But only the smallest beginning has been made on the science of the heavens, and to quote a memorable appeal of SIR WILLIAM HERSCHEL: "The subject promises so rich a harvest that I cannot help inviting every lover of astronomy to join with me in observations that must inevitably lead to new discoveries."

§ 243. *Table of Spectroscopic Binary Stars Whose Period and Eccentricities Have Been Determined.*

Name	α 1900.0	δ 1900.0	Mag. Vis. Phot.	P	e	Authority	Source
	h m	° ′		d			
α *Andromedae*	0 3.0	+28 33	2.2 ; ..	96.67	0.525	R. H. BAKER	*Publ. All. Obs.*, I, No. 3
α *Ursae Min.*	1 22.6	+88 46	2.1 ; 4.4	3.9683	0.2001	HARTMANN	CAMPBELL, *L.O.B.*, 79
β *Arietis*	1 49.1	+20 19	2.8 ; ..	107.0	0.88	LUDENDORFF	*Aph. J.*, June, 1907
β *Persei*	3 2.	+40 34	2.2–3.7 ; ..	2.87	0.031	SCHLESINGER	*Publ. All. Obs.*, I, No. 5
π^4 *Orionis*	4 46.	+ 5 26	.. ; 4.0	9.5191	0.027	R. H. BAKER	*Publ. All. Obs.*, I, No. 15
α *Aurigae*	5 9.3	+45 54	0.2 ; ..	104.022	0.0164	REESE	*L.O.B.*, No. 6, 1901
β *Orionis*	5 10.	− 8 17	0.3 ; ..	21.90	0.296	J. S. PLASKETT	*Aph. J.*, July, 1909, p. 31
η *Orionis*	5 19.4	− 2 29	3.4 ; 3.9	7.9896	0.016	ADAMS	*Aph. J.*, 17, p. 68
ψ *Orionis*	5 21.6	+ 3 1	.. ; 4.5	2.52588	0.065	J. S. PLASKETT	*Aph. J.*, Nov., 1908, p. 272
δ *Orionis*	5 26.9	− 0 23	2.6 ; var	5.7333	0.103	HARTMANN	*Aph. J.*, 19, 268, 1904
ι *Orionis*	5 30.5	− 5 59	.. ; 3.4	29.136	0.74	PLASKETT & HARPER	*Aph. J.*, Dec., 1909, p. 381
ζ *Tauri*	5 31.7	+21 5	3.0 ; 3.4	138.	0.180	ADAMS	*Aph. J.*, Sept., 1905
B.D.−1° 1004	5 36.0	− 1 11	5.1 ; ..	27.160	0.76	PLASKETT & HARPER	*Aph. J.*, Dec., 1909, p. 381
*R T Aurigae**	6 22.1	+30 34	4.9–5.9 ; ..	3.73	0.36	DUNCAN	*L.O.B.*, 151
ζ *Geminorum*	6 58.2	+20 43	3.8–4.3 ; ..	10.154	0.22	CAMPBELL	*Aph. J.*, 13, 90, 1901
α_1 *Geminorum*	7 28.2	+32 7	3.7 ; ..	2.92835	0.07	CURTIS	CAMPBELL, *L.O.B.*, 79
α_2 *Geminorum*	7 28.2	+32 7	2.7 ; ..	9.21883	0.503	CURTIS	CAMPBELL, *L.O.B.*, 79
V Puppis	7 55.3	−48 58	4.1 ; 4.8	1.454	0.00	ROBERTS	*Aph. J.*, 13, 177
κ *Cancri*	9 2.0	+11 4	5.0 ; 5.3	6.393	0.149	ICHINOHE	*Aph. J.*, June, 1907, p. 318
α *Carinae*	9 8.4	−58 33	3.5 ; ..	6.744	0.18	CURTIS	*L.O.B.*, 122, 1907
κ *Velorum*	9 19.0	−54 35	2.6 ; ..	116.65	0.19	CURTIS	*L.O.B.*, 122
*l Carinae**	9 42.5	−62 3	3.6–5.0 ; ..	35.53	0.3±	WRIGHT & ALBRECT	*L.O.B.*, 151
η *Virginis*	12 15.1	− 0 9	3.7 ; ..	71.9	0.25	ICHINOHE	*Aph. J.*, *Nov.*, 1907
α *Virginis*	13 20.	−10 38	1.2 ; ..	4.01426	0.10	R. H. BAKER	*Publ. All. Obs.*, I, No. 10
RR Centauri	14 9.4	−59 27	6.0–9.8 ; ..	160.5	0.01	ROBERTS	*M.N.*, June, 1903
δ *Librae*	14 56.	− 8 7	4.8–6.2 ; ..	2.32735	0.054	SCHLESINGER	*Publ. All. Obs.* I, No. 20
α *Coron. Bor.*	15 30.0	+27 3	2.3 ; ..	17.36	0.387	F. C. JORDAN	*Publ. All. Obs.* I, No. 12
θ *Draconis*	16 0.1	+58 50	4.1 ; 4.8	3.0708	0.0141	CURTIS	*L.O.B.*, 122
β *Herculis*	16 26.0	+21 42	2.9 ; 4.2	410.58	0.55	REESE	*L.O.B.*, 79
u Herculis	17 14.	+33 12	4.6–5.4 ; ..	2.05102	0.053	R. H. BAKER	*Publ. All. Obs.*, I, No. 11
ω *Draconis*	17 37.5	+68 48	4.9 ; ..	5.27968	0.0107	A. B. TURNER	*L.O.B.*, 123, 1907
*X Sagittarii**	17 41.3	−27 47.6	4.4–5.0 ; ..	7.01185	0.40	J. H. MOORE	*L.O.B.*, 157
*Y Ophiuchi**	17 47.3	− 6 7	6.2–7.1 ; ..	17.1207	0.10	ALBRECHT	*L.O.B.*, 118
*W Sagittarii**	17 58.6	−29 35	4.3–5.1 ; ..	7.5946	0.32	CURTISS	*L.O.B.*, 62 and 79
μ *Sagittarii*	18 8.0	−21 5	4.1 ; ..	180.2	0.441	ICHINOHE	*Aph. J.*, Oct., 1907
*Y Sagittarii**	18 15.5	−18 55	5.4–6.2 ; ..	5.7734	0.3	CURTISS	*L.O.B.*, 62, 1904
χ *Draconis*	18 22.8	+72 42	3.7 ; 4.2	281.8	0.423	WRIGHT	*Aph. J.*, 11, 131, 1900
ζ^1 *Lyrae*	18 41.	+37 30	....	4.29991	0.00	F. C. JORDAN	*Publ. All. Obs.* I, No. 17
β *Lyrae*	18 46.3	+33 15	3.4–4.1 ; ..	12.908	0.07	MYERS	*Aph. J.*, 7, 1, 1898
κ *Pavonis**	18 46.6	−67 21	3.8–5.2 ; ..	9.09	....	WRIGHT	*L.O.B.*, 151
*U Aquilae**	19 24.0	− 7 15	6.2–6.9 ; ..	7.02	....	ALBRECHT	*L.O.B.*, 151
*SU Cygni**	19 40.8	+29 1	6.5–7.2 ; ..	3.85	0.21	MADDRILL	*L.O.B.*, 151
η *Aquilae**	19 47.4	+ 0 45	3.7 ; 4.5	7.176	0.489	WRIGHT	*Aph. J.*, 9, 59, 1899
*S Sagittae**	19 51.5	+16 22	5.4–6.1 ; ..	8.38	0.35	MADDRILL	*L.O.B.*, 151
θ *Aquilae*	20 6.	− 1 7	3.4 ; ..	17.117	0.685	R. H. BAKER	*Publ. All. Obs.* I, No. 7
α *Pavonis*	20 17.7	−57 3	2.0 ; ..	11.753	0.01	CURTIS	*L.O.B.*, 122
*T Vulpeculae**	20 47.2	+27 52	5.5–6.5 ; ..	4.43578	0.43	ALBRECHT	*L.O.B.*, 118
β *Cephei*	21 27.4	+70 9	3.1 ; ..	0.187	0.0±	FROST	*Aph. J.*, Nov., 1906, p. 260
ι *Pegasi*	22 2.3	+24 51	4.0 ; 4.4	10.2131	0.0085	CURTIS	*L.O.B.*, 53, 1904
2 *Lacertae*	22 17.	+46 2	.. ; 4.6	2.6164	0.015	R. H. BAKER	*Publ. All. Obs.*, I, No. 13
δ *Cephei**	22 25.4	+57 54	3.7–4.6 ; ..	5.367	0.46	BELOPOLSKI	*Aph. J.*, 3, 227, 1896
η *Pegasi*	22 38.3	+29 42	3.0 ; 4.2	818.0	0.1548	CRAWFORD	*L.O.B.*, 5, 1901
λ *Andromedae*	23 32.6	+45 56	3.9 ; 5.0	20.54	0.086	BURNS	*L.O.B.*, 105

* Variable Stars of the *Cepheid* and *Geminid* type and known to be Binary. Average eccentricity for 53 orbits = 0.23.

§ 244. *The Orbits of Spectroscopic Binary Stars and Their Distribution as Respects the Region of Eccentricity.*

In the foregoing table we have condensed the most important data now available* in regard to the periods and eccentricities of spectroscopic binaries. There are in all fifty-three orbits, with an average eccentricity of 0.23, about the same as that of the planet *Mercury*. The number of orbits with small eccentricity is very striking. The distribution of orbits as respects the region of eccentricity is as follows:

	No. of Orbits
Between 0.00 and 0.10	21
" 0.10 " 0.20	8
" 0.20 " 0.30	5
" 0.30 " 0.40	6
" 0.40 " 0.50	6
" 0.50 " 0.60	3
" 0.60 " 0.70	1
" 0.70 " 0.80	2
" 0.80 " 0.90	1
" 0.90 " 1.00	0
Total,	53

By means of the accompanying figure it becomes easy to understand this distribution graphically. The corresponding curve found by the author for the distribution of visual binaries in 1896 was given in Vol. I of these *Researches*, Chapter III. This would not be materially changed by the revision of the eighty orbits which now admit of determination.

In the work of 1896 the distribution of orbits as respects eccentricity was as follows:

	No. of Orbits		No. of Orbits
Between 0.00 and 0.10	0	Between 0.50 and 0.60	9
" 0.10 " 0.20	2	" 0.60 " 0.70	2
" 0.20 " 0.30	4	" 0.70 " 0.80	4
" 0.30 " 0.40	8	" 0.80 " 0.90	2
" 0.40 " 0.50	9	" 0.90 " 1.00	0

PROFESSOR R. G. AITKEN of the Lick Observatory has recently given considerable attention to the orbits of double stars, and published in *L.O.B.* 84, a catalogue of the best available elements. Although the author of the present work does not concur in all the elements which AITKEN adopts, the final mean results

* Since this table was prepared three important memoirs on Spectroscopic Binaries have appeared, as follows: 1. By W. W. CAMPBELL, *Publications Astron. Soc. of Pacific*, April, 1910; 2. By SCHLESINGER and BAKER, *Publications of Allegheny Observatory*, Vol. I, No. 21; 3. By H. LUDENDORFF, in *Astron. Nachr.*, Nos. 4415–16. The above discussion was written in December, 1909, and we leave it unchanged, but simply call attention to the rapid progress of this important line of research.

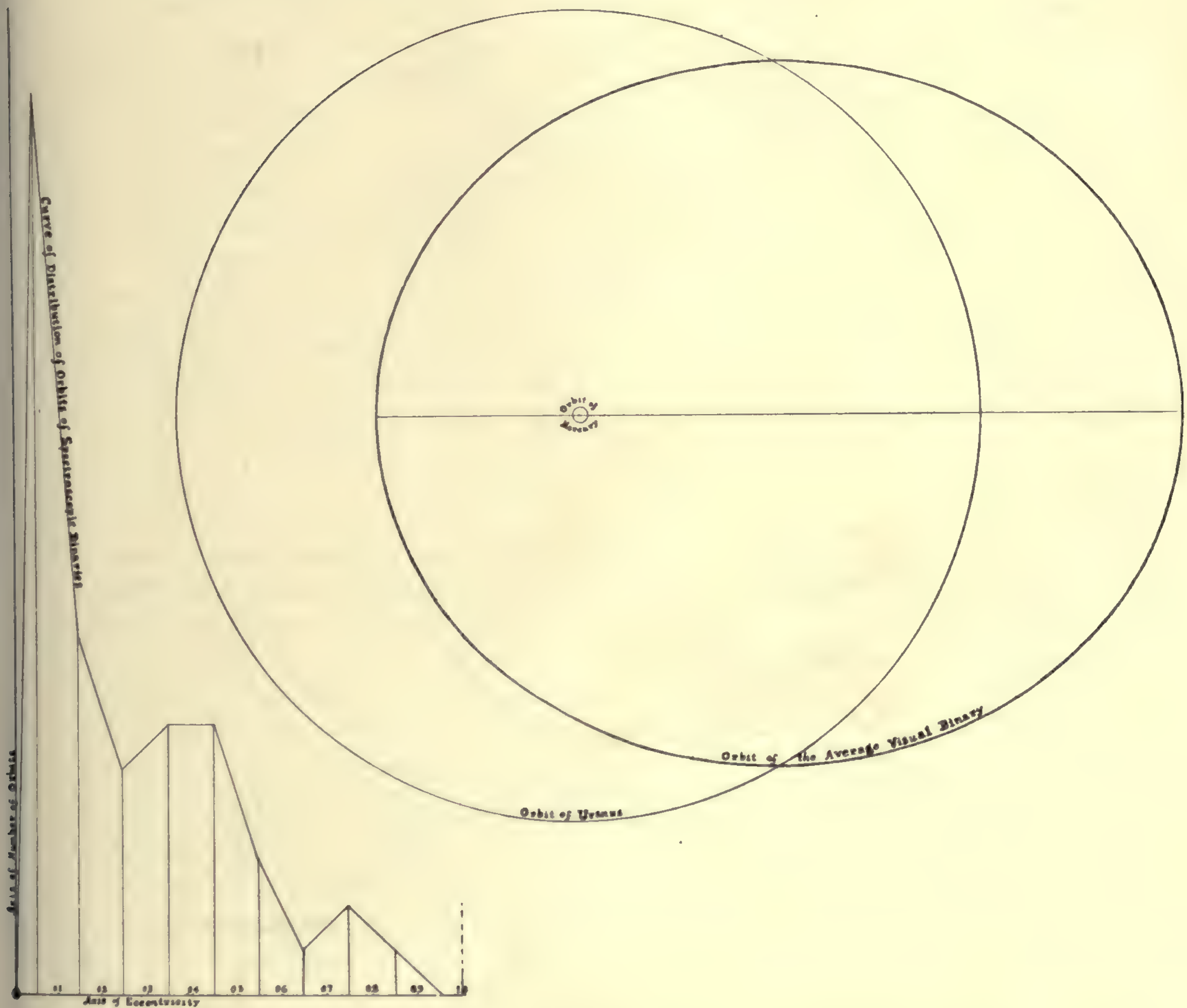

FIG. 41. ILLUSTRATION OF THE SMALL DIMENSIONS OF THE ORBITS OF SPECTROSCOPIC BINARIES, WHICH ON THE AVERAGE ARE LESS THAN THAT OF THE PLANET MERCURY; AND OF THE TENDENCY TO SMALL ECCENTRICITIES, AS SHOWN GRAPHICALLY BY THE DIAGRAM ON THE LEFT.

are but very slightly influenced thereby, and the difference between these authorities is of little importance in our present discussion. According to the forty-nine orbits given in AITKEN's list of well determined systems the distribution is as follows:

				No. of Orbits
Between	0.00	and	0.10	0
"	0.10	"	0.20	2
"	0.20	"	0.30	3
"	0.30	"	0.40	9
"	0.40	"	0.50	13
"	0.50	"	0.60	10
"	0.60	"	0.70	5
"	0.70	"	0.80	3
"	0.80	"	0.90	3
"	0.90	"	1.00	0

Thus the distribution is practically the same as that found by the author for visual binaries fourteen years ago, though the mean eccentricity, or maximum of the curve of distribution, is slightly increased by AITKEN's selections.

§ 245. *Masses of Binary Systems.*

When the visual orbit is known and also the parallax of the system, so that the dimensions of the orbit becomes known in astronomical units, the mass may be found from the formula resulting from the generalization of KEPLER's third law:

$$M_1 + M_2 = \frac{a^3}{R^3} \cdot \frac{T^2}{P^2}(M + m) \; ; \tag{540}$$

where $M + m$ is the mass of the Sun and Earth, usually taken as unity; $T = 1$ year, $R =$ semi-axis major of the Earth's orbit, both ordinarily taken as unity; and a and P the semi-axis major and periodic time of the stellar system. Accordingly, for ordinary double stars, when the year, the combined mass of the Sun and Earth, and the Sun's mean distance are taken as units, the formula becomes:

$$M_1 + M_2 = \frac{a^3}{P^2}. \tag{541}$$

When the parallax is not known by the ordinary method of direct measurement, but the orbit is determined by visual observations, and the relative movement of the components in the line of sight at any time can be found by the spectrographic method, the parallax is fixed by a single spectrographic determination. In this case the absolute dimensions and mass of the system becomes known with accuracy (cf. Vol. I of these *Researches*, p. 30).

If the orbit cannot be seen visually, so as to disclose its dimensions in arc, and give its visual elements, spectrographic observations alone are seldom sufficient to give the dimensions and mass of the system. In some cases, however, where the plane of motion is in the line of sight, as in the *Algol* variables, the dimensions of the orbit of the bright component about the centre of gravity of the system becomes known from spectrographic observations alone, without any observations of the visual dimensions of the orbit. The number of systems thus admitting of partial determination of mass and dimensions is only a small fraction of the whole; yet it will, in time, become very considerable.

In general, spectroscopic binaries, which can not be seen visually separated, do not give us their mass and dimensions. Spectrographic observations give us only the motion in the line of sight, and the total range of this variation during a revolution; but as the inclination of the orbit, as well as the angular separation of the components, is unknown, the absolute dimensions and mass of the system can not be determined. The chief hope of extending our knowledge of this numerous class of systems, lies in the increase of the resolving power of telescopes. This is not very promising at present, but eventually it will enable observers to separate some of them.

In the foregoing list of spectroscopic binaries we have given only those systems which have not been visually resolved. It is true that PROFESSOR CAMPBELL includes in his first *Catalogue of Spectroscopic Binary Stars, L.O.B.* 79, seven visual systems, namely: α *Canis Majoris*, α *Geminorum*, ϵ *Hydrae*, AB, γ *Virginis*, α *Centauri*, ζ *Herculis*, δ *Equulei*; yet as the elements in each case are deduced not from spectroscopic observations, but from micrometer measures, they properly belong to the class of visual binaries. And as the orbits are included in the latter class, they should not be repeated in the list of spectroscopic binaries, though they form an interesting connecting link between the two types of systems.

§ 246. *Spectroscopic Binaries Differ from Visual Binaries Chiefly in the Smaller Semi-Axes of Their Orbits and the Shorter Periodic Times.*

Heretofore no spectroscopic binary has been resolved telescopically, though a few visual binaries with known orbits have been found by spectrographic observations to be also spectroscopic binaries; so that the two classes of systems are merged into one common whole. If our telescopic power could be increased one hundred-fold, or even ten-fold, the connection would become much closer than it is now. In that case a good many spectroscopic binaries would be resolved into their visual components.

But the resolving power of telescopes is not sensibly increasing at present, nor are the future prospects very hopeful in this respect; and we must therefore reverse the problem and verify the visual binaries by spectrographic observations. This was recommended by the author in 1896, and has since been done in a perfectly satisfactory manner, in such cases as δ *Equulei*, α *Centauri*, *Sirius*, etc.; so that we may feel entirely confident that spectroscopic binaries differ from visual ones chiefly in their smaller semi-axes and shorter periodic times, with the resulting greater velocity in the line of sight. The mean distances of spectroscopic binaries are so small that we can not separate the components in any except a few cases where the relative motion is comparatively slow, and the orbits so large that they were already known as visual binaries before the application of the spectrographic method.

Since the spectroscopic binaries *on the average* probably are about as massive as the visual binaries, *as a class*, the formula for the mass of a system $M_1 + M_2 = \frac{a^3}{P^2}$, enables us to infer from the known periods about what *the average distance* is also.

Thus for the 53 systems previously considered we find the average period to be about 37 days. But if we exclude a few long-period stars, the average period of all the rest is about ten days. Taking $M_1 + M_2 = 1$, as in the case of the Sun and Earth, we find the average value of the mean distance to be 0.2173 astronomical unit, when $P = 37$ days; and only 0.09088 astronomical unit, when $P = 10$ days. In the case where $P = 37$ days the orbits are therefore a little over half the size of that of the planet *Mercury*, which has a mean distance of 0.387. But in the more typical case where $P = 10$ days, the orbits are only one-fourth that size, or have a mean distance equal to about ten times the Sun's diameter.

This result is remarkable and full of meaning of great significance for the general interpretation of the Universe. The average dimensions of the visual binary systems may be compared to those of our major planets in the solar systems, from *Jupiter* to *Neptune*, where the period ranges from twelve to one hundred and sixty-five years. An orbit like that of *Jupiter* corresponds to the more rapid class of visual binaries found of late years by BURNHAM and other modern observers. It is safe to infer that *as a class* the spectroscopic binary orbits are smaller than the visual ones in a ratio comparable to that found between the orbits of the planet *Mercury* and those of *Saturn* or *Uranus*. Their periods naturally stand in a similar ratio, according to KEPLER's law. But in many cases the spectroscopic binaries have mean distances very much smaller than that of *Mercury*, and in some cases only a few times larger than the diameter of our Sun.

It follows therefore that the spectroscopic binaries *as a class* have much smaller orbits and shorter periods than the visual binaries; just as when a binary is tele-

scopically resolved into a closer triple system, the closer component has the smaller orbit and the shorter periodic time. And if the dimensions of the orbits are increasing under the influence of the secular action of tidal friction, the spectroscopic binaries are in an earlier stage of development. Under this influence, in the course of millions of years, their orbits will grow more and more like those of the visual binaries. If, however, the resisting medium is the more dominant cause, as appears to be the case, then the secular effects will be just the reverse of those here indicated.

Where the period is short and the expansion or contraction of the orbit rapid, there is good reason to hope that changes in the periodic time, as well as in the eccentricity and motion of the line of apsides, may become sensible to observation before many years. Under tidal friction the mean distance and eccentricity will steadily increase with the time, while the line of apsides will progress, as in the well known case of the Fifth Satellite of *Jupiter,* where this movement has been found to afford the most exact method of evaluating the oblateness of that planet. In fact ADAMS and COHN determined *Jupiter's* oblateness by this theoretical method, and the author of these *Researches* afterwards found by a series of careful micrometer measures of this planet that the observed and theoretical oblatenesses are in entire accord (cf. *A.N.*, 3670, p. 406).

In the same way the study of the motion of the line of apsides for certain double stars may give us a very accurate means of determining their oblateness of figure, although the diameter of a fixed star may never be sensible in any telescope, unless designed on the principle recommended by PROFESSOR MICHELSON (cf. "Light Waves and Their Uses," *Decennial Publications of the University of Chicago,* pp. 143-144). There is thus opened to our contemplation not only the possibility of great progress in the study of the forms and dimensions of binary orbits, but also in the theory of the figures of the stars themselves, which may be considered indeed a crowning achievement of Astronomical Science.

In this connection attention should be called to the important researches of DR. A. W. ROBERTS of South Africa, whost methods promise better results for variable stars than all others put together, because they rest on a true physical cause.

§ 247. *Investigation of the Form of the Orbits of Variables.*

In the *Monthly Notices* for June, 1903, DR. ROBERTS shows how to find the form of the orbit of *RR Centauri* from the variations of the light. PROFESSOR G. W. MYERS had previously treated the variable star *β Lyrae* in a somewhat analogous manner (*Aph. J.*, Vol. 7, 1898), in accordance with the conceptions laid down in the author's *Inaugural Dissertation* of 1892.

ROBERTS shows that in the case of *RR Centauri* "the eccentricity can not in any case be greater than 0.01, and of course may be much less. There is conclusive evidence therefore that *RR Centauri* revolves in a practically circular orbit." He finds the density to be one-third that of our Sun.

In the *Monthly Notices* for May, 1905, DR. ROBERTS treats several other variables with great ingenuity, and deduces their distances apart as well as oblateness. He finds the prolateness of *RR Centauri* to be 0.78, of *R Vulpeculae* 0.39; the latter being "composed of nearly equal masses, one of which is two and a half times brighter than the other." The space between the globes is 0.24, when the radius of the orbit is taken as unity. In the same paper DR. ROBERTS gives this remarkable table:

System	Distance Between Components	Prolateness Observation	Prolateness Theory
RR Centauri	−0.01	0.78	0.66
β Lyrae	+0.01	0.57	0.58
V Vulpeculae	+0.24	0.37	0.35

"The nearer the components are to one another the greater their prolateness, and the amount of prolateness is in fair conformity with what theory alone would indicate."

The Report of the British Association for the Advancement of Science, for 1905, pp. 249-256, contains an important report by DR. ROBERTS on "Apsidal Binary Star Systems," in which it is proved that "the component stars of close binary systems move in almost circular orbits. In no case is the eccentricity of motion greater than 0.1; the average eccentricity of the six stars considered is one-fourth of this." In the case of *β Lyrae* he finds evidence of a recession between the two components, and that the recession is gradually diminishing in rate; also that the line of apsides of *β Lyrae* is revolving. The eccentricities found by ROBERTS for the orbits of these six stars are as follows:

Name of Star	Eccentricity e	Period	Prolateness	Density, Sun's Density Being Unity
V Puppis	0.018	$1^d\ 10^h\ 54^m\ 27^s$	0.50	0.025
X Carinae	0.020	1 1 59 0	0.48	0.050
RR Centauri	0.017	0 14 32 7	0.78	0.250
β Lyrae	0.022	12 21 59 10	0.50	0.0003
V Vulpeculae	....	75 days	..	0.00002
U Pegasi	0.034	$0^d\ 8^h\ 49^m\ 41^s$	0.51	0.36

Ever since the publication of the author's *Inaugural Dissertation* at the University of Berlin, in 1892, many have believed that variable star phenomena must

be referred principally to attendant bodies, with the resulting tidal phenomena, and a resisting medium. It is agreeable to notice, that recent investigators usually adopt the theory of tides and occultations by attendant bodies and the action of a resisting medium; and it seems certain that the early work of MYERS and ROBERTS and the later work of CURTISS, ALBRECHT and DUNCAN will mark an epoch in this important subject. The results brought out in the present work seem to show that the resisting medium plays the most important part in the phenomena of variable stars, which will be further discussed in Chapter XXIII.

§ 248. *On the Dynamical Relationship Between the Mass-ratio and the Efficiency of the Secular Action of Tidal Friction and of the Resisting Medium in the Stellar Systems.*

The possible expansion of the orbits of double stars under the secular action of tidal friction, suggested by the study of the comparative eccentricities of the orbits of spectroscopic and visual binary stars, will naturally increase the interest in certain problems connected with the action of this great physical cause, the exact investigation of which we owe principally to PROFESSOR SIR G. H. DARWIN (cf. *Philosophical Transactions and Proceedings of the Royal Society*, 1878-1882).

Bodily tides certainly exist wherever the force of gravity operates between neighboring globes of fluid matter, and thus tidal friction is a physical agency as universal as gravitation itself.

The only question remaining is whether tidal friction or the action of the resisting medium is the more dominant cause. The latter agency indeed seems to be the most important, and we believe its influence has generally been paramount. Nevertheless it will be worth while to examine the secular effects of tidal friction; and especially since, by reversing the secular effects of this cause, we obtain very nearly the corresponding effects of the resisting medium. Thus one examination will serve to throw light on the effects of both causes in systems with varying mass-ratio.

We shall begin with the consideration of the effect of varying mass-ratio upon the moment of momentum of orbital motion, when all the other conditions are supposed to be uniform and unchanged. If we have two masses, M and m, revolving in circular orbits about their common centre of inertia, with a distance between their centres of figure equal to ϱ, and an angular velocity Ω, the moment of momentum of orbital motion will be given by the expression

$$M\left(\frac{m\varrho}{M+m}\right)^2\Omega + m\left(\frac{M\varrho}{M+m}\right)^2\Omega = \frac{Mm}{M+m}\varrho^2\Omega\,. \tag{542}$$

In the right member of this fundamental equation, we find that ϱ and Ω will be unaltered by a change in the mass-distribution within the system; for the distance is fixed by hypothesis, and the period depends wholly on the combined mass, not on its distribution; so that the only change taking place will depend on the ratio $\frac{Mm}{M+m}$, which becomes one-fourth when M and m are equal, and each equal to one-half of the total mass of the system.

Now suppose the equable distribution of mass to be changed, so that one mass becomes smaller than the other, as implied by the symbols M and m. As the total mass $M + m = C$, a constant, and mean distance of the system remains the same, the periodic time, or angular velocity Ω, is unaltered, according to KEPLER'S law. Under the conditions here assumed, the mass lost by m is gained by M; and thus we may suppose the mass-ratio of the smaller to the larger body to be any fraction less than unity. When $m = \frac{1}{2}M$, or $m = \frac{1}{3}(M + m)$, the expression $\frac{Mm}{M+m} = \frac{2}{9}$; and this function decreases rapidly with decrease in the relative value of m; so that when the companion or satellite becomes a mere particle, the moment of momentum of orbital motion vanishes. Accordingly we see that the moment of momentum of orbital motion is a maximum when the masses are equal, and a minimum when one is a mere particle. This conclusion holds for an elliptic as well as for a circular orbit, as will be seen hereafter.

§249. *Inferences from the Dynamical Theory of Tidal Friction.*

In view of the consideration here set forth we may draw the following conclusions:

(1) In systems of given radius vector ϱ and angular velocity Ω about the common center of inertia, the moment of momentum of orbital motion will be a maximum when the masses are equal, and a minimum when the smaller mass is zero.

(2) Between these two extremes all possible values of the moment of momentum of orbital motion are included, so that any system in nature may be represented.

(3) If the total mass of the system be kept constant, but supposed to vary in distribution, between the extremes here mentioned; and the rotational and orbital moment of momentum of the system be also taken as constant; then the smaller moment of momentum of orbital motion will correspond to the greater moment of momentum of axial rotation. For according to well known mechanical laws the moment of momentum of the entire system when once set in motion is

constant, whatever be the mutual action of the bodies. It is made up of three parts, the two axial rotations, which may be called conveniently y and z, and the moment of momentum of orbital motion x, so that $x + y + z = H$, a constant, however the bodies may interact upon each other. If therefore the mass be so distributed that x is small, y and z will have to be correspondingly larger, in order to keep the moment of momentum of the system constant, as here assumed.

(4) It will follow that when the companion is small and the radius vector remains of the initial length, the orbital moment of momentum will be small compared to the moment of momentum of axial rotation, especially of the larger mass. Hence reduction of axial rotation under the secular action of tidal friction, by which moment of momentum of axial rotation is transferred to moment of momentum of orbital motion, might vastly expand the dimensions of the orbit. But as the tides would be small, owing to the feeble attraction exerted between such unequal masses, the process of change would be excessively slow; and as witnessed at any given epoch, while the stars are still self luminous, the orbit probably would not be highly eccentric.

(5) When the masses are equal, the moment of momentum of orbital motion would be a maximum to begin with; tidal action has also maximum power and gives maximum rate of change; and maximum effect on the major axis and eccentricity of the orbit would be produced in any given interval of time.

(6) For the tide-generating potential is

$$V = \frac{3mR^2}{2\rho^3}(\cos^2 z - \tfrac{1}{3}) \,, \tag{543}$$

where ρ is the distance, m the disturbing mass, and R the radius of the central body in which the tide is raised. Now the height of the tide varies directly as the tide-generating force, which may be obtained by differentiating this expression so as to get the component in any desired direction. It is well known that the height of the tide depends principally on the horizontal component (cf. AIRY's *Tides and Waves*, p. 68), which may be obtained by differentiating V with respect to z:

$$\frac{\partial V}{\partial Z} = -\frac{3mR^2}{\rho^3}\cos z \sin z \,. \tag{544}$$

(7) The moment of the tidal frictional couple operating against the axial rotation depends on the height of the tide, and also on the length of the arm on which it acts. As each of these varies as the intensity of the tide-generating force, it follows that the intensity of the tidal frictional couple varies as the square of the tide-generating force; and therefore inversely as the sixth power of the distance,

but directly as the square of the disturbing mass, and as the fourth power of the radius of the central body.

(8) The tangential component alone produces the expansion of the orbit of the companion, and this varies inversely as the seventh power of the distance, directly as the fourth power of the radius of the central body and the square of the disturbing mass. Thus when the disturbing body is regarded as acting as if its mass were collected at its centre of gravity, the secular change varies as $\frac{m^2 R^4}{\rho^4}$.

(9) In the case of double stars a double tidal action exists, and the total effect is a summation of two separate actions, given by an expression which for our present purposes may be reduced to the form

$$F = c\frac{m^2 R^4}{\rho^7} + c'\frac{M^2 r^4}{\rho^7}, \tag{545}$$

where c and c' are constants. It will be seen that ρ^7 is common to the two terms of (545), and since c and c' are constants which do not change with the variation of the mass-distribution, but may be taken to be identical, when all the conditions in the two masses are similar, and the matter homogeneous, it follows that the expression will be a maximum when $m^2 R^4 + M^2 r^4$ is a maximum, or when $r = R$, and the two masses are equal. This may be most satisfactorily shown as follows.

(10) Take a given volume of incompressible fluid matter of uniform density and suppose it divided first into two equal masses set revolving in circular orbits as before described. Then imagine the one mass to be decreased by cutting away shells corresponding to tenths of the radius, the matter lost by one body at each stage being transferred to the other. As the total mass of the system is constant, the periodic time will remain unchanged; and hence we may find the effects of unequal mass distribution on tidal efficiency under any supposed initial conditions. The results obtained by numerical calculation of the changes of the function $F' = r^6 R^4 + R^6 r^4$. are shown in the table given in (12) below.

(11) It will be seen that the efficiency of tidal friction rapidly diminishes with decrease of the radius or mass of the disturbing body; and as soon as the masses become very unequal the efficiency of tidal friction becomes wholly insensible. Hence equal masses given maximum efficiency, or the greatest effect in a given time. All these results obviously apply to ideal systems of homogeneous matter, in which all the conditions are exactly the same; but no doubt many actual systems approximate these conditions sufficiently to give results of considerable interest in connection with the stellar universe.

(12) Table showing the efficiency of Tidal Friction for varying mass-ratio:

r	R	m	M	Mass-Ratio $= \frac{m}{M}$	Efficiency of Tidal Friction
1.0	1.00000	1.000	1.000	1 : 1	2.000
0.9	1.08322	0.729	1.271	1 : 1.7435	1.79156
0.8	1.14175	0.512	1.488	1 : 2.9062	1.35223
0.7	1.18333	0.343	1.657	1 : 4.8306	0.88991
0.6	1.21283	0.216	1.784	1 : 8.26	0.53142
0.5	1.23311	0.125	1.875	1 : 15.00	0.255854
0.4	1.24634	0.064	1.936	1 : 32.50	0.105834
0.3	1.25423	0.027	1.973	1 : 73.08	0.033335
0.2	1.25824	0.008	1.992	1 : 249	0.006364
0.1	1.25971	0.001	1.999	1 : 1999	0.0004025
0.0	1.25992	0.000	2.000	0.00000	0.0000000

§250. *Comparison of Theory with the Observations of Actual Systems.*

In the examination of the orbits of about eighty visual binary stars, the writer has noticed a number of systems, in which the eccentricity is high and the components about equal in brightness, and presumably also in mass. In the following list of binaries, with eccentricities of about 0.6 or above, the components differ in brightness less than two magnitudes; the eccentricities indicated in parentheses are provisional, but no doubt they will be found to be essentially correct.

13 *Ceti* (0.74); β 395 (0.85); Σ 186 (0.67); γ *Andromedae BC* (0.85); 20 *Persei* (0.75); 9 *Puppis* (0.70); β 208 (0.85); ϵ *Hydrae AB* (0.70); ψ *Velorum* (0.61); Σ 1639 (0.70); γ *Centauri* (0.80); γ *Virginis* (0.90); 25 *Canum Venaticorum* (0.75); Σ 1879 (0.65); γ *Lupi* (0.70); $O\Sigma$ 298 (0.58); Σ 1989 (0.66); ξ *Scorpii AB* (0.76); Σ 2026 (0.70); λ *Ophiuchi* (0.68); τ *Ophiuchi* (0.59); 73 *Ophiuchi* (0.84); β 639 (0.80); Σ 2525 (0.95); β 80 (0.72) — 25 pairs in which the components are fairly equal.

The following list includes nine stars in which the inequality of the components always exceeds two magnitudes, and has an average difference of more than five magnitudes:

η *Cassiopeiae* (0.45); α *Canis Majoris* (0.58); α *Canis Minoris* (0.40); ξ *Boötis* (0.54); γ *Coronae Borealis* (0.48); ζ *Herculis* (0.50); τ *Cygni* (0.37); 85 *Pegasi* (0.40); 99 *Herculis* (0.78).

There are among the 80 systems about nine times as many stars with components approximately equal as there are stars with components very unequal. In nine very unequal pairs we find only one star with eccentricity above 0.6. In some 75 nearly equal pairs we find 25 with eccentricity of 0.6 or above. This gives an average ratio of one in three; or three times as many with high eccentricity as would appear among an equal number of very unequal pairs. Hence, although

very unequal pairs are only one-ninth as frequent as approximately equal pairs, there is only one-third the probability of a high eccentricity occurring among the same number of such unequal systems.

The ratio of one-third, for the probability of an equal pair having high eccentricity, is founded on 25 systems out of 75, a number sufficiently large to indicate that the ratio will prove approximately correct, however great be the total number of bodies considered. In the case of the unequal pairs, however, the ratio of one-ninth is not so well established, because the number of bodies is too small to give a clear indication. Yet it seems unlikely that more than one in nine of such pairs will have high eccentricity, and it may be that the ratio will prove to be even smaller than this. Thus on the basis of observed facts it seems highly probable that three times as many equal pairs have high eccentricity as unequal pairs, when the same number of the two classes of systems is considered. And as equal pairs are about nine times more numerous than unequal pairs, probably owing to the difficulty of seeing very close unequal objects with the telescope, it follows that in practice the probability of a high eccentricity occurring among the equal pairs is at least 25 times greater than that it will occur among the very unequal pairs, which accords with observation.

Accordingly among the equal pairs large eccentricity is so frequent as to attract attention, while among the unequal pairs the eccentricity usually is small. There naturally are many equal pairs with small or moderate eccentricity, and according to theory this ought to take place, since many of them are in very different stages of development. But the preponderance of high eccentricities among the equal pairs would seem to be an indication of the *higher efficiency* of Tidal Friction or of the *lesser importance* of the Action of a Resisting Medium in such systems, as explained theoretically in the foregoing discussion and in the section below.

If this inference be justifiable, it will follow that in deducing from the roundness and small size of the orbits of spectroscopic binaries a direct confirmation of the Tidal Theory and of the Theory of the Action of a Resisting Medium, as explained elsewhere in this work, we may at the same time find additional verification of the Theory in the larger eccentricities occurring among binary stars with nearly equal components.

§251. *Resisting Medium and Variation in the Mass Distribution.*

If now we consider the effects of the action of a resisting medium under a variation of the mass distribution, we shall find that the moment of momentum of orbital motion about the common centre of gravity of the system is

$$M\left(\frac{m\varrho}{M+m}\right)^2\Omega\sqrt{1-e^2}+m\left(\frac{M\varrho}{M+m}\right)^2\Omega\sqrt{1-e^2}=\frac{Mm}{M+m}\Omega\varrho^2\sqrt{1-e^2}\ . \qquad (546)$$

This too is a maximum when $m = M$, and $e = o$, or the orbit is circular. If, however, the value of e be regarded as fixed by some primordial condition of the system, $m = M$ will still give the maximum moment of momentum of orbital motion.

It only remains to consider the changes in e when the other elements are unaltered. Since the moment of momentum of orbital motion is a maximum when $m = M$, it is evident that a minimum change of the mean distance and therefore of the eccentricity will result from a given amount of resistance when the masses are equal. *Accordingly, just as the rate of increase of the eccentricity due to tidal friction is a maximum when the masses are equal, so, conversely, the rate of decrease of the eccentricity due to the resisting medium is a minimum under the same condition. In the action of a resisting medium an equable distribution of mass tends to preserve an original high eccentricity.*

If the known history of the solar system did not afford a criterion as to the relative efficiency of these two causes, we might be unable to distinguish between them, but in view of the paramount part played by the action of the resisting medium in the development of the planets and satellites, as shown by the roundness of their orbits, there would seem to be no doubt that the secular effects of the resisting medium are generally preponderant in nature.

Accordingly the larger eccentricities of the orbits of visual binaries compared to those of spectroscopic binaries is the natural outcome of resistance which has left the larger orbits with the larger eccentricities, and *vice versa*. In the case of equal components the effect of resistance has produced a minimum decrease of the primitive eccentricity, and visual binaries with equal components therefore frequently move in paths so very eccentric as to attract attention. It would thus appear that observation and theory are in good agreement.

§ 252. *Indications Furnished by Triple and Quadruple Stars Regarding Motion Near a Common Plane.*

Although triple, quadruple and other multiple stars have been studied since the time of Sir William Herschel (cf. *Phil. Trans.*, 1802), their components in general are too far apart to give evidence of rapid motion; and it is chiefly since the epoch-making work of Burnham in discovering very close double stars that our knowledge of multiple stars has been rapidly advanced. The study of spectroscopic binaries within the past twelve years has also contributed to our knowledge, by

showing that many of the visual double stars have one or both components spectroscopically double, so that the systems formerly classed as binary are really ternary and quaternary. But whether the systems be resolved into closer components by telescopic or spectroscopic research, it is generally found that the distance between the close pair is small compared to that of the wide pair with which it is associated. Thus the distance of the wide pair is often from twenty to fifty times that of the close pair, so that the latter may largely escape the perturbations of the third body, or at least revolve in comparative safety and stability. If the relative distances were not thus arranged, stability and permanence could not be assured, and the systems might not long endure. But as we observe vast multitudes of such systems arranged on this plan, we conclude that they are stable, and have originated under conditions which ensure permanency. When a double star had been formed in the usual way by the growth of separate centres in a widely diffused nebula, one or both components likewise captured and developed companions, and the result was a triple or quadruple star. So far as we yet know, these physical systems have the components revolving in a common direction, and in any given case the orbits are not greatly inclined. The motion is essentially analogous to that of the Earth and Moon around the Sun. As examples of such multiple stars, we may mention the following visual pairs: ζ *Cancri*, ξ *Scorpii*, ϵ *Lyrae*, ϵ *Hydrae*, ν *Scorpii*, β 581, etc.; and spectroscopically resolved systems such as α *Geminorum*, ξ *Ursae Majoris*, κ *Pegasi*, α *Ursae Minoris*, etc. The present theory of the development of the stellar systems is thus confirmed by the indications furnished by triple and quadruple stars, and one can not doubt that the process is general throughout nature.

§ 253. *The Positions of the Axes of Rotation in Double Star Systems.*

In the earlier investigations of the author it was supposed that the separation of binary stars had taken place by a process of *nebular fission* closely resembling the fission of rotating masses of fluid in hydrostatic equilibrium under the pressure and attraction of their parts. On this hypothesis it was correctly inferred that the axes of rotation would be nearly perpendicular to the plane of the orbit. In this volume we have substituted the dynamical process of nebular fission for the hydrostatic process of *fluid fission* heretofore imagined as a rough approximation to nature's mode of Cosmical Evolution.

It is sufficient to say that in dealing with planetary obliquities we have shown that the tendency arising from the capture of satellites is to produce zero obliquity, or cause the bodies to rotate about axes nearly perpendicular to the orbit plane.

A similar tendency is obviously at work in all stellar systems; and it may reasonably be assumed that as a rule the obliquity probably is small. Thus the components have direct rotation, as was assumed in my *Inaugural Dissertation* and subsequent researches on the evolution of these binary systems.

Accordingly whilst the process of nebular fission is different from that provisionally imagined in 1892, the resulting rotations and axial positions of the bodies remain unchanged, so that tidal friction would operate very nearly as originally supposed. Tidal friction seems to be much less effective than was formerly believed, while the importance of the secular effects of the resisting medium is correspondingly augmented. And whilst it seems more probable that the eccentricities of the orbits of binary stars are survivals of larger values dating back to the nebular stage, than that they have been developed from small initial values by the secular effects of tidal friction; yet it is certain that both causes are at work, and usually have diametrically opposite tendencies.

If therefore the earlier work is somewhat incomplete, it is nevertheless very satisfactory to find that most of the underlying hypotheses are verified by the present investigation. It seems to be quite certain that the rotations are direct, and the obliquities small, in the great majority of the stellar systems. As this result could never be verified by direct observation, owing to the enormous distances of the stars, but is deduced from the mathematical theory of the motion of three bodies, and confirmed by well established phenomena in the solar system, it is the more satisfactory to the human mind. Indeed it may fairly be said to be one of the most interesting inferences yet made by the deductive process in any branch of Physical Science.

§ 254. *Augmentation of the Destructive Forces in an Eccentric Double Star System.*

In treating of the problem of three bodies, Chapter VIII, we have pointed out that in eccentric systems the destructive tendency is much greater than in systems with nearly circular orbits. The third body suffers such great variation in the action depending on the other two that anything approaching even temporary stability is nearly impossible. The result apparently would be that the third body would soon be brought into collision with one of the two large masses; or it would be driven from the system never to return. It seems probable therefore that eccentric double star systems would be cleared of satellites much quicker than planetary systems with nearly circular orbits. It is impossible to treat this subject fully, because our knowledge is still but little developed, but this general result seems to be reasonably well established.

If this line of reasoning be admissible, it will follow also that satellites survive longest in planetary systems. Moreover the great duration of planetary systems enables their orbits eventually to be worn quite round by the resistance at work against comparatively small masses. And finally since the double star systems as a class are quite *eccentric*, this general fact shows that the same property held true of the primordial orbits of the planets, before the resisting medium had modified the original state of the system. This is another and final answer to the question of DR. G. W. HILL, which we have discussed at the close of Chapter VII. It thus seems to be well established that as a rule the primordial orbits of planets were originally quite eccentric, like those of double stars, but that this eccentricity decreases with the development of the system; and in a mature system such as that of the planets revolving about our Sun, the orbits may become so very circular as to excite the wonder of the astronomer, the geometer and the natural philosopher, as has constantly happened during the past 2,000 years.

§ 255. *Comparison of the Binary Systems with the Solar System.*

The two most important characteristics of the binary systems are: (1) the large mass-ratios of their components, and (2) the high eccentricities of their orbits. In the solar system, on the other hand, the attendant bodies are very small compared to the large central bodies which now govern their motions, and moreover the orbits are nearly circular. This latter effect, the roundness of the orbits of the planets and satellites, has been traced the secular action of a resisting medium; and it is shown by exact calculations based on the mechanical principle of the conservation of areas that it has arisen from this cause and no other.

Multiple stars are not only remarkable for the large masses of the companions but also for the corresponding restriction in the number of bodies in a system. The great majority of the stellar systems are binary, either visual or spectroscopic; but in a certain, rather small, percentage of the total number of cases we find triple and quadruple systems, and multiple stars of more complex character. What connection, then, according to the principle of continuity, are we to imagine between systems of the type of our solar system and these stellar systems?

We have seen that a companion star may gather in satellites, just as *Jupiter* did in the Solar System; and that if the supply of material is properly distributed in the nebula, the mass of the companion may in time become large, as in the systems of binary and multiple stars. On the other hand the eccentricity of the orbit will tend to disappear. If, however, the original eccentricity was quite high, it may easily happen that a large eccentricity will still survive, after most of the

satellites are gathered in. This seems to be the case with our actual visual binary systems.

The problem therefore reduces itself to the following: In a fraction of the total number of systems, perhaps about one-fifth of the whole, the constitution of the primordial spiral nebulae was such as to give binary systems rather than planetary systems. And in a case of this kind the resisting medium, made up of nebulosity, cosmical dust and satellites, is least effective in reducing the eccentricity of the primordial orbit, owing to the preponderant moment of momentum of orbital motion, arising from the relatively great mass of the companion.

When, therefore, a nebula is so constituted as to give rise to a double star, a system with a mass distribution which is essentially double, the result will be the survival of a large eccentricity, such as is indicated by observation. On the other hand, when the primordial nebula is free from a single preponderant companion revolving about the central star, but characterized by a multitude of nearly equal planets, all quite small, the result will be the development of a planetary system such as our own.

This seems to be the general tendency in nature, as indicated by the fact that about four-fifths of all the stars appear to be single, whereas in reality they are surrounded by planetary systems made up of bodies so small as to be invisible in our telescopes. From this line of reasoning it follows that there is a connection, conformable to the principle of continuity, between the planetary systems and the double star systems of the universe. Both have been formed by the same process, but the dominant forces have been in different proportion, according to the original states of the primordial nebulae from which they have arisen.

§ 256. *Since All Cosmical Systems Develop from Nebulosity Expelled from the Stars, the Orbit Planes of Binary Stars Ought to Lie at All Angles with Respect to the Plane of the Milky Way.*

We have seen that the nebulae are clouds of cosmical dust expelled from the stars by the radiation pressure of their light, and by electric and other repulsive forces; and we have shown how the falling together of this nebulosity necessarily gives rise to spiral nebulae, which develop into cosmical systems. Some of the resulting systems resemble the planetary system, while others become double and multiple stars; and it appears that the largest of all the nebulae develop into clusters and clouds of stars.

Now a cosmical system is produced by the agglomeration of cosmical dust, and by the capture of bodies already partially developed; but the cosmical dust

is expelled from the stars lying in every direction and drifts about hither and thither till it collects into a nebula; and thus it follows that when it collects into a vortex and begins to whirl about the centre, the fundamental plane of motion will have no definite relationship to the plane of the Milky Way, but may be inclined to it any angle whatever. And since it appears by observation that about one-fifth of the cosmical vortices produce binary systems, the other four-fifths giving planetary systems with attendant bodies too small to be seen in our telescopes, it follows that the orbit planes of these binary systems should have no definite relationship to the plane of the Milky Way, but should be tilted at all possible angles. This seems to be the case, as was shown in the third chapter of the first volume of these *Researches*; and consequently we conclude that, on this point, theory and observation are in good accord.

As the planetary systems among the fixed stars are wholly invisible, and the situation of their fundamental planes cannot be investigated by any means now known or likely to become available hereafter to the scientific investigator, we are reduced to the neccessity of falling back on the binary stars, which alone have attendant bodies large enough and bright enough to be visible in our telescopes, or sufficiently massive to give a variable velocity in the line of sight, which can be measured by means of the spectrograph. And although we are thus restricted to one-fifth of all the stellar systems known to exist in the universe, and in practice to but a small fraction of this fraction; yet it is evident that the distribution of the inclinations found for the small number of systems visible to us, will also hold for the vastly greater number of systems which are invisible but known to exist. Accordingly to assure ourselves of the haphazard law of the inclinations for all existing systems it is sufficient to study the situations of the orbits of binary stars, on which a considerable beginning has already been made, as set forth in the first volume of these *Researches.*

§ 257. *Difference in Color of Binary Components Due to Selective Separation of Certain Chemical Elements.*

The difference in the color of the components of binary stars has long been a subject of remark and investigation. So far as we know no satisfactory explanation of this phenomenon has been obtained; and whilst we cannot yet hope to reach a final solution of the problem, it seems probable that the general cause at work may be indicated.

It was for a long time believed that the apparent contrast in color was largely or wholly subjective; but after the various physical experiments and the extensive

observations of Herschel and Struve and other eminent astronomers, this idea was given up, and it was admitted that the apparent contrast in color is real and inherent.

The brighter component as a rule inclines to orange or red, the fainter one to bluish or purple; in other words the dominant light of the companion lies near the more refrangible end of the spectrum, while that of the principal star lies near the opposite extreme, at the red end of the spectrum. This general tendency has been explained by selective absorption, the gases in the atmosphere of the companion being such as to allow the short wave lengths to pass, while the reverse is true of the brighter component, the heavier absorption there cutting down the shorter vibrations. Within certain limits there is a considerable amount of truth in this idea, but the explanation is still very incomplete. Why should the emitting and absorbing elements be thus distributed?

The most probable answer to this question is the following: Since both components have been augmented by the capture of satellites, but the fainter component as a rule has the smaller mass, it has probably gained an undue proportion of certain chemical elements of small atomic weight, such as Hydrogen, Helium, Coronium, etc. This separation might result from the elastic expansion of these lighter gaseous elements to greater distance from the large mass, owing to very high temperature or their repulsion under electric charges, so that they come under the control of the smaller mass — the two bodies probably having atmospheres with opposite electric charges — or to some such process of separation, into the details of which we need not go at present.

It is evident that difference in color and spectrum rests essentially upon a difference in chemical constitution. And this is not a question of the relative ages of the components, as was once believed; for the two masses have had separate origins, in a common nebula, and there is no proof that difference in color is a sign of unequal rate of development due to difference of mass.

It is rather a chemical difference due to the differentiation of the elements; with the result that the radiation is in striking contrast. This adds greatly to the beauty of coupled double stars, and such striking contrast of color is an almost infallible sign of physical connection into true binary systems, so that this may generally be predicted in advance of the development of orbital motion.

§ 258. *The Probable Cause of the Extreme Darkness of the Companions of Certain Double Stars.*

It has long been remarked that the companions of certain binary stars are singularly dark. Thus the companions of both *Sirius* and *Procyon* were detected by Bessel from the perturbations they produced long before either of these faint

objects were discovered by observation. But later investigation of these systems indicates that the satellite of *Sirius* has half the mass of the large star, though it gives only one ten-thousandth as much light; while in the case of the equally obscure companion of *Procyon* the mass is at least one-fifth that of the principal star. This contrast of coupled light and dark bodies is further illustrated by examples among the *Algol* Variables, and by some of the variables of the type of β *Lyrae*, where two stars of unequal luminosity are nearly in contact and revolving in short periods.

Moreover, this extreme type of dark companions has been observed among the double stars visible in our large telescopes. The older observers, such as the Herschels, the Struves and Alvan Clark, have called attention to the coupling of very unequal stars in such systems as ζ *Herculis*, 99 *Herculis* and 95 *Ceti*, while the modern observations of Burnham, See, Aitken and others, confirm this contrast of brightness in certain close binaries, as 85 *Pegasi*, λ_1 2, λ_1 23, etc.; and also in binaries of a wider class where the companions are excessively faint.

As an example of the latter class of dull companions, it may be mentioned that during the writer's survey of southern double stars at the Lowell Observatory, 1896-1898, he found over half a dozen remarkable objects which were so dark and obscure as to present the aspect of shining by reflected light. Thus the stars numbered λ_1 76, 88, 113, 289, 311, 429, 455, 459, in the First Catalogue of New Double Stars published in the *Astronomical Journal*, Nos. 431-432, appeared exceedingly remarkable for the faintness and obscurity of their companions. The author's assistant, Mr. Cogshall, used to remark, when he found one of these dull objects, that the *color was black*; meaning a dull color of extreme darkness, but with sufficient contrast to the blackness of the background of the sky that we could just see it.

My experience led me to the view that in these objects the predominant color is ultra violet, so that the dull light of the star has a little purplish tinge to it; and I ascribed our detection of these objects on the black background of the sky to the contrast afforded by this ultra violet light. When I mentioned these remarkable objects to Professor E. C. Pickering, he too expressed surprise, but thought the light most likely to be of ultra violet character. This would make them simply extreme cases of the contrast of color usually prevailing in double star systems, and not a separate and isolated class of bodies, as their appearances would almost indicate.

Accordingly it seems probable that the cause which produces general contrast of color in binary stars may also produce extreme cases so obscure usually as to be almost black. This separation of certain chemical elements doubtless is the cause

of the ultra violet light of certain visual binaries and also of the dark companions of *Sirius* and *Procyon*, where the large stars are so bright that we can not see the color of the companions, but merely recognize that they are very faint. And from the relatively large mass of the companions in such systems as *Sirius* and *Procyon*, we may infer that even in the case of the dull satellites, mentioned above as presenting the aspect of shining by reflected light, the relative masses are considerable, and possibly of nearly the same order of magnitude as in the cases of *Sirius* and *Procyon*, which have been investigated by the observed motions of the systems about their centres of gravity.

The great obscurity of certain companions, therefore, appears to be an extreme case of the general tendency to contrast of color; and does not imply that the faint and dark companions may not be quite massive. The explanation already given in the last section appears to be the only rational account of these contrasts; and whilst this is far from complete, it gives us some idea of the causes at work, and shows that we may expect a considerable number of stars to have large companions which are so dull as to pass unnoticed in our telescopes.

This is a development of BESSEL's conception of the astronomy of the invisible, and directly deducible from the two celebrated cases of *Sirius* and *Procyon*, which have been under investigation for over seventy years. BESSEL's announcement to HUMBOLDT in 1844 was as follows: "I adhere to the conviction that *Procyon* and *Sirius* form real binary systems, consisting of a visible and an invisible star. There is no reason to suppose luminosity an essential quality of cosmical bodies. The visibility of countless stars is no argument against the invisiblity of countless others" (WOLF, *Gesch. d. Astron.*, p. 743, note). And now we have more substantial grounds upon which this conclusion could be generalized and shown to be applicable to stars of several distinct classes.

§ 259. *The Use of Binary Stars for Measuring the Distances of Clusters and of the Milky Way.*

In Vol. I of these *Researches* we estimated the distances at which direct measurement might be possible at 1,000 light-years. If we wish to inquire what is the maximum depth of the Milky Way to which the spectroscopic method would enable us to penetrate, we may reason as follows: At present it is not admissible to assume that any star has a mass exceeding 1,000 times that of the Sun. Bodies revolving around such a powerful centre of attraction might have the same absolute velocity as in the solar system, at over 31 times the corresponding absolute distance. The effect, therefore, of a large mass is to increase the distance at which

a given absolute velocity can be maintained by the factor $\sqrt{M+m}$, which cannot exceed 31. This makes it possible to place the system as much as 31 times farther away than would be possible for a system having the mass of our Sun. Now it is shown in § 7, Chapter I, of Vol. I, of these *Researches*, that the spectroscopic method of determining distance might be applied to *α Centauri* at a distance of 800 light-years. This double star has just twice the mass of the Sun. Consequently a maximum star with 500 times the mass of *α Centauri* might be investigated spectroscopically at about 22 times greater distance, or 17,600 light-years.

Accordingly it does not seem possible to extend the method of direct measurement to distances exceeding some 20,000 light-years; and our ability to measure even that distance will depend on the existence of double stars with enormous masses. Up to the present time no such systems have been discovered, and it is impossible to say how far direct measurement may be extended; but it seems likely that the method will be applicable chiefly to the stratum of stars comparatively near the Sun. It appears probable, therefore, that only an indirect method, such as that used by HERSCHEL, will ever enable us to measure the distance of the remotest stars, which may be removed from us by several million light-years.

§ 260. *Significance of the Absence of Binaries from Clusters.*

In view of the new light thrown upon clusters by the researches of Boss on the moving cluster in *Taurus*, of which an account is given in the next chapter, we cannot be quite certain what the absolute dimensions of the average cluster is. In the first volume of these *Researches*, we have given an exact method by means of which the spectroscope may sometime be used to determine the distance of a cluster or of the Milky Way. It depends on the finding within the cluster or cloud of the Milky Way of a binary star with an orbit which can be determined both visually and spectroscopically. This will eventually enable us to determine the distances of such groups of stars by actual measurement.

Up to the present time, however, it has not been possible to apply this valuable method, because no visual binary is known to be revolving in any cluster; at least no orbit of such a binary has yet been determined, *and* BURNHAM *and other double star observers have remarked on the comparative poverty of clusters in binary stars.* This fact is becoming more and more generally recognized and has a deeper significance than has been suspected. Let us see if we can fathom the meaning of the almost total absence of visual binaries from clusters.

According to the results of Boss, relative to the moving cluster in *Taurus*, all clusters are of immense dimensions absolutely and at enormous distances from the

Earth. Therefore binary systems of the usual angular separation could not possibly be seen at such immense distances, and the result would be that the clusters should be nearly if not quite devoid of true visual binaries. If this should prove to be true, when the clusters come to be more closely studied, it will confirm the large scale of construction and the immense distances of the clusters in general. Only a few visual pairs of close double stars have been found in clusters; and it will take considerable time to show whether they are mere groups of perspective or true physical systems.

On the whole, the indications now are that they are due to perspective, the probability of such juxtaposition increasing rapidly as the density of stars on the background of the sky augments, as is always true in a dense cluster of stars.

Nothing is more needed in observational astronomy at present than a study of clusters, for the purpose of finding true binary systems. Perhaps the stars, if found, might prove to be too faint to be studied spectroscopically with existing telescopes; but with the growth of large reflectors during the next half century, such investigation of the distances of the clusters ought to become possible.

At present we can only say that the meager evidence we have is inadequate to solve the problem; but, on the whole, it tends to magnify our views of the clusters and of the Milky Way to about the scale imagined by Herschel, or even beyond it, while it decreases our hope of finding the distance of these gigantic systems by direct measurement. Therefore, whilst we may find spectroscopic binaries by the thousand, in the greatest clusters, it will in all probability be most difficult to discover visual binaries which can be measured in our telescopes from a point so remote as the Earth. The finding of spectroscopic binaries is independent of the distance, while the difficulty of the measurement of visual binaries increases directly with the distance. Perhaps such measurements will never be possible among the remotest clusters observed in the sidereal heavens.

CHAPTER XXI.

Theory of Star Clusters.

§ 261. *On the Earliest Discovery of Clusters and on the Explorations of* Herschel.

The existence of dense masses of stars in different regions of the heavens was one of the earliest results disclosed by the invention of the telescope. For although the star-clouds of the Milky Way had been noticed by the Greeks, the telescopic clusters are smaller and denser groups of stars, and cannot be distinctly seen by the naked eye. From the first these telescopic swarms of stars were as deeply mysterious as their discovery was unexpected. Yet even Galileo's announcement that he had proved the great arch of the Galaxy to be composed of clouds of stars too small to be seen individually, excited great wonder among those who first viewed the heavenly bodies through the telescope, and beheld the resolution of the Milky Way into stars and clusters.

It is true, as already remarked, that Democritus and Anaxagoras had assured the Greeks 2,000 years before that the Milky Way was composed of dense clouds of stars too small to be seen individually and so numerous as to give the effect of a milky light. But although this might be believed by a few philosophers, it could not be demonstrated to the masses of mankind before the time of Galileo; and when tangible proof did come, the astonishment was so great that observers could hardly believe the evidence of their senses. The wonder over the constitution of the universe naturally increased with the improvement of the telescope; but it was only after Herschel's memorable exploration of the heavens that astronomers realized the immense number and variety of clusters scattered throughout the sidereal universe.

As to the origin of these masses of stars, most of the early astronomers remained silent, in simple amazement at the facts disclosed by the telescope, without seriously trying to explain them. Some few of the earlier investigators probably felt that the stars might perhaps be aggregating together under the power of gravity. Yet it was only after Herschel's epoch-making explorations that even he saw clearly the effect of a clustering power which was gradually breaking up the Milky Way. This was the beginning of our physical theory of clusters; and

these masses of stars have since been viewed as transition phenomena in the development of the universe. In fact, HERSCHEL considered that the observed state of the heavens attained under the continuous action of the clustering power, which was breaking up the Milky Way, afforded a kind of chronometer for measuring the past and future duration of the present order of things. He believed that he could look backward to a time when the clustering power had not yet begun to operate, and forward to a time when its ravages on the structure of the heavens must be vastly greater than at present; so that the sidereal universe could not endure forever, otherwise the clustering power could not already have made such visible inroads upon the assumed uniform structure of the primordial Milky Way.

The views of HERSCHEL are so important, and now so inaccessible, owing to the scarcity of his original papers — a collected edition of his works never having been published,* although it has been repeatedly urged by STRUVE, BESSEL, ARGELANDER, NEWCOMB, PRITCHARD, and many other eminent astronomers — that at the end of this chapter we have departed from the usual plan, and have quoted his theories at considerable length. This seemed to be the only way of doing justice to HERSCHEL'S work, and of making it accessible to the modern reader. It is too important to pass over; too original and too distinctly Herschelian to mar by an attempt at condensation. If these lengthy quotations shall be the means of diffusing a juster estimate of HERSCHEL'S great labors, there will be no criticism of this mode of procedure for which the author may not unhesitatingly assume full responsibility.

It is said that in the last days of his life, LAPLACE was occupied with the correction and reprinting of a new edition of his *Exposition du Systéme du Monde*, and often expressed the view that correction of the works of authors after their death was not permissible, because it altered the original thought, and always to the prejudice of the history of science. Similar views were expressed by NEWTON, and have been generally held by all eminent men of science. For that reason, abridgement or restatement of views is difficult, and always accompanied with some sacrifice. If, therefore, the views can be given in the original language of the author, it is always preferable. Those who object to this method of bringing out the conclusions of investigators whose works are inaccessible have inferior grounds upon which to base their claims. Clearness, accuracy and authenticity, within moderate limits of exposition, are the essential elements of sound knowledge, which is the ultimate object of all scientific effort.

* Since this was written the Royal Society and the Royal Astronomical Society have appointed a Joint Committee on the publication of the "Collected Works of SIR WILLIAM HERSCHEL," and it is now hoped that the reissue of these celebrated papers, which will have a living interest to the remotest ages, may soon be undertaken under the auspices of these learned Societies, on which the discoveries of HERSCHEL shed such imperishable renown. The last service of SIR WILLIAM HUGGINS to Astronomy was connected with the republication of these immortal works of SIR WILLIAM HERSCHEL.

We treat of the Milky Way in the next chapter, and shall at present content ourselves with the theory of clusters; but we may observe that the two subjects are connected by the clustering power of gravitation, which is everywhere at work, and has already given the Milky Way the appearance of a vast collection of clusters and larger clouds of stars. The present chapter and the next one are therefore closely related, and both represent particular aspects of the Capture Theory, as applied to the great swarms of bodies constituting the sidereal universe.

§ 262. *Relation of the Nebulae to Clusters.*

We have seen in Chapter IV that in the main the clusters are distributed along the path of the Milky Way. This arrangement has a fundamental significance, in showing that the clusters are closely connected with the principal stratum of stars. Along the same zone we find various large, diffuse and irregular nebulae, as if these objects were formed from cosmical dust recently expelled from the starry stratum and still diffused in that general direction.

On the other hand, we have seen that the larger nebulae, such as that in *Andromeda*, are in all probability made up of a cluster of small stars surrounded by nebulous fog. The star-cluster theory of the white nebulae is strongly supported by spectroscopic evidence, and by the demonstrated prevalence of multitudes of moons in the nebula which developed into our solar system. For it is now clearly established that the solar system originated from a spiral nebula; and the history of our own system is therefore typical of the development of the vast majority of the systems existing throughout nature.

There are some cases in which the nebulae are unmistakably passing into star clusters, by visible processes of transformation; though as a rule the denser globular clusters have been found by PERRINE (*Lick Observatory Bulletin* No. 155) to be essentially devoid of visible nebulosity. Thus the whirlpool nebula MESSIER 33 *Trianguli* is evidently forming a cluster, and it has a conspicuous spiral movement. A similar remark applies to MESSIER 101 *Ursae Majoris,* and especially to MESSIER 63 *Canum Venaticorum.* This latter object is not only forming a cluster, but also has a conspicuous spiral movement and is already greatly condensed towards the centre; so that in time it will become a dense globular cluster, quite devoid of nebulosity of any kind.

From this line of reasoning it appears that the nebulae and clusters are everywhere closely connected; but that by the time a cluster becomes compact, the nebulosity is absorbed by the component stars, and the globular clusters, as a class, therefore, are almost devoid of nebulosity, as found by PERRINE from observations

taken at the Lick Observatory. Yet, notwithstanding this gradual disappearance of nebulosity in the older clusters, there are some still in the transition stage, showing their origin from nebulae; while many large objects now classed as nebulae, such as the nebula of *Andromeda*, are in reality clusters in an early stage of development. In fact, all white nebulae are clusters, in the sense that they are literally filled with moons and planets, some of which are more or less self-luminous from internal collisions or impacts of various kinds.

In general, therefore, there is just as intimate a connection between clusters and nebulae as between nebulae and planetary systems. The one class of objects passes by gradual development into the other, and the phenomena observed in the heavens represent all stages of development of the various classes of nebulae. It is evident that many nebulae of immense extent and large mass develop into clusters, because each centre of attraction in such a large mass is more or less independent of the others. In time the nebulosity disappears by absorption; and this leaves us a star cluster, which is certainly the most beautiful object presented to our contemplation in the physical universe, as was long ago remarked by the elder Herschel (*Phil. Trans.*, 1802, p. 497).

It is worth while recalling in this connection that nebulae are formed by the gathering together of the elements of cosmical dust, the particles of which thus become captured and attached to larger masses of nebulosity. The condensation of the resulting nebula produces stars, planets, and moons, as well as an infinite number of smaller masses. In the course of immense ages an orderly system, such as our solar system, may result. And now, just as small masses of all kinds are captured by the nebula, and serve some purpose in the development of the resulting cosmical system, so also stars and smaller bodies are often captured by clusters, and made to serve some purpose in the development of these glorious systems. Accordingly the clusters develop not only from nebulae, but also from the gathering in of isolated bodies, including stars with planetary systems revolving about them. This subject will be discussed more fully when we come to treat of the views of Sir William Herschel, to whom our theory of clusters is so largely due.

§ 263. *Significance of the Distribution of Clusters Along the Milky Way.*

The fact disclosed by observation showing that the clusters as a rule are distributed along the Milky Way has some physical meaning which heretofore has not been clearly made out. We have seen that of all the systems presented to our contemplation in the sidereal heavens, the globular cluster is the most beautiful and also the oldest; because, in true globular clusters, the nebulosity has nearly

all disappeared by absorption, and this takes immeasurable eons for its accomplishment. The Milky Way itself is essentially a spiral made up of clusters and streams; and therefore we may infer that it is in the starry stratum of the Galaxy that we see the oldest and most perfect development. The nebulae on either side are built up of cosmical dust expelled from the central stratum of stars.

Accordingly, so far as one can now determine, the ordering of the universe, with the clusters and masses of stars in the centre and the nebulae on either side, is the inevitable effect of attractive and repulsive forces operating over vast periods of time. It was already noticed by the elder HERSCHEL that the Milky Way is breaking up under the continued action of the clustering power, which has given that great arch of light the aspect of a series of clusters rather than that of a uniform band of milky light. HERSCHEL referred to the clustering tendency noticed in the Milky Way as early as 1785 (*Phil. Trans.*, 1785, p. 255), and he there concluded that some parts of our sidereal system had suffered more from the ravages of time than other parts. He cites, in confirmation of this idea, the absolutely dark space, nearly four degrees broad, in the constellation *Scorpio*, while on its western border the cluster *Messier* 80 is found to be one of the richest and most compressed clusters in the heavens.

In his paper of 1802 (*Phil. Trans.*, 1802, p. 495), HERSCHEL again treats of the tendency to sidereal aggregation noticed in the Milky Way. He found the brightness greatest in the centre of the clusters; and this indicated to him that the stars of each group were clustering towards a common centre. From the aspect of the stars between β and γ *Cygni* he concluded that they are clustering towards opposite regions of a space about five degrees wide; and having found by counting that there were 331,000 stars in the area, he inferred that each cluster would contain at least 165,000 stars.

In a paper published in 1814 (*Phil. Trans.*, 1814, pp. 248 *et seq*), HERSCHEL again considers the tendency of the stars of the Milky Way to arrange themselves into separate systems, and examines the consequences of the continued operation of this clustering process. As the clusters are now much denser in the Milky Way than in regions remote from the Galaxy, he concludes that, under the influence of the mutual attraction of the stars, this clustering process would continue until it would ultimately result in the complete breaking up of the Milky Way, and the formation of a number of sidereal systems totally distinct from one another. "The grandeur of this conclusion," says GRANT,* "was worthy of the genius of HERSCHEL."

* *History of Physical Astronomy*, p. 576.

In his paper of 1802 Herschel examines our Sun and its stellar associates, and says they are in a "magnificent collection of innumerable stars, called the Milky Way, which must occasion a very powerful balance of opposite attractions to hold the intermediate stars at rest. For though our Sun, and all the stars we see, may truly be said to be in the plane of the Milky Way, yet I am now convinced, by long inspection and continued examination of it, that the Milky Way itself consists of stars very differently scattered from those which are immediately about us." . . . . "This immense aggregation is by no means uniform. Its component stars show evident signs of clustering together into many separate allotments" (*Phil. Trans.*, 1802, pp. 479-495).

These views of the elder Herschel will always deserve the consideration of philosophers in dealing with the grandest problem presented to our contemplation by the sidereal universe. Yet, after all, the question may perhaps be raised whether the Milky Way has not always been more or less aggregated into clusters, which have simply grown denser or become more dispersed with the flight of ages.

If these clusters have arisen from the condensation of nebulosity, and this diffused cosmical dust everywhere acts as a resisting medium, it would necessarily follow that all clusters would in time become more and more dense and globular, unless the aggregations thus going on are eventually dispersed by external forces due to the action of other clusters. These two causes together would fully account for the clustered and also for the fragmentary and dispersed aspect of the Milky Way seen in so many portions of that great arch of light.

§ 264. *The Clusters Generally Follow the Milky Way Because the Sidereal Universe is Made Up of Streams of Stars Lying in That Plane.*

After what we find out in dealing with the moving cluster in *Taurus*, dealt with in § 266, it is evident how great an effect the perspective of distance exerts on clusters of stars. As seen from a point near by, they may appear quite diffuse, but when removed further away they become a condensed cluster. Now since the stars are confined mainly to the fundamental plane of the Galaxy, it follows that even if they are quite distant from one another, many groups of them, from the mere effect of distance, will be made to appear as clusters, some of them quite dense, others comparatively open. The state of apparent condensation depends on the distance of the group, and its form: and if not globular, whether the aggregation is seen end-on or sidewise. These various possibilities give the varied structures of clusters, streams and clouds of stars seen in the Milky Way.

In such a complex aggregate it could not be expected that the phenomena

would all be simple. Many clusters and clouds and streams are here and there superposed, and observation does not enable us to disentangle one group from the other. However these details be explained, it is clear that the mere fact of the stars being collected near a place, whether in one or more superposed streams, or distributed in a more or less continuous stratum, would have the effect of giving us a series of clusters grouped along the general course of the Milky Way. The clusters therefore are necessarily found by observation to be most intimately connected with the Galaxy. Any other result is manifestly impossible, if we admit the wide extension of the Milky Way in its own plane. And, conversely, the clusters being there, and the nature of these masses of stars being known from the observation of groups such as the *Hyades*, we necessarily conclude that the Milky Way is vastly extended in the direction in which the clusters are gathered.

§ 265. *Spiral Movement in Clusters and in the Milky Way.*

The spiral movement inferred to exist generally in clusters is directly traceable by visible lines of nebulosity in such spiral nebulae as *Messier* 33 *Trianguli, Messier* 63 *Canum Venaticorum, Herschel IV*, 76 *Cephei*, etc. And as it is not possible to draw a hard and fast distinction between nebulae and clusters, it is justly inferred that the movement in all clusters whatsoever is similar to that observed in those particular clusters which still retain traces of nebulosity. This latter movement is similar to the movement seen in nebulae, and is of a spiral character.

As certain large nebulae pass by gradual development into clusters, we may be sure that the movements of all clusters are like those which still retain a nebulous aspect. The nebulosity enables us to make out the nature of the movement in special cases; but, on dynamical grounds, it must always be of the same general character, whether the nebulosity once pervading such a mass has disappeared or not.

This conclusion that the clusters are animated by spiral movement is of vast importance for our interpretation of the phenomena of the physical universe. We may be sure that all clusters have circulatory movement of a spiral character, and the same conclusion applies to the individual star-clouds of the Milky Way, and no doubt also to the Milky Way as a whole. Throughout the vast extent of the most gigantic cosmical systems observed in space the movement is sure to be of spiral character. This has been proved for the larger Magellanic Cloud from photographs taken by Russell, of Sydney, in 1890, which showed traces of a spiral including almost the whole of this immense mass of nebulae and stars.

In such spiral movement, there is circulation around the centre of gravity of the system, and orbital motion of the individual stars, but the orbits are of

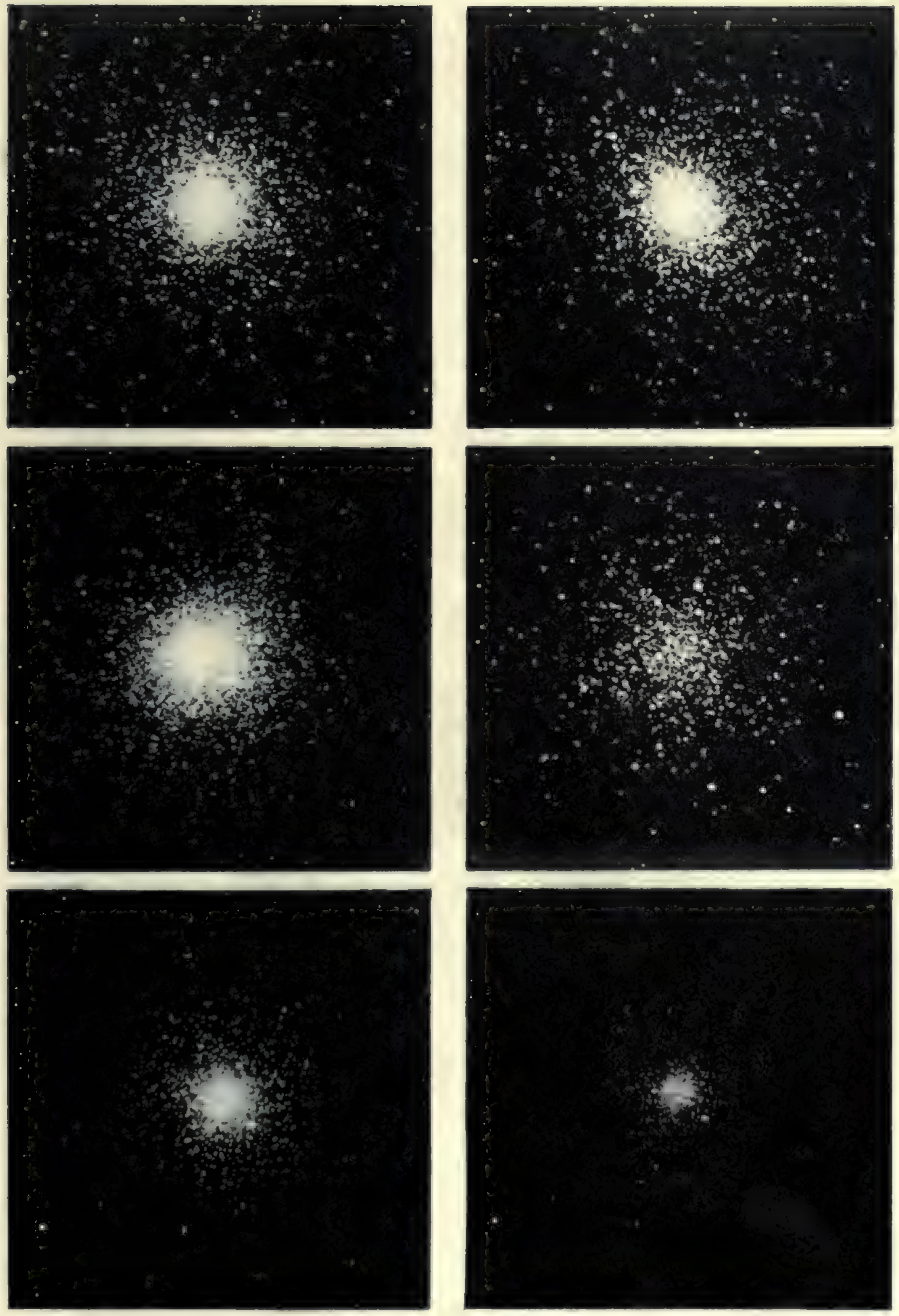

Plate F.

PHOTOGRAPHS OF STAR CLUSTERS TAKEN AT THE LICK OBSERVATORY AND THE ROYAL OBSERVATORY, CAPE OF GOOD HOPE:

M. 3, CANUM VENATICORUM; M. 5, LIBRAE;

M. 13, HERCULIS; M. 12, OPHIUCHI;

47 TOUCANI, GENERAL VIEW (CAPE); 47 TOUCANI, INTERNAL STRUCTURE (CAPE).

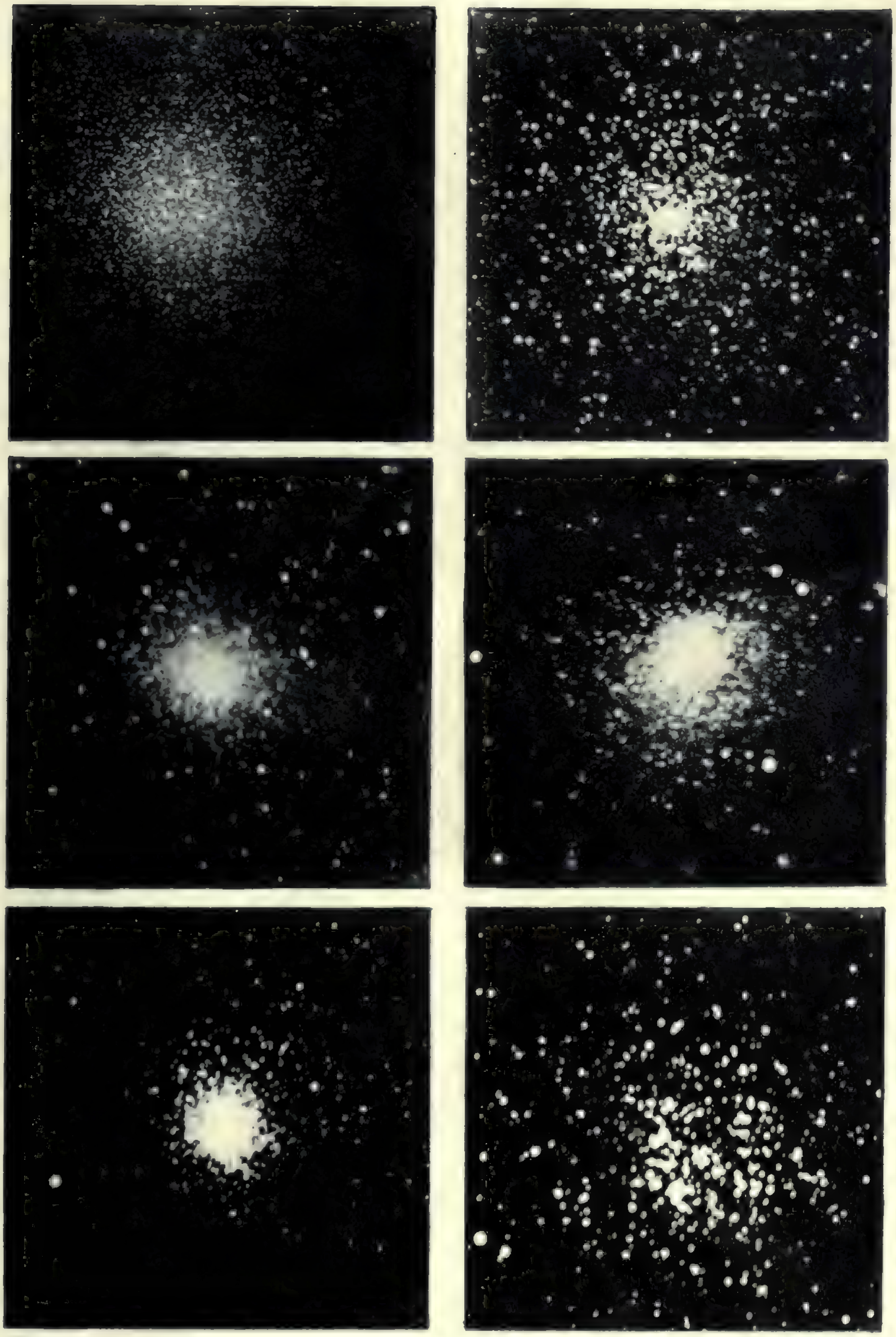

Plate G.

PHOTOGRAPHS OF STAR CLUSTERS TAKEN AT LICK, YERKES, ROBERTS, AND HARVARD OBSERVATORIES:

OMEGA CENTAURI (LICK); M. 15, PEGASI (YERKES);

M. 14, OPHIUCHI (ROBERTS); M. 2, AQUARII (ROBERTS);

M. 62, SCORPII (HARVARD); M. 11, ANTINOI (HARVARD).

Plate H.

PHOTOGRAPH OF THE GREAT STAR CLUSTER OMEGA CENTAURI,
TAKEN AT THE ROYAL OBSERVATORY, CAPE OF GOOD HOPE,
MAY 24, 1905 (J. LUNT), EXPOSURE 1h.

Plate 1.

GRAVURE ANDERSEN LAMB CO. N.Y.

PHOTOGRAPH OF THE GREAT STAR CLUSTER OMEGA CENTAURI,
TAKEN AT THE D.O. MILLS BRANCH OF LICK OBSERVATORY, SANTIAGO DE CHILE,
BY H.D. CURTIS, EXPOSURE 2h 30m

immense extent and never closed. A cluster as a whole may also experience gravitational oscillation, contracting sensibly as the stars fall towards the centre, and subsequently expanding as they recede from it. Some stars may even be driven out to great distances, and thus give rise to the outlyers noticed in many clusters. It is scarcely necessary to remark that none of these oscillatory movements have been observed within historical times. Our inferences rest on purely theoretical grounds, and POINCARÉ concludes from theory that in certain cases the outlyers give evidence of expulsion; yet it is probable that condensation due to the clustering power noticed by the elder HERSCHEL would equally well explain the observed phenomena.

§ 266. Boss's *Investigation of the Moving Cluster Observed in the Hyades.*

In the *Astronomical Journal*, No. 604 (Vol. 26, 1908), PROFESSOR LEWIS Boss, of Albany, has discussed a remarkable moving cluster in *Taurus*, comprising many of the stars known as the *Hyades*. The magnitudes of the forty stars treated vary from 3.5 to 7, and some of the more central members of the group have been under investigation by Boss for some twenty-five years; because it was long ago noticed that their proper motions were nearly identical. On completing recently his "Preliminary General Catalogue" of the brighter stars, he was enabled to recognize thirty-nine and possibly forty-two stars whose directions of motion converge to a common but very distant point, at which the apparent velocities will enable them to arrive almost simultaneously after the lapse of some sixty-five million years.

The cluster as now observed is spread over a total space of 15°, but is condensed towards the centre; and Boss shows that this apparent convergence towards a distant point is due to the recession of the cluster from us in space. In fifty thousand years the indications are that the cluster will be a little more condensed than at present; but Boss finds that the motions of the stars probably are in parallel lines, and after sixty-five million years, the cluster will have become a globular cluster 20′ in diameter, composed largely of stars from the 9th to the 12th magnitude, and considerably condensed towards the centre.

Since the outermost stars of the group as seen in the heavens are separated by 15°, their minimum distance apart must be greater than one-fourth $\left(\frac{15}{57.3}\right)$ of the distance of the cluster from the Earth. From the spectroscopic observations of KÜSTNER, at Bonn, it has been concluded that the average velocity of the entire cluster towards its vanishing point is 45.6 kms. per second, and the average parallax found to be 0″.025. From this it follows that the most widely separated stars of this cluster have a mutual parallax less than four times 0.025, or 0″.1.

Considering the probable significance of this result for sideral astronomy, Boss remarks that more than thirty of the stars nearest the Sun have parallaxes exceeding 0″.1. Accordingly it thus appears that the vast space occupied by the solar cluster has a diameter only twice the minimum diameter of the space occupied by the cluster in *Taurus*.

It has been remarked by CAMPBELL and other spectroscopic workers that 15 kms. of the recessional velocity of this cluster is due to the motion of the solar system away from the region of the *Hyades*. PROFESSOR FROST, Director of the Yerkes Observatory, has found by observations with the BRUCE spectrograph, that eight out of fourteen of the stars investigated in this moving cluster are spectroscopic binaries (cf. *Astrophysical Journal*, April, 1909, p. 237). All but one of the fourteen stars examined are of the *Orion* type, in which the proportion of spectroscopic binaries is usually not less than one to three.

This investigation of the *Hyades* cluster has excited wide interest among astronomers, as tending to restore in some respects the views of the elder HERSCHEL on the immense extent of the sidereal universe. For if this is a typical cluster, we readily appreciate how immensely distant most of the clusters must be; and though this cluster at present stands somewhat alone, there is no reason to consider it an exception in the general arrangement of clusters. As long ago as 1784 (*Phil. Trans.*, 1784, p. 450), SIR WILLIAM HERSCHEL remarked that the scattered aspect of the stars in the open cluster of *Coma Berenices* depends on its proximity to the solar system. As we shall see later, it is just possible that even HERSCHEL's views were too restricted to correspond to the reality in regard to the distribution of stars in space, though his views have long been considered extreme.

It was justly inscribed on the tomb of this unrivaled explorer of the heavens that he broke through the barriers of the heavens (*Coelorum perrupit claustra*), yet a repetition and extension of his grand labors by a modern successor may alone give us an adequate grasp of the problem of the construction of the universe. In a recent publication NEWCOMB estimated the distance of the remotest stars at some 3,000 light-years, or about 800 times that of the nearest star, α *Centauri*. HERSCHEL's researches show that these estimates of NEWCOMB are entirely inadequate and may have to be magnified a thousand times to give us an approximation to the actual dimensions of the Milky Way.

§ 267. *Owing to the Large Solid Angle Subtended by a Cluster, a Neighboring Star Will Almost Necessarily be Drawn Into It.*

This may be inferred from the aspects of a cluster such as the *Hyades* and *Coma Berenices*. For unless the star is passing by with considerable relative

velocity, it will be drawn into the group; and once within the cluster, the chances are that it will abide there. The dimensions of such a cluster are immense, and in the long interval of time required to traverse it, the attractive forces of the stars of the cluster will make themselves felt. In many cases, therefore, the star will be captured; and this probably happens in nearly all cases where the star penetrates the cluster to any appreciable depth. Now, if we consider the immense size of the clusters, we shall perceive that in the course of ages many stars will thus be gathered in; and the effect will be to give the Milky Way the aspect of a group of clusters rather than that of a uniform band of milky light. This leads to the breaking up of the Milky Way, as remarked by SIR WILLIAM HERSCHEL.

If, on the other hand, certain mutual actions occur between the stars of a cluster, it may be that one or more of them will be expelled from the clusters or driven out to a considerable distance from the centre. The chances of complete expulsion are small, because this could only arise from the action of a portion of the cluster, and the effect would have to be powerful enough to overcome the gravitation of the whole mass of stars. Moreover, just as a nebula is formed by the collection or capture of an infinite number of particles or small bodies, and afterwards slowly condenses, so also with clusters — they are formed by the principle of capture and condensation. Isolated bodies passing near these aggregations are almost necessarily drawn into them. The presence of a resisting medium aids the process of capture by causing the isolated masses to drift towards the more powerful centres of attraction. *The size of all orbits is thus reduced, and the state of compression of a cluster constantly accelerated;* so that in time the globular clusters become more and more compressed towards the centre, as long ago observed by HERSCHEL in his extensive explorations of the heavens.

§ 268. *Capture is Possible by Gravity Alone, but the Presence of a Resisting Medium Accelerates the Operation of This Clustering Power.*

Imagine a star to plunge into a cluster and traverse its diameter with such velocity that it will be carried through to the other side, after which it will oscillate back and forth along the same general path. If no collision or close approaches occurred, this movement might become *periodic*; but under the circulation of the other stars of the cluster, the period could not be constant, because the exact configuration would never be the same during two successive oscillations. Any supposed *periodic orbit* would therefore be temporary.* And if a resisting medium

* A considerable number of Periodic Orbits were discussed by HERSCHEL as long ago as 1802. In his celebrated paper on "Binary Stars," *Phil. Trans.*, 1802, he has outlined with great clearness the nature of the movement for several types of multiple stars.

were at work, even if the action of this cause be slight, the effect would be to work the stars closer together and introduce into the cluster a state of greater and greater compression. The amplitude of all oscillations would be reduced, just as in the case of a resisted pendulum; and finally the stars would be collected near the centre, and have their excursions confined within narrow limits. The star, therefore, which once oscillated through a wide space, would do so no more, but have its movement greatly reduced in amplitude.

To produce this effect, it is not necessary that the resisting medium be dense; a feeble resistance operating over a very long time will produce the same effect. And since millions of ages have been required to produce the observed aspects of the universe, we must look upon the observed phenomena as the outcome of these several forces. Gravitation alone might gather the stars into a globular cluster, because gravitation of the whole mass will tend to make it assume that form; but the rate of compression is accelerated by the secular action of a resisting medium, and those clusters which are compressed to a perfect blaze of light in the centre, give evidence of great age under the action of these two forces. The path of a star in a cluster is obviously non-reëntrant, and while the streams of stars probably have a spiral movement, and the system as a whole rotates about some axis, the average tendency will be for the stars to contract their paths to smaller and smaller dimensions. Herschel therefore justly observed that the more compressed clusters are the oldest; and that a central accumulation to a blaze of light is an indication of advanced development. Thus 47 *Tucani* is a cluster which exhibits the effects of great age, but ω *Centauri* still has a more youthful aspect.

§ 269. *Mechanism of Clusters and Changes of Distribution of the Stars Under the Power of Gravity.*

Not only may we infer that the stars exhibit the clustering power, but also understand how the stars eventually may become quite evenly distributed over the space now occupied by the cluster, though they may have been originally in certain streams. Let the original coils of a cluster be imagined to be parts of elliptic paths; then it is obvious that the streams may lie not in one plane, but in any direction from the origin, and thus in any of the eight equal parts into which the solid angle is divided by the coordinate planes when the origin is situated at the centre of gravity of the cluster. We cannot give a general theory of the movement, in the present state of science, but the following particular solution by Sir John Herschel (*Phil. Trans.*, 1833) will throw some light on the problem of the mechanism of clusters: "A quiescent spherical form may subsist as the bounding

outline of an immense number of equal stars, uniformly distributed through its extent. In such a state of things each star might describe an ellipse in any plane, and in any direction in that plane, about the common centre, without the possibility of collision. If the form be not spherical, and the distribution of the stars not homogeneous, the dynamical relations become too complicated to be distinctly apprehended."

As having an analogy with the mechanism of clusters we may pause a moment to recall what is shown on a vast scale in the two Magellanic Clouds, which present the appearance of detached portions of the Milky Way. In her excellent work on *The Herschels and Modern Astronomy*, pp. 166-7, the late Miss Clerke says: "The first photographs of the Magellanic Clouds were taken in 1890-91 by Mr. Russell of Sydney. They contained an extraordinary revelation. Both objects came out in them as gigantic spirals. Their miscellaneous contents are then arranged according to the dictates of a prevalent, though unexplained cosmical law. The Nubecula Major is a double vortex, and the extent of its outlying portions, invisible except to the camera, is at least eight times that of the central mass; but they conform to the same helical lines."

In view of the law established in this volume, a spiral structure of the Magellanic Clouds is not remarkable, but naturally to be expected, and it is highly probable that the circulation of the Milky Way also is everywhere of a spiral character.

Resuming now the subject of spiral circulation in clusters, it is evident that whatever be the original distribution of the stars, after a great number of revolutions the perturbations will have dispersed the stars of the streams, and spread them over the whole of the space now occupied by the cluster, just as the elements of meteoric dust diffused along the path of a comet are spread out and diffused over the solar system by the action of the planets. The even distribution is the outcome of vast age. Accordingly those clusters presenting a uniform but very condensed aspect, such as 47 *Tucani*, and most of the dense globular clusters, are old; those still showing dark lanes and other irregularities, like the great cluster in *Hercules* and some others, probably are relatively younger, because as yet the perturbations have not produced such a uniform diffusion of the bodies.

Thus in time the distribution may become more uniform, so that collisions would become more difficult; but the condensation slowly augments and no doubt the swarm of stars experiences a secular shrinkage, especially under the influence of a resisting medium. It would seem that in time a cluster must unite to form one vast mass; but as yet this does not seem to have occurred very generally in

the universe, unless such giant stars as *Canopus* and *Arcturus* have been thus produced.

Accordingly by the study of the forms of spiral nebulae we may recognize the nature of the movement in clusters and in the Galaxy. As HERSCHEL remarked, this great stratum of stars is largely breaking up under the continued effect of a clustering power, which can be nothing else than universal gravitation. In general we may say that nebulae are least advanced in development; then come scattered and nebulous clusters, and finally dense globular clusters, which show the maximum effect of the clustering power that is gradually breaking up the Milky Way. This majestic band of milky light is largely made up of clusters and streams of stars, moving in several directions, but gradually showing more and more the effect of the clustering power which is most clearly seen in the spiral nebulae.

§ 270. LORD KELVIN'S *Comparison of the Stars of the Milky Way to the Molecules of a Mass of Gas.*

In a well known paper on the Clustering of Gravitational Matter in any part of the Universe, LORD KELVIN has compared the stars of the Milky Way to the molecules in a mass of gas. The grains of dust are in this case no longer atoms, but stars, separated by wide intervals and moving with considerable velocities. In fact they are so wide apart that the path of each star is nearly rectilinear, but in some instances two may come near enough together to have their paths curved in the passage. No such deflection of the path, it is true, has been observed within the historical period, but after long ages such an effect may be found somewhere.

LORD KELVIN'S idea was that to the eyes of a giant for whom the stars would be what atoms are to us, the Milky Way would seem to behave as a bubble of gas. This is a useful suggestion, but it is not entirely accurate. For in the mass of gas, the velocities of the particles are such that they traverse very quickly, in an infinitesimal fraction of a second, the spaces between the average molecule, because the velocity of the molecule is large compared to the spaces between the molecules. In the case of the stars, however, this is no longer true; on the contrary the stars move with such small velocities that it would take long ages for one of them to traverse the average space between the component stars of the Milky Way; yet after all to such a giant as LORD KELVIN imagined, this interval might seem very brief.

Thus while there is a close analogy between the stars of the Galaxy and the Theory of Gases, there is also some fundamental difference. No doubt the laws

of the gases relative to the average effects of proper motions and to the statistical method of calculating by the theory of probability the mean results of distribution and movement would be applicable, and there are some other results which are quite clear, and will be considered in the following sections.

§ 271. *The Kinetic Energy of Stars in a Cluster, as of Molecules in a Mass of Gas in Equilibrium, Greatest at the Centre.*

In the theory of a mass of gas in equilibrium under the pressure and attraction of its parts, it is well known that the density, temperature and pressure are greatest at the centre. The adiabatic law gives the state of a gas, and was first applied to the heavenly bodies by J. Homer Lane in 1869 (*Am. Journal of Science*, July, 1870). It has since been extended by Lord Kelvin, Ritter, See and others, and at length we have a fairly perfect theory of the gaseous nebulae and stars. See developed especially the theory of a monatomic gas in *Astron. Nachr.*, 4053, and Lord Kelvin has considered other gases in *Phil. Mag.*, April, 1908, and *Proc. Royal Society* of Edinburgh, Vol. XXVIII, March 9, 1908.

Just as the velocity of molecules is greatest in the centre of a mass of gas, so also the velocity of stars is greatest at the centre of a cluster. The reason of this in the case of the cluster is that the centre of the cluster is its centre of gravity and centre of attraction, and in falling into this point, the velocity steadily increases.* On going outward again, it as steadily decreases, and becomes small, when a great distance from the centre has been attained. The velocity and temperature and pressure are greatest at the centre of such a mass of gas. So also in a cluster the velocity is greatest at the centre; but as the stars are not in collision, there is properly speaking no pressure, and the only temperature is that of the individual stars. But what corresponds to temperature in a gas, namely average kinetic energy of the stars, evidently is a maximum at the centre of the cluster.

It is clear that as the velocities are a maximum at the centre of a cluster, some of the stars might acquire there such velocities as to carry them beyond the gravitative control of the cluster, and they would speed through the Milky Way as run-away stars, and the cluster would lose some of its members. On the other hand stars passing through might have their velocities correspondingly reduced

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* If a star *formed* in the centre of a cluster, or reached there by any process which did not develop large velocity, the symmetry of the cluster and the mutual balancing of opposite forces might leave it without much velocity, in spite of the central situation. Several such inert stars in a cluster, however, would tend to unite into a single mass, and as large central stars are not usual in clusters, the inference is that nearly all are moving with considerable velocities, so that such large stars do not ordinarily develop, or develop only in the oldest clusters.

and they would thus become attached to the cluster. As the resisting medium is everywhere at work, the number of stars captured would in the long run exceed those lost, and the clusters gradually gather in more and more stars. It is this effect which HERSCHEL foresaw in the clustering power at work in the Milky Way.

§ 272. *Paths of the Fixed Stars Nearly Straight Except in the Clusters, Where They are Curved and Tangled in Every Possible Manner.*

In his suggestive address on the Milky Way and the Theory of Gases (cf. *Bulletin de la Société Astronomique de France*, April, 1906; translated in *Popular Astronomy* for Oct., 1906), POINCARÉ points out that in general the paths of the stars will resemble those of the radiant matter seen in CROOKES' tubes, where the free path is very long and straight. When two molecules pass near one another, the trajectory is suddenly curved and the whole path is made up of straight lines joined by short, curved arcs, corresponding to the points of closest approach of the molecules. It is very similar with the stars of the Milky Way. In order to curve the path of a star appreciably, the approach has to be very close. Now such close approaches seldom occur in the general course of the Milky Way; and it is even calculated that a star might traverse the Galaxy many times without coming near enough to another star to cause the path to be curved perceptibly.

But in the denser clusters the case is somewhat different. Here a star is always partially under the influence of its neighbors. And in moving through the denser parts of the cluster, it will happen that the approach may be comparatively close, and the path will curve more or less at every point; nor will the path lie in a plane, but become a curve of double curvature.

In regard to the chances of a star being swerved from its course in traversing the Milky Way, POINCARÉ has the following interesting suggestion in his address on the Milky Way and the Theory of Gases, which grew out of the paper of LORD KELVIN on the same subject. "What then happens to the Milky Way?" asks POINCARÉ. "It is a mass of gas whose density is very low but whose dimensions are very great; does a star have chances to cross it without suffering a shock, *i.e.*, without passing near enough to another star to be perceptibly swerved from its course? What do we understand by *near enough?* This is necessarily a little arbitrary; let us say that it is the distance from the Sun to *Neptune*, which would represent a deviation of about ten degrees, then let us suppose each one of our stars enveloped in a protecting sphere of this radius; will a straight line be able to pass between these spheres? At the average distance of the stars of the Milky Way the radius of one of these spheres will be seen at an angle of about one-tenth

of a second, and we have a thousand million stars. Let us place on the celestial sphere a thousand million circles with a radius of a tenth of a second. Is there any chance that these circles will cover the celestial sphere a great number of times? Far from it, they will only cover the sixteen-thousandth part. Thus the Milky Way is not the image of gaseous matter, but of the radiant matter of CROOKES."

§ 273. POINCARÉ'S *Remarks on the Theory of Clusters.*

After discussing the Milky Way as if it were a spherical mass of stars, POINCARÉ considers what modification should be introduced to take account of actual conditions, and then treats of the theory of clusters (*Popular Astronomy*, October, 1906):

"The Milky Way is not spherical and up to this time we have reasoned as though it were, since that is the form of equilibrium which a gas isolated in space would take. In support of this theory there exist star clusters whose form is globular and to which what has just been said applies better. HERSCHEL had already endeavored to explain their remarkable appearance. He supposed that the stars of the clusters are uniformly distributed in such a manner that one cluster would be a homogeneous sphere; each star then would describe an ellipse and all these orbits would be traveled over in the same time, so that at the end of a period the cluster would find again its primitive configuration and that this form would be stable. Unfortunately the clusters do not appear homogeneous; one observes condensation at the centre, one would observe it even if the sphere were homogeneous, since it is thicker at the centre, but it would not be so marked. We can then rather compare a cluster to a gas in adiabatic equilibrium and which takes the spherical form because that is the form of equilibrium of a gaseous mass.

"But, you will say, these clusters are much smaller than the Milky Way of which they probably even form a part, and although they are denser they will still give us something more analagous to radiant matter; now gases reach their adiabatic equilibrium only by a succession of innumerable shocks of the molecules. There might be some way of arranging that. Let us suppose that the stars of the cluster have just enough energy so that their velocity is annulled when they reach the surface; then they will be able to pass through the mass without shock, but having reached the surface they will turn backward and cross it again; after a great number of crossings they will end by being turned aside by a shock; in these conditions we should still have a matter that could be considered as gaseous; if by chance there had been in the cluster stars whose velocity was greater, they have

left it long ago, they have gone away to return there no more. For all these reasons it would be interesting to examine the known clusters, to try and explain the law of their densities and to see if it is the adiabatic law of gases."

In view of these considerations, it is clear that stars occasionally may be lost from clusters, but still oftener drawn into these dense masses and captured. This explains the tendency of the stars to form swarms as noticed by the elder HERSCHEL in various parts of the universe; and it justifies his inference that the Milky Way is breaking up under the continued action of the clustering power, and already has ceased to be a continuous band of milky light, and become a clouded belt of star dust, such as now spans the midnight sky with such unspeakable grandeur.

The Rotation of the Milky Way. In the address above cited POINCARÉ concludes that the Milky Way is not in equilibrium and that the stars have a tendency to quit its plane "so that the system will tend toward a spherical form of equilibrium of an isolated mass of gas. Or else the entire system is animated by a common rotation, and it is for this reason that it is flattened like the Earth, like *Jupiter*, like all bodies which revolve. Only, as the flattening is considerable, it must be that the rotation is rapid; rapid indeed, but we must agree on the meaning of this word. The density of the Milky Way is 10^{25} times less than that of the Sun; a velocity of rotation which is $\sqrt{10^{25}}$ times less than that of the Sun would then have its equivalent from the point of view of the flattening; a velocity 10^{12} times slower than that of the Earth, that is a thirtieth of a second of arc in a century, would be a very rapid rotation, almost too rapid for stable equilibrium to be possible."

"In this hypothesis the observable proper motions will appear to us uniformly distributed and there will no longer be any preponderance for the components parallel to the Galactic plane. They will tell us nothing about the rotation itself, since we are a part of the revolving system."

From these considerations it is clear that if the Milky Way has no rotation, it is not in equilibrium and will eventually round up and become more and more spherical under the mutual attraction of its component stars. On the other hand, if it is in rotation, there would be little or no tendency to round up, and the present form of the Galaxy might be maintained, though it would become locally condensed or disintegrated under the continued action of the clustering power noticed by HERSCHEL, the ravages of which are already very apparent in the congestion of the stars in many parts of the Milky Way, which thus presents the aspect of a luminous band of star-clouds. Thus what is to us mortals the sublimest of all the grand phenomena of nature will in time tend to pass away, and we shall have instead only a disconnected series of clusters.

§ 274. Sir William Herschel's *Views on the Nature of Clusters*, 1785.

In the *Philosophical Transactions* for 1785, pp. 213-266, Herschel discusses the Construction of the Heavens, and states his views on clusters as follows:

"*Theoretical View.*

"Let us then suppose numberless stars of various sizes, scattered over an indefinite portion of space in such a manner as to be almost equally distributed throughout the whole. The laws of attraction, which no doubt extend to the remotest regions of the fixed stars, will operate in such a manner as most probably to produce the following remarkable effects.

"*Formation of Nebulae.**

"Form I. In the first place, since we have supposed the stars to be of various sizes, it will frequently happen that a star, being considerably larger than its neighboring ones, will attract them more than they will be attracted by others that are immediately around them; by which means they will be, in time, as it were, condensed about a centre; or, in other words, form themselves into a cluster of stars of almost a globular figure, more or less regularly so, according to the size and original distance of the surrounding stars. The perturbations of these mutual attractions must undoubtedly be very intricate, as we may easily comprehend by considering what Sir Isaac Newton says in the first book of his *Principia*, in the thirty-eighth and following problems; but in order to apply this great author's reasoning of bodies moving in ellipses to such as are here, for a while, supposed to have no other motion than what their mutual gravity has imparted to them, we must suppose the conjugate axes of these ellipses indefinitely diminished, whereby the ellipses will become straight lines.

"Form II. The next case, which will also happen almost as frequently as the former, is where a few stars, though not superior in size to the rest, may chance to be rather nearer each other than the surrounding ones; for here also will be formed a prevailing attraction in the combined centre of gravity of them all, which will occasion the neighboring stars to draw together; not indeed so as to form a regular or globular figure, but however in such a manner as to be condensed towards the common centre of gravity of the whole irregular cluster. And this construction admits of the utmost variety of shapes, according to the number and situation of the stars which first gave rise to the condensation of the rest.

* At this stage of his explorations Herschel supposed nebulae to be nothing but clusters of stars. This must be borne in mind throughout the discussion based on his early papers.

"Form III. From the composition and repeated conjunction of both the foregoing forms, a third may be derived, when many large stars, or combined small ones, are situated in long extended, regular, or crooked rows, hooks, or branches; for they will also draw the surrounding ones, so as to produce figures of condensed stars coarsely similar to the former which gave rise to these condensations.

"Form IV. We may likewise admit of still more extensive combinations; when, at the same time that a cluster of stars is forming in one part of space, there may be another collecting in a different, but perhaps not far distant quarter, which may occasion a mutual approach towards their common centre of gravity.

"Form V. In the last place, as a natural consequence of the former cases, there will be formed great cavities or vacancies by the retreat of the stars towards the various centres which attract them; so that upon the whole there is evidently a field of greatest variety for the mutual and combined attractions of the heavenly bodies to exert themselves in. I shall, therefore, without extending myself farther upon this subject, proceed to a few considerations, that will naturally occur to every one who may view this subject in the light I have here done.

"*Objections Considered.*

"At first sight then it will seem as if a system, such as has been displayed in the foregoing paragraphs, would evidently tend to a general destruction, by the shock of one star's falling upon another. It would here be sufficient answer to say that if observation should prove this really to be the system of the universe, there is no doubt but that the great Author of it has amply provided for the preservation of the whole, though it should not appear to us in what manner this is effected. But I shall moreover point out several circumstances that do manifestly tend to a general preservation; as, in the first place, the indefinite extent of the sidereal heavens, which must produce a balance that will effectually secure all the great parts of the whole from approaching to each other. There remains then only to see how the particular stars belonging to separate clusters will be preserved from rushing on to their centres of attraction. And here I must observe that, though I have before by way of rendering the case more simple considered the stars as being originally at rest, I intended not to exclude projectile forces; and the admission of them will prove such a barrier against the seeming destructive power of attraction as to secure from it all the stars belonging to a cluster, if not forever, at least for millions of ages. Besides, we ought perhaps to look upon such clusters, and the destruction of now and then a star, in some thousands of ages, as perhaps the very means by which the whole is preserved and renewed. These clusters may be the *Laboratories* of the universe, if I may so express myself,

wherein the most salutary remedies for the decay of the whole are prepared" (pp. 214-217).

Continuing his discussion of clusters, HERSCHEL concludes (p. 259) that some of them are six or eight thousand times the distance of *Sirius*.

"My opinion of their size is grounded on the following observations. There are many round nebulae, of the first form, of about five or six minutes in diameter, the stars of which I can see very distinctly; and on comparing them with the visual ray calculated from some of my long gauges, I suppose, by the appearance of the small stars in those gauges, that the centres of these round nebulae may be 600 times the distance of *Sirius* from us. In estimating the distance of such clusters I consulted rather the comparatively apparent size of the stars than their mutual distance; for the condensation in these clusters, being probably much greater than in our own system, if we were to overlook this circumstance and calculate by their apparent compression, where, in about six minutes diameter, there are perhaps ten or more stars in the line of measures, we should find, that on the supposition of an equal scattering of the stars throughout all nebulae, the distance of the centre of such a cluster from us could not be less than 6,000 times the distance of *Sirius*. And, perhaps, in putting it, by the apparent size of the stars, at 600 times only, I may have considerably underrated it; but my argument if that should be the case, will be so much the stronger."

On page 266, HERSCHEL adds a remark on the origin of new stars:

"If it were not perhaps too hazardous to pursue a former surmise of a renewal in what I figuratively called the laboratories of the universe, the stars forming these extraordinary nebulae, by some decay or waste of nature, being no longer fit for their former purposes, and having their projectile forces, if any such they had, retarded in each other's atmosphere, may rush at last together, and either in succession, or by one general tremendous shock, unite into a new body. Perhaps the extraordinary and sudden blaze of a new star in *Cassiopea's* Chair, in 1572, might possibly be of such a nature."

§ 275. HERSCHEL'S *Views on the Arrangement of the Stars in a Cluster*, 1789.

In the *Philosophical Transactions* for 1789, pp. 214-226, HERSCHEL has a "Catalogue of a Second Thousand of New Nebulae and Clusters of Stars, with a Few Introductory Remarks on the Construction of the Heavens," in which he develops his views of the Constitution of Clusters:

"But first of all it will be necessary to explain what is our idea of a cluster of stars, and by what means we have obtained it. For an instance, I shall take

the phenomenon which presents itself in many clusters. It is that of a number of lucid spots, of equal lustre, scattered over a circular space, in such a manner as to appear gradually more compressed towards the middle; and which compression, in the clusters to which I allude, is generally so far, as by imperceptible degrees, to end in a luminous centre, of a resolvable blaze of light. To solve this appearance, it may be conjectured, that stars of any given, very unequal magnitudes may easily be so arranged, in scattered, much extended, irregular rows, as to produce the above described picture; or, that stars, scattered about almost promiscuously within the frustrum of a given cone, may be assigned of such properly diversified magnitudes as also to form the same picture. But who, that is acquainted with the doctrine of chances, can seriously maintain such improbable conjectures? To consider this only in a very coarse way, let us suppose a cluster to consist of 5,000 stars, and that each of them may be put into one of the 5,000 given places, and have one of 5,000 assigned magnitudes. Then, without extending our calculation any further, we have five and twenty millions of chances out of which only one will answer the above improbable conjecture, while all the rest are against it. When we now remark that this relates only to the given places within the frustrum of a supposed cone, whereas these stars might have been scattered all over the visible space of the heavens; that they might have been scattered, even within the supposed cone, in a million of places different from the assumed ones, the chance of this apparent cluster's not being a real one, will be rendered so highly improbable that it ought to be entirely rejected.

"Mr. Michell computes, with respect to the six brightest stars of the *Pleiades* only, that the odds are near 500,000 to 1 that no six stars out of the number of those which are equal in splendor to the faintest of them, scattered at random in the whole heavens, would be within so small a distance from each other as the *Pleiades* are.

"Taking it then for granted that the stars which appear to be gathered together in a group are in reality thus accumulated, I proceed to prove also that they are nearly of an equal magnitude.

"The cluster itself, on account of the small angle it subtends to the eye, we must suppose to be very far removed from us. For, were the stars which compose it at the same distance from one another as *Sirius* is from the Sun; and supposing the cluster to be seen under an angle of ten minutes, and to contain fifty stars in one of its diameters, we should have the mean distance of such stars twelve seconds; and therefore the distance of the cluster from us about seventeen thousand times greater than the distance of *Sirius*. Now, since the apparent magnitude of these stars is equal, and their distance from us is also equal — because we may safely

neglect the diameter of the cluster, which, if the centre be seventeen thousand times the distance of *Sirius* from us, will give us seventeen thousand and twenty-five for the farthest, and seventeen thousand wanting twenty-five for the nearest star of the cluster — it follows that we must either give up the idea of a cluster, and recur to the above refuted supposition, or admit the equality of the stars that compose their clusters. It is to be remarked that we do not mean entirely to exclude all variety of size; for the very great distance, and the consequent smallness of the component clustering stars, will not permit us to be extremely precise in the estimation of their magnitudes; though we have certainly seen enough of them to know that they are contained within pretty narrow limits; and do not, perhaps, exceed each other in magnitude more than in some such proportion as one full-grown plant of a certain species may exceed another full-grown plant of the same species.

"If we have drawn proper conclusions relating to the size of stars, we may with still greater safety speak of their relative situations, and affirm that in the same distances from the centre an equal scattering takes place. If this were not the case, the appearance of a cluster could not be uniformly increasing in brightness towards the middle, but would appear nebulous in those parts which were more crowded with stars; but, as far as we can distinguish, in the clusters of which we speak, every concentric circle maintains an equal degree of compression, as long as the stars are visible; and when they become too crowded to be distinguished, an equal brightness takes place, at equal distances from the centre, which is the most luminous part.

"The next step in my argument will be to show that these clusters are of a globular form. This again we rest on the sound doctrine of chances. Here, by way of strength to our argument, we may be allowed to take in all round nebulae, though the reasons we have for believing that they consist of stars have not as yet been entered into. For, what I have to say concerning their spherical figure will equally hold good whether they be groups of stars or not. In my catalogues we have, I suppose, not less than one thousand of these round objects. Now, whatever may be the shape of the group of stars, or of a nebula, which we would introduce instead of the spherical one, such as a cone, an ellipsis, a spheroid, a circle or a cylinder, it will be evident that out of a thousand situations, which the axes of such forms may have, there is but one that can answer the phenomenon for which we want to account; and that is, when those axes are exactly in a line drawn from the object to the place of the observer. Here again we have a million chances of which all but one are against any other hypothesis than that which we maintain, and which, for this reason, ought to be admitted.

"The last thing to be inferred from the above related appearances is that these clusters of stars are more condensed towards the center than at the surface. If there should be a group of stars in the spherical form, consisting of such as were equally scattered over all the assigned space, it would not appear to be very gradually more compressed and brighter in the middle; much less would it seem to have a bright nucleus in the centre. A spherical cluster of an equal compression within — for that such there are will be seen hereafter — may be distinguished by the degrees of brightness which take place in going from the centre to the circumference. Thus, when a is the brightness in the centre, it will be $\sqrt{a^2 - x^2}$ at any other distance x from the center. Or, putting $a = 1$ and $x =$ any decimal fraction; then, in a table of natural sines, where x is the sine, the brightness at x will be expressed by the cosine. Now, as a gradual increase of brightness does not agree with the degrees calculated from a supposition of an equal scattering, and as the cluster has been proved to be spherical, it must needs be admitted that there is indeed a greater accumulation towards the centre. And thus, from the above-mentioned appearances, we come to know that there are globular clusters of stars nearly equal in size, which are scattered evenly at equal distances from the middle, but with an increasing accumulation towards the centre. . . . .

"Having then established that the clusters of stars of the first form, and round nebulae, are of a spherical figure, I think myself plainly authorized to conclude that they are thus formed by the action of central powers. To manifest the validity of this inference, the figure of the Earth may be given as an instance; whose rotundity, setting aside small deviations, the causes of which are well known, is without hesitation allowed to be a phenomenon decisively establishing a centripetal force. Nor do we stand in need of the revolving satellites of *Jupiter, Saturn*, and the *Georgium Sidus*, to assure us that the same powers are likewise lodged in the masses of these planets. Their globular figure alone must be admitted as a sufficient argument to render this point uncontrovertible. We also apply this inference with equal propriety to the body of the Sun, as well as to that of *Mercury*, *Venus*, *Mars* and the Moon; as owing their spherical shape to the same cause. And how can we avoid inferring, that the construction of the clusters of stars and nebulae likewise, of which we have been speaking, is as evidently owing to central powers?

"Besides, the step that I here make in my inference is in fact a very easy one, and such as ought freely to be granted. Have I not already shown that these clusters cannot have come to their present formation by any random scattering of stars? The doctrine of chance, by exposing the very great odds against such hypotheses, may be said to demonstrate that the stars are thus assembled by some

power or other. Then, what do I attempt more than merely to lead the mind to the conditions under which this power is seen to act?

"In case of such consequences I may be permitted to be a little more diffuse, and draw additional arguments from the internal construction of spherical clusters and nebulae. If we find that there is not only a general form, which, as has been proved, is a sufficient manifestation of a centripetal force, what shall we say when the accumulated condensation, which everywhere follows a direction towards a centre, is even visible to the very eye? Were we not already acquainted with attraction, this gradual condensation would point out a central power, by the remarkable disposition of the stars tending towards a centre. In consequence of this visible accumulation, whether it may be owing to attraction only, or whether other powers may assist in the formation, we ought not to hesitate to ascribe the effect to such as are *central*; no phenomena being more decisive in that particular, than those of which I am treating. . . . .

"I shall now extend the weight of my argument, by taking in likewise every cluster of stars or nebula that shows a gradual condensation, or increasing brightness, towards a centre or certain point; whether the outward shape of such clusters or nebulae be round, extended, or of any other given form. What has been said with regard to the doctrine of chance, will or course apply to every cluster, and more especially to the extended and irregular-shaped ones, on account of their greater size. It is among these that we find the largest assemblages of stars, and most diffusive nebulosities; and therefore the odds against such assemblages happening without some particular power to gather them, increase exceedingly with the number of the stars that are taken together. But if the gradual accumulation either of stars or increasing brightness has before been admitted as a direction to the seat of power, the same effect will equally point out the same cause in the cases now under consideration. There are besides some additional circumstances in the appearance of extended clusters and nebulae, that favor very much the idea of a power lodged in the brightest part. Although the form of them be not globular, it is plainly to be seen that there is a tendency towards sphericity, by the swell of the dimensions the nearer we draw towards the most luminous place, denoting as it were a course or tide of stars, setting towards a centre. And — if allegorical expressions may be allowed — it should seem as if the stars thus flocking towards the seat of power were stemmed by the crowd of those already assembled, and that while some of them are successful in forcing their predecessors sideways out of their places, others are themselves obliged to take up with lateral situations, while all of them seem equally to strive for a place in the central swelling, and generating spherical figure.

"Since then almost all the nebulae and clusters of stars I have seen, the number of which is not less than three and twenty hundred are more condensed and brighter in the middle; and since, from every form, it is now equally apparent that the central accumulation or brightness must be the result of central powers, we may venture to affirm that this theory is no longer an unfounded hypothesis, but is fully established on grounds which cannot be overturned. . . . . .

"Let us then continue to turn our view to the power which is molding the different assortments of stars into spherical clusters. Any force, that acts uninterruptedly, must produce effects proportional to the time of its action. Now, as it has been shown that the spherical figure of a cluster of stars is owing to central power, it follows that those clusters, which, *ceteris paribus*, are the most complete in this figure, must have been longest exposed to the action of these causes. This will admit of various points of views. Suppose for instance that 5,000 stars had been once in a certain scattered situation, and that other 5,000 equal stars had been in the same situation, then that of the two clusters which had been longest exposed to the action of the modelling power, we suppose, would be most condensed, and more advanced to the maturity of its figure. An obvious consequence that may be drawn from this consideration is that we are enabled to judge of the relative age, maturity, or climax of a sidereal system, from the disposition of its component parts; and, making the degrees of brightness in nebulae stand for the different accumulation of stars in clusters, the same conclusions will extend equally to them all. But we are not to conclude from what has been said that every spherical cluster is of an equal standing in regard to absolute duration, since one that is composed of a thousand stars only must certainly arrive to the perfection of its form sooner than another, which takes in a range of a million. Youth and age are comparative expressions; and an oak of a certain age may be called very young, while a contemporary shrub is already on the verge of its decay" (pp. 213-225).

§ 276. Herschel's *Theory of Clusters*, 1802.

In the "Catalogue of 500 New Nebulae, Nebulous Stars, Planetary Nebulae, and Clusters of Stars, with Remarks on the Construction of the Heavens," published in the *Philosophical Transactions* for 1802, pp. 477-528, Herschel establishes the existence of double and multiple stars and the nature of their orbits, and then considers clusters as follows:

"IV. *Of Clustering Stars, and the Milky Way.*

"From the quadruple, quintuple, and multiple stars, we are naturally led to a consideration of the vast collections of small stars that are profusely scattered

over the Milky Way. On a very slight examination, it will appear that this immense starry aggregation is by no means uniform. The stars of which it is composed are very unequally scattered, and show evident marks of clustering together into many separate allotments. By referring to some one of these clustering collections in the heavens, what will be said of them will be much better understood, than if we were to treat of them merely in a general way. Let us take the space between β and γ *Cygni* for an example, in which the stars are clustering with a kind of division between them, so that we may suppose them to be clustering towards two different regions. By a computation, founded on observations, which ascertain the number of stars in different fields of view, it appears that our space between β and γ ,taking an average breadth of about five degrees of it, contains more than three hundred and thirty-one thousand stars; and, admitting them to be clustering two different ways, we have one hundred and sixty-five thousand for each clustering collection. Now, as a more particular account of the Milky Way will be the subject of a separate chapter, I shall only observe that the above-mentioned milky appearances deserve the name of clustering collections, as they are certainly brighter about the middle, and fainter near their undefined borders. For, in my sweeps of the heavens, it has been fully ascertained that the brightness of the Milky Way arises only from stars; and that their compression increases in proportion to the brightness of the Milky Way.

"We may indeed partly ascribe the increase, both in brightness and of apparent compression, to a greater depth of the space which contains these stars; but this will equally tend to show their clustering condition; for, since the increase of brightness is gradual, the space containing the clustering stars must tend to a spherical form, if the gradual increase of brightness is to be explained by the situation of the stars.

"V. *Of Groups of Stars.*

"From clustering stars there is but a short transition to groups of stars; they are, however, sufficiently distinct to deserve a separate notice. A group is a collection of closely, and almost equally compressed stars, of any figure or outline; it contains no particular condensation that might point out the seat of an hypothetical central force; and is sufficiently separated from neighboring stars to show that it makes a peculiar system of its own. It must be remembered that its being a separate system does not exclude it from the action or influence of other systems. We are to understand this with the same reserve that has been pointed out, when we explained what we called insulated stars.

"The construction of groups of stars is perhaps, of all objects in the heavens, the most difficult to explain; much less can we now enter into a detail of the

numerous observations I have already made upon this subject. I therefore proceed in my enumeration.

"VI. *Of Clusters of Stars.*

"These are certainly the most magnificent objects that can be seen in the heavens. They are totally different from mere groups of stars, in their beautiful and artificial arrangement. Their form is generally round; and the compression of the stars shows a gradual, and pretty sudden accumulation towards the centre, where, aided by the depth of the cluster, which we can have no doubt is of a globular form, the condensation is such that the stars are sufficiently compressed to produce a mottled lustre, nearly amounting to the semblance of a nucleus. A centre of attraction is so strongly indicated, by all the circumstances of the appearance of the cluster, that we cannot doubt a single moment of its existence, either in a state of real solidity, or in that of an empty centre, possessed of an hypothetical force, arising from the joint exertion of the numerous stars that enter into the composition of the cluster.

"The number of observations I have to give relating to this article, in which my telescopes, especially those of high space-penetrating power, have been of the greatest service, of course can find no room in this enumeration."

§ 277. HERSCHEL'S *Theory of a Clustering Power Which is Breaking Up the Milky Way*, 1814.

In a paper included in the *Philosophical Transactions* for 1814 (pp. 248-284), HERSCHEL devotes considerable attention to the clustering power which is collecting the stars into swarms of various forms and thus gradually breaking up the Milky Way:

"*Connoiss.* 30 is 'A brilliant cluster, the stars of which are gradually more compressed in the middle. It is insulated, that is, none of the stars in the neighborhood are likely to be connected with it. Its diameter is from 2′ 40″ to 3′ 30″. Its figure is irregularly round. The stars about the centre are so much compressed as to appear to run together. Towards the north are two rows of bright stars, four or five in a line.'

"In this accumulation of stars, we plainly see the exertion of a central clustering power, which may reside in a central mass, or, what is more probable in the compound energy of the stars about the centre. The lines of the bright stars, although by a drawing made at the time of observation, one of them seems to pass through the cluster, are probably not connected with it.

"14. *Of Differently Compressed Clusters of Stars.*

"I have hitherto only considered the arrangement of stars in clusters with a view to point out that they are drawn together by a clustering power, in the same manner as the nebulous matter has, in my former paper, been proved to be condensed by the gravitating principle; but in the forty-one clusters of the following two collections we shall see that it is one and the same power uniformly exerted which first condenses the nebulous matter into stars, and afterwards draws them together into clusters, and which by a continuance of its action gradually increases the compression of the stars that form the clusters.

"15. *Of the Gradual Concentration and Insulation of Clusters of Stars.*

"The existence of a clustering power is nowhere so visibly pointed out as in the thirty-nine clusters which are given in the following collection. My remarks upon them will come with more clearness when applied to a particular description of some of them.

"VI, 5 is 'A beautiful cluster of very compressed small stars of several sizes. It is of an irregular round form, about 12′ or 15′ in diameter, and the stars are gradually most compressed in the middle.'

"Here the gradually increasing compression of the stars points out the central situation of the clustering power; the form is also that of a solid, not much differing from a globular figure; and by the outline of the cluster we may consider it as already in an advanced state of insulation; from these circumstances we may therefore conclude that this cluster has been long under the influence of the clustering power. . . . . .

"*Connoiss.* 68 is 'A beautiful cluster of stars, extremely rich, and so compressed that most of the stars are blended together; it is near 3′ broad and about 4′ long, but chiefly round, and there are very few scatted stars about.'

"This oval cluster is also approaching to the globular form, and the central compression is carried to a high degree. The insulation is likewise so far advanced that it admits of an accurate description of the contour.

"The clusters of this class are beautiful, but can hardly be seen to any advantage without a twenty-feet telescope.

"16. *Of Globular Clusters of Stars.*

"The objects of this collection are of a sufficient brightness to be seen with any good common telescope, in which they appear like telescopic comets, or bright nebulae, and, under this disguise, we owe their discovery to many eminent astronomers; but in order to ascertain their most beautiful and artificial construction,

the application of high powers, not only of penetrating into space but also of magnifying, are absolutely necessary; and as they are generally but little known and are undoubtedly the most interesting objects in the heavens, I shall describe several of them by selecting from a series of observations of thirty-four years some that were made with each of my instruments, that it may be a direction for those who wish to view them to know what they may expect to see with such telescopes as happen to be in their possession" (pp. 272-273).

"It will not be necessary to add that the two last mentioned globular clusters, viewed with more powerful instruments, are of equal beauty with the rest; and from what has been said it is obvious that here the exertion of a clustering power has brought the accumulation and artificial construction of these wonderful celestial objects to the highest degree of mysterious perfection" (pp. 277-278).

Οἱ δὲ περὶ Ἀναξαγόραν καὶ Δημόκριτον, φῶς εἶναι τὸ γάλα λέγουσιν ἄστρων τινῶν · . . . . Οὗτος δ' ὁ κύκλος ἐν ᾧ τὸ γάλα φαίνεται τοῖς ὁρῶσιν, [ὅ, τε] μέγιστος ὢν τυγχάνει, καὶ τῇ θέσει κείμενος οὕτως, ὥστε πολὺ τοὺς τροπικοὺς ὑπερβάλλειν. Πρὸς δὲ τούτοις ἄστρων ὁ τόπος πλήρης ἐστὶ τῶν τε μεγίστων καὶ λαμπροτάτων, καὶ ἔτι τῶν σποράδων καλουμένων. Τοῦτο δ' ἐστὶ καὶ τοῖς ὄμμασιν ἰδεῖν φανερόν. Ὥστε διὰ ταῦτα συνεχῶς καὶ ἀεὶ ταύτην πᾶσαν ἀθροίζεσθαι τὴν σύγκρισιν · σημεῖον δὲ · καὶ γὰρ ἀυτοῦ τοῦ κύκλου πλεῖον τὸ φῶς ἐστιν ἐν θατέρῳ ἡμικυκλίῳ τῷ τὸ δίπλωμα ἔχοντι · ἐν τούτῳ γὰρ πλείω καὶ πυκνότερά ἐστιν ἄστρα ἢ ἐν θατέρῳ, ὡς ὀυ δι' ἑτέραν τινὰ ἀιτίαν γιγνομένου τοῦ φέγγους , ἢ διὰ τὴν τῶν ἄστρων φοράν · ἐι γὰρ ἔν τε τῷ κύκλῳ τούτῳ γίνεται ἐν ᾧ τὰ πλεῖστα κεῖται τῶν ἄστρων , καὶ ἀυτοῦ του κύκλου ἐν ᾧ μᾶλλον φαίνεται καταπεπυκνῶσθα καὶ μεγέθει καὶ πλήθει ἀστέρων , ταύτην ἐικὸς ὑπολαβεῖν οἰκειοτάτην ἀιτίαν εἶναι τοῦ πάθους.

They say concerning Anaxagoras and Democritus that they held that the Milky Way is due to the light of certain stars . . . . But the circle in which the Milky Way appears to our observations is immense; and the position is such that it extends much beyond the tropics. Moreover the place is full of the largest and most brilliant stars and also of those called sporadic. This is clearly visible to our eyes, in such a way as to suggest that these stars continually and always produce this entire combination of phenomena. Here is the proof. The most brilliant light of the circle appears in that one of the two hemispheres which contains the bifurcation. Now then, in this part, there are moreover many stars, and they are more crowded together than in the other; as if the movement of the stars might have been the sole cause of the brilliancy of the Milky Way. For if this brilliancy is in the circle which shows the greatest number of stars, and even in the part of the circle where the stars are largest and the greatest number of them gathered together and condensed, it is natural to suppose that this is the most probable and most direct cause of the phenomenon.

Aristotle, *De Meteorologica*, Lib. I, Chap. VIII, §§ 4, 15-17.

CHAPTER XXII.

General Theory of the Milky Way and of the Construction of the Sidereal Heavens.

§ 278. *Views of the Milky Way Held by the Greeks.*

The Science of the physical universe begins with the Greeks, and it will therefore be of interest to examine their theories of the Milky Way, although it would be unreasonable to expect more than sound fundamental principles from the greatest philosophers who lived before the invention of the telescope. The naked eye discloses to us only the general aspects of the Milky Way, yet the lucid stars are related to the Galaxy in such a way as to suggest that this great arch of light is due to the presence of countless multitudes of faint stars lying in that general direction in space. In fact, the bright stars in such constellations as *Cygnus*, *Sagittarius*, and *Centaurus* are not only related to the general path of the Milky Way, but also to the great star-clouds which conspicuously illuminate its course, in several of the constellations embraced within these limits; and thus a connection between the milky light of the Galaxy and the brighter naked-eye stars becomes so evident as to be almost irresistible. Accordingly, it is not remarkable that Anaxagoras and Democritus, as Aristotle tells us (*Meteorology*, Lib. I, Chap. VIII, § 4), should have concluded that the Milky Way is due simply to the light of certain stars. Similar views are credited to these great thinkers by Diogenes Laertius, in his "Lives of Celebrated Philosophers" (Lib. II).

Aristotle tells us, however, that some of the Pythagorian philosophers claimed that the Milky Way was the route of one of the stars which followed the course of the fall of *Phaeton*; that it was the Sun itself which of old followed the path of the Galaxy, so that the space was in some way scorched and made of luminous character by the passage of this star. The difference in the color of the sky along the circle of the Galaxy is thus attributed to the radiations of a body which once pursued this path. Aristotle then points out the absurdity of these traditional Pythagorian doctrines, and says there is still more reason why the Zodiac should be in this condition, since not only the Sun, but also all the

planets, move near the plane of the ecliptic. And yet the only luminous part of the Zodiac is that where it crosses the Milky Way at the point of intersection.

It is not worth while to dwell on the inadequate views of some of the Greek philosophers who held that the Milky Way was caused by the light of certain stars lying in the Earth's shadow, which were not entirely put out by the light of the Sun. Yet we may remark that this is the modern view* of the *Gegenschein*, which appears as an illumination of the sky not unlike the milky light of the Galaxy, but confined to a small elliptical area, and not diffused along a great circle of the heavens, as in the case of the Milky Way. In refuting these suggestions regarding the luminosity of stars in the shadow of the Earth, ARISTOTLE remarks that if the Milky Way depended on the position of the Sun, its situation would change with the movement of that luminary; whereas it remains fixed among the stars, and, moreover, that the cone of the Earth's shadow is not long, owing to the fact that the diameter of the Sun so greatly exceeds that of the Earth.

The view of the Milky Way finally adopted by ARISTOTLE is that this great circle is composed of a luminous substance occupying an intermediate position between the terrestrial atmosphere and the region of the fixed stars, and having the power and the property of fire (*Meteorology*, Lib. I, Chap. VII, § 11); so that by the motion of the higher air the comets are produced from it by exhalations, or spontaneous combustion, according to the curious conceptions on this subject prevalent among the ancients.

ARISTOTLE made the tails or emanations observed to proceed from comets the means of bringing these erratic bodies into a singular relationship to the Milky Way. According to this great philosopher, the innumerable multitude of stars which compose this starry zone give out a self-luminous, incandescent matter; and the circle of the Galaxy which divides the heavens into two slightly unequal hemispheres was regarded as a large comet, the substance of which was being incessantly renewed (cf. *Meteorology*, Lib. I, §§ 19-21; and HUMBOLDT's *Cosmos*, Vol. I, p. 88). ARISTOTLE expressly says that the Milky Way is the emanation or *chevelure* of a great number of stars lying along the course of the Galaxy, and he therefore defines the milky light as the emanation of this Galactic circle.

Opposite to the opening page of this chapter we give a singularly beautiful passage extracted from the theory developed in the *Meteorology* of ARISTOTLE,

* In *Nature*, of June 16, 1910, MR. R. T. A. INNES, Director of the Transvaal Observatory, suggests that the earth is bombarded with meteorites which are throwing off corpuscles; that these are repelled by the earth and sun and thus produce in the part of the sky opposite to the sun a faint tail less extensive than that of a comet, but bright enough to be visible on a dark night as the *Gegenschein*. This new theory has much to commend it, and will deserve the serious consideration of investigators.

which enables one to form a high estimate of his logical power and philosophic penetration into the great secrets of Nature.

PTOLEMY does not enter into a detailed discussion of the nature of the Milky Way, but from the allusions in the *Almagest* (Lib. VIII, Ch. 2), it appears certain that his opinion is the same as that of ANAXAGORAS and DEMOCRITUS — that the Milky Way is due to a multitude of small stars so very near to each other that their light is blended together to produce the appearance of a luminous zone (cf. also PLUTARCH, *De Placit,* Lib. III, Ch. I). This latter view was quite current among the Greeks, and is adopted by MANILIUS in his poem on the "Sphere" (Lib. I, Cap. IX). And any one so familiar with the celestial sphere as ARISTYLLUS, TIMOCHARIS and HIPPARCHUS must have been, would almost necessarily have reached this conclusion from the general aspects of the starry heavens. It is highly probable that this opinion had been uniformly accepted in the Alexandrian School since the days of TIMOCHARIS and ARISTYLLUS, and was therefore not a subject of much difference of opinion in the time of PTOLEMY; so that he assumes the sidereal constitution of the Milky Way, and does not consider it necessary to discuss it.

§ 279. *Views of Modern Writers on the Nature of the Milky Way, Prior to the Epoch of* SIR WM. HERSCHEL.

The views of the Greeks of the classic period were somewhat divided by the rival theories of ANAXAGORAS and DEMOCRITUS, on the one hand, and of ARISTOTLE on the other; and a similar difference of opinion continued to prevail among the Roman and Greek writers of later times. Thus it became a subject of debate, about the beginning of the Christian era, whether the Milky Way belonged to the domain of the *higher atmosphere*, as held by ARISTOTLE, or to the remoter region of the fixed stars proper. In the Middle Ages ARISTOTLE's views were somewhat misunderstood, and generally held to be that the Milky Way was a part of the *lower or terrestrial atmosphere.*

The celebrated Arabian philosopher, AVERROES, who died A.D. 1206, considered it advisable to refute this popularly misunderstood meteoric theory of the Milky Way. No doubt this was because Aristotleian writings were then almost universally used among the Saracens, who had taken up the cultivation of the sciences founded by the Greeks. This eminent Arabian justly pointed out, as in fact ARISTOTLE himself had already done, that the Milky Way must be at the distance of the stars, because the projection among them was the same, whether the Galaxy was viewed from Cordova or Morocco; so that it had no sensible parallax, as seen from different parts of the Earth, and must necessarily be in the region of the fixed stars (cf. HOUZEAU's *Vade Mecum de l'Astronomie,* p. 918). Similar discussions

were carried on by other Arabian writers who cultivated the sciences of the Greeks. And during the 14th century DANTE speaks of the view that the Milky Way is made up of a great mass of small stars as offering a high degree of probability.

In his great work *De Revolutionibus Orbium Celestium*, 1543, COPERNICUS expounds the true theory of the system of the world without considering the constitution of the Milky Way. But TYCHO BRAHÉ's account of the new star of 1572, and the speculations he has left us on this subject (cf. *Progymnasmata*, p. 795), indicate that he believed the Milky Way to be formed of a nebulous substance similar to that imagined by ARISTOTLE. And the speculations of KEPLER on the new star of 1604 imply that he entertained similar views (cf. *De Stella Nova*, Cap. XXIII, p. 115). These disputes and uncertainties, after the lapse of nearly twenty centuries, were finally settled by GALILEO's invention of the telescope in January, 1610.

In an announcement subsequently published in the *Sidereus Nuncius*, GALILEO says: "It is a truly wonderful fact that to the vast number of fixed stars which the eye perceives, an innumerable multitude before unseen, and exceeding more than ten fold those hitherto known, have been rendered discernible. Nor can it be regarded as a matter of small moment that all disputes respecting the nature of the Milky Way have been brought to a close, and the nature of the zone made manifest not to the intellect only, but to the senses" (*Opere di* GALILEO, Tome II, p. 4). Later telescopic observers verified and extended the pioneer work of GALILEO, and one of the great objects always in view when laboring to improve the telescope was to increase the power for exploring the wonders of the starry heavens.

Among the early observers of the Milky Way, mention should be made of HUYGHENS, who used a 23-ft. refractor. In 1656 he declared that the milky whiteness of the Galactic zone was not to be ascribed to irresolvable nebulosity or scantily interspersed nebulae, as some still believed, in accordance with the teachings of ARISTOTLE; but was owing solely to accumulated strata of small stars. After describing the nebula of *Orion*, and the stars in that region, he adds how different it is from other fixed stars: "For those stars which have generally been considered as nebulae, and even the Milky Way itself, when seen through a telescope, are found to have nothing nebulous about them, but are merely a multitiude of stars collected together into clusters."

§ 280. *Views of* WRIGHT *and* KANT *Regarding the Nature of the Milky Way.*

In his excellent "History of Physical Astronomy," GRANT gives the following account of the theories of WRIGHT and KANT: "Although the nature of the Milky Way was now well understood, no attempt was made for a long time to

investigate the particulars of its structure, and to connect its appearance with the distribution of the stars throughout the other parts of the visible heavens. This important object was at length accomplished by THOMAS WRIGHT, in his 'Theory of the Universe,' a work to which we have already had occasion to allude. The author has given an exposition of his theory in nine letters addressed to a friend. Alluding to the current opinion respecting the Milky Way, that it is composed entirely of stars, he asserts that this view of its nature was supported by his own observations with a reflector of one foot focal length. The following statement embraces the more important points of his theory.

"If we judge of the Milky Way by phenomena only, we must conclude it to be a vast ring of stars, scattered promiscuously round the celestial regions in the direction of a perfect circle. This view of its structure, however, does not accord with the aplanatic position and irregular distribution of multitudes of other stars of the same nature, dispersed throughout the celestial regions. It is not consistent with the harmony which pervades all of the other arrangements of nature, that one portion of the stars should be disposed with the most perfect regularity, while all the others were scattered about in the utmost confusion, without any regard to symmetry. It is more probable that the whole visible creation of stars forms one vast system, the parts of which are adjusted with the most perfect harmony, and that its incongruous aspect is due to the eccentric position in which it is viewed, and to the motions of the constituent bodies relatively to each other. When we reflect upon the various configurations of the planets, and the changes which they perpetually undergo, we may be assured that nothing but a like eccentric position of the stars could occasion such confusion among bodies otherwise so regular. In like manner, we may conclude that, as the planetary system, if viewed from the Sun, would appear perfectly symmetrical, so there may be some place in the universe where the arrangement and motions of the stars may appear most beautiful.

"If we suppose the Sun to be plunged in a vast stratum of stars, of inconsiderable thickness compared with its dimensions in other respects, it is not difficult to see that the actual appearance of the heavens may be reconciled with a harmonious arrangement of the constituent bodies of such a system, relative to some common centre, provided it be admitted, at the same time, that the stars have all a proper motion. In such a system it is manifest that the distribution of the stars would appear more irregular the farther the place of the spectator was removed from the centre of the stratum towards either of the sides. It is also evident that the stars would appear to be distributed in least abundance in the opposite directions of the thickness of the stratum, the visual line being shortest in either of those directions, and that the number of visible stars would increase as the stratum

was viewed through a greater depth, until at length, from the continual crowding of the stars behind each other, it would ultimately assume the appearance of a zone of light. According to this hypothesis, then, the whole of the visible stars, including the Sun, form part of the system of the Milky Way, their irregular distribution being occasioned by the eccentric position of the Sun, combined with their own proper motions.

"There are, in all probability, various systems resembling the Milky Way; but it is not unreasonable to suppose that there may be systems of stars differing as much in the order and distribution of their constituent bodies as the zones of *Jupiter* do from the rings of *Saturn*. We may, in fact, suppose that some systems of stars move in perfect spheres, at different inclinations and in different directions; while others, again, may revolve like the primary planets, in a general zone, or more probably in the manner of *Saturn's* ring; nay, perhaps, ring within ring, to a third or fourth order.

"In propounding his theory of the Milky Way, of which the foregoing is a brief sketch, WRIGHT does not recognize the existence of systems of stars subordinate to that great system. He indeed asserts that those cloudy spots which are resolvable into stars, might be explained by the principles which he laid down, but he does not formally assign to them a place in his theory. In the concluding letter, however, he appears to admit the existence of a multitude of sidereal systems within the boundaries of the visible universe, subordinate, of course, to the great system of the Milky Way. We also find, in the same letter, an interesting expression of his opinion respecting those nebulae which had hitherto proved irresolvable even in the best telescopes. Taking into consideration the multitude of sidereal systems included within the confines of the visible universe, it appeared to him not improbable that the immensity of space is occupied by endless succession of systems analogous in their structure to the great system (the Milky Way) of which the visible universe is composed. 'That this in all probability, may be the real case,' says he, 'is in some degree made evident by the many cloudy spots, just perceivable by us, as far without our starry regions, in which, although visibly luminous spaces, no one star or particular constituent body can possibly be distinguished; *those, in all likelihood, may be external creation, bordering upon the known one, too remote for even our telescopes to reach.*'

"The speculations of WRIGHT on the Milky Way," continues GRANT, "are so consistent with sound philosophy and the results of observation, that they cannot fail to obtain the sanction of every person who submits them to a careful examination. At the time of their original promulgation, however, the attention of mathematicians had become deeply engrossed with the development of the

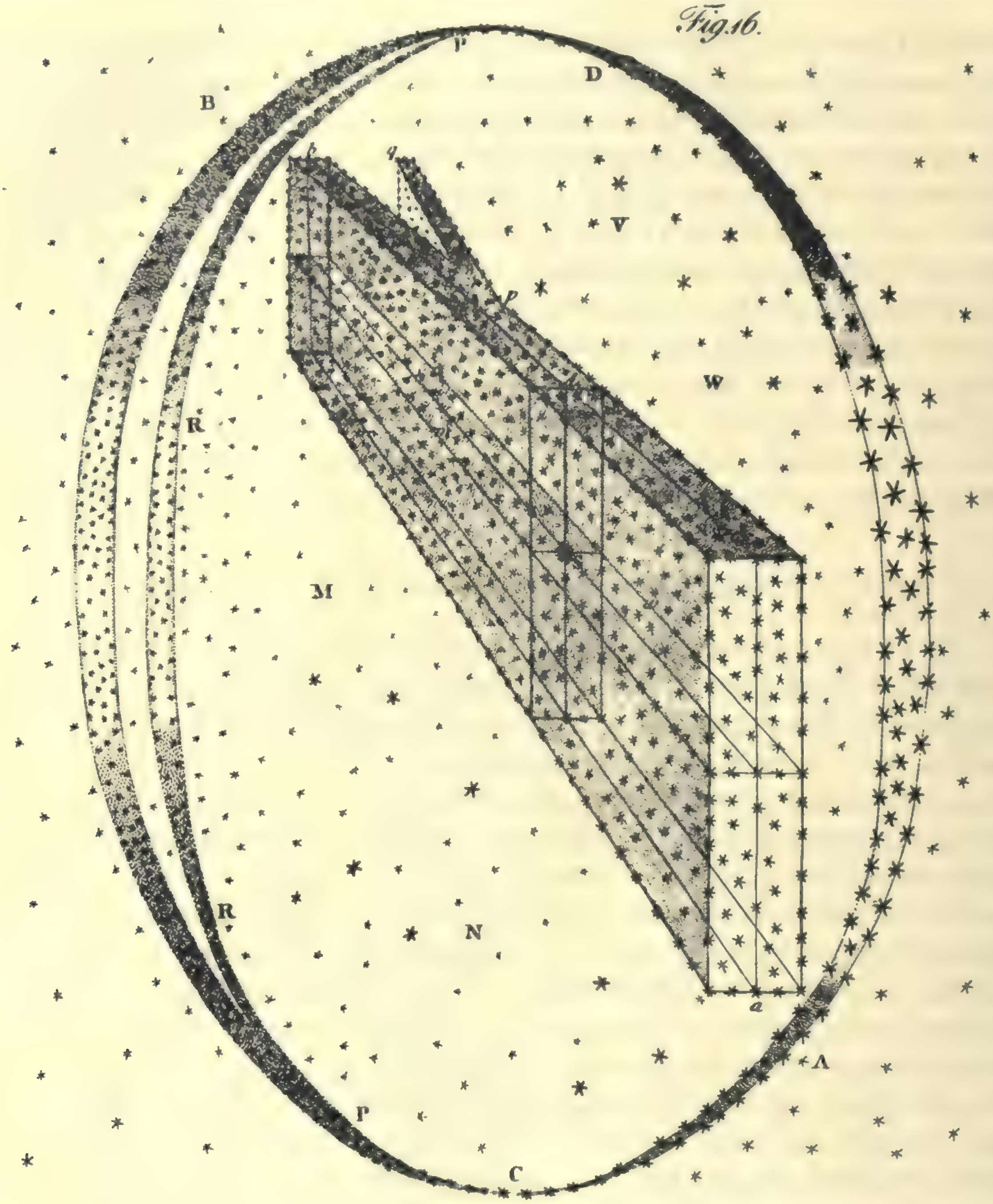
theory of gravitation, while astronomers, on the other hand, were impressed with the necessity of introducing a corresponding degree of refinement into their observations, and establishing with the utmost possible accuracy the elements of the planetary movements. It happened, from this cause, that only individuals of the same speculative turn of mind as WRIGHT himself were induced to adopt his theory as the basis of further inquiry. Such was the case with respect to KANT, the celebrated German metaphysician, who, in the year 1755, published a work containing an exposition of his views respecting the cosmical arrangement of the celestial bodies. In the introduction to this work, he acknowledges that the germ of his ideas on the distribution of the stars was suggested to him by the speculations of WRIGHT on the subject. His system, indeed, does not materially differ from that of the English philosopher. A system bearing close affinity to either, was also propounded a few years afterwards, by LAMBERT, in his 'Cosmological Letters.'"

§ 281. *The Beginning of the Explorations of* SIR WM. HERSCHEL.

As we have just seen, the views of the Milky Way, afterwards developed and founded on extensive exploration of the heavens by SIR WM. HERSCHEL, were first suggested by WRIGHT and KANT. But the observational basis which HERSCHEL gave to these theories by his incomparable explorations of the starry heavens was so much more complete than anything previously available, that we must consider it essentially a new development. The work of WRIGHT and KANT was speculative rather than observational; HERSCHEL'S work, on the other hand, included by far the most elaborate observations ever made, and enough of the speculative spirit to render the data philosophically intelligible. In the *Philosophical Transactions* for 1784, HERSCHEL has an "Account of Some Observations Tending to Investigate the Construction of the Heavens," in which he outlines his plan for studying the various nebulous* strata:

"In future, therefore, we shall look upon these regions into which we may now penetrate by means of such large telescopes, as a naturalist regards a rich extent of ground or chain of mountains, containing strata variously inclined and directed, as well as consisting of very different materials. A surface of a globe or map, therefore, will but ill delineate the interior parts of the heavens" . . . . "On applying the telescope to a part of the via lactea, I found that it completely resolved the whole whitish appearance into small stars, which my former telescopes had not light enough to effect" (p. 438). He then goes on to explain

* In his earlier work HERSCHEL always supposed the nebulae to be of stellar character, and therefore by *nebulous strata* he means *beds of stars*.



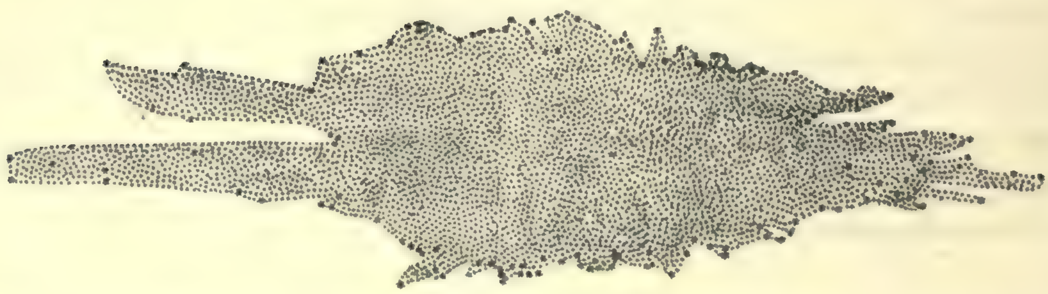


FIG. 42. THE UPPER FIGURE GIVES HERSCHEL'S THEORY OF THE CONSTRUCTION OF THE SIDEREAL HEAVENS, 1784. THE LOWER FIGURE REPRESENTS HERSCHEL'S SECTION OF THE SIDEREAL STRATUM PERPENDICULAR TO THE PLANE OF THE MILKY WAY, 1785. IN THIS FIGURE THE SCALE OF THE UNIVERSE IS SO IMMENSE THAT THE DISTANCE OF SIRIUS IS ONLY 1:160TH OF AN INCH, AND ALL THE NAKED EYE STARS ARE INCLUDED WITHIN A CIRCLE OF RADIUS 1:16TH OF AN INCH.

his method of gauging by counting the number of stars in ten fields and taking the mean.

"It is very probable that the great stratum, called the Milky Way, is that in which the Sun is placed, though perhaps not in the very centre of its thickness. We gather this from the appearance of the Galaxy, which seems to encompass the whole heavens, as it certainly must do if the Sun is within the same. For, suppose a number of stars arranged between two parallel planes, indefinitely extended every way, but at a given considerable distance from each other; and calling this a sidereal stratum, an eye placed somewhere within it will see all the stars in the direction of the planes of the stratum projected into a great circle, which will appear lucid on account of the accumulation of the stars; while the rest of the heavens, at the sides, will only seem to be scattered over the constellations, more or less crowded, according to the distance of the planes or number of stars contained in the thickness or sides of the stratum" (p. 443).

"From appearances then, as I observed before, we may infer, that the Sun is most likely placed in one of the great strata of the fixed stars and very probably not far from the place where some smaller stratum branches out from it. Such a supposition will satisfactorily, and with great simplicity, account for all the phenomena of the Milky Way, which, according to this hypothesis, is no other than the appearance of the projection of the stars contained in this stratum and its secondary branch. As a farther inducement to look on the Galaxy in this point of view, let it be considered that we can no longer doubt of its whitish appearance arising from the mixed lustre of the numberless stars that compose it. Now, should we imagine it to be an irregular ring of stars, in the centre nearly of which we must then suppose the Sun to be placed, it will appear not a little extraordinary that the Sun, being a fixed star like those which compose this imagined ring, should just be in the centre of such a multitude of celestial bodies, without any apparent reason for this singular distinction; whereas, on our supposition, every star in this stratum, not very near the termination of its length or height, will be so placed as also to have its own Galaxy, with only such variations in the form and lustre of it, as may arise from the particular situation of each star" (p. 445).

"If the Sun should be placed in the great sidereal stratum of the Milky Way, and, as we have surmised above, not far from the branching out of a secondary stratum, it will very naturally lead us to guess at the cause of the probable motion of the solar system; for the very bright, great node of the via lactis, or union of the two strata about *Cepheus* and *Cassiopeia*, and the *Scorpion* and *Sagittarius*,

points out a conflux of stars manifestly quite sufficient to occasion a tendency towards that node in any star situated at no very great distance; and the secondary branch of the Galaxy not being much less than a semi-circle seems to indicate such a situation of our solar system in the great undivided stratum as the most probable" (pp. 447-448).

§ 282. Herschel's *Views of the Arrangement of the Stars in Space.*

Already in his paper of 1784, Herschel had arrived at the foregoing figure as the best available representation of the arrangement of the stars in space. This figure is very well known, and need not be dwelt upon here. As the outcome of his paper of 1785, his gauges enabled him to represent a section of the sidereal stratum in a plane perpendicular to the Milky Way. This is also reproduced just below the other figure for convenience of reference. The only explanation we need give of this section of the universe perpendicular to the plane of the Milky Way is that Herschel tried to avoid clusters in taking his gauges; so that it represents the outcome of his method of studying the distribution of the stars from seven hundred gauges. He adds that in the section as here drawn, the distance of *Sirius* is represented by no more than one-eightieth part of an inch (corresponding to one hundred and sixtieth of an inch as the figure is here reproduced); and that all the naked eye stars are comprised within a sphere of only an eighth (one-sixteenth) of an inch in radius.

"From this figure, however, which I hope is not a very inaccurate one, we may see that our nebula,* as we observed before, is of the third form; that is: *A very extensive, branching, compound Congeries of many millions of stars;* which most probably owes its origin to many remarkably large as well as pretty closely scattered small stars, that may have drawn together the rest. Now, to have some idea of the wonderful extent of this system, I must observe that this section of it is drawn upon a scale where the distance of *Sirius* is no more than the eightieth part of an inch; so that probably all the stars, which in the finest nights we are able to distinguish with the naked eye, may be comprehended within a sphere, drawn round the large star near the middle, representing our situation in the nebula,* of less than half a quarter of an inch radius" (*Phil. Trans*, 1785, p. 254).

Dividing the heavens into zones of 15° in width, parallel to the plane of the Galaxy, Herschel finally obtained as the outcome of his counts of the stars in 3,400 telescopic fields the following mean results:

* To be understood in the sense of cluster, as already explained.

Zones about the North Pole of the Milky Way	Stars per Field W. Herschel	Corresponding Result for Southern Side of Galaxy, Found by Sir John Herschel
First Zone 90°–75° Galactic Latitude	4	6
Second Zone 75°–60° " "	5	7
Third Zone 60°–45° " "	8	9
Fourth Zone 45°–30° " "	14	13
Fifth Zone 30°–15° " "	24	26
Sixth Zone 15°– 0° " "	53	59

As discussed by W. Struve, in his *Études d'Astronomie Stellaire*, Petersbourg, 1847, pp. 71-72, these gauges of the Herschels give the following results:

Galactic Latitude	No. of Stars per Field
$\phi = 90°$	4.15
75°	4.68
60°	6.52
45°	10.36
30°	17.68
15°	30.30
0°	122.00

In his *Études d'Astronomie Stellaire*, Struve has calculated the following table of density of the stars on either side of the main stratum of the Milky Way, the unit of distance being the maximum distance to which Herschel's twenty-inch telescope could penetrate:

Distance from Principal Plane	Density	Mean Distance Between Neighboring Stars
0.00	1.000	1.000
0.05	0.48568	1.272
0.10	0.33288	1.458
0.20	0.23895	1.611
0.30	0.17980	1.772
0.40	0.13020	1.973
0.50	0.08646	2.261
0.60	0.05510	2.628
0.70	0.03079	3.190
0.80	0.01414	4.136
0.866	0.00532	5.729

This table gives the average condensation near the central plane, and a gradual thinning out on either side. Of course, this table does not apply to clusters or dense star-clouds, but only to portions of the sidereal stratum with average density. By taking the cube roots of these numbers, we get the relative distances of the zones of the starry stratum, on Herschel's hypothesis. Herschel's telescope

had an aperture of twenty inches, and covered a field of view of about 15′, or about one-fourth of the apparent surface of the Moon. SIR JOHN HERSCHEL's gauges in the southern hemisphere were made with a similar instrument, and corresponding field of view, so as to render the results comparable, and the general agreement is excellent. The clefts in the accompanying illustrations of SIR WM. HERSCHEL correspond to the bifurcation of the Milky Way, extending from *Cygnus* to *Centaurus;* thus HERSCHEL concluded that the universe extended about five times as far in the plane of the Milky Way as in the direction perpendicular to this plane. This gives a density of stars one hundred and twenty-five times as great in the plane of the Galaxy as in its poles, which does not seem excessive, but may easily be too small.

§ 283. *The Increase in the Density of Small Stars Towards the Plane of the Milky Way.*

A simple examination of the sky on a clear night shows that, while some of the bright stars are remote from the Milky Way, there is a very noticeable preponderance of the brighter stars towards this fundamental plane of the universe. This is not surprising, but a natural result of the arrangement giving greater extent of the universe in the direction of the plane of the Galaxy; for the brighter stars are made up of two classes: (1) The stars included within the cluster to which the Sun belongs, and others moving past us at no great distance; (2) Stars larger and brighter than the average, lying necessarily nearer and nearer the Milky Way, as the distance of that extended stratum increases. This will easily explain the obvious preponderance of bright stars near the Milky Way. PROCTOR and GORE have each examined this subject from the statistical point of view; and it is found that of 32 stars brighter than 2.0 magnitude, 12 lie upon the Milky Way, namely: *Vega, Capella, Altair, α Orionis, Procyon, α Cygni, α Persei, Sirius, α* and *β Centauri* and *α* and *β Crucis.* Now, the area of the Galaxy does not exceed one-seventh of the whole celestial sphere; but 12 is 37.5 per cent. of the 32 brightest stars, which are condensed into 14.3 per cent. of the space of the sky. According to GORE, 33 stars brighter than 3.0 magnitude lie upon the Milky Way, out of a total of 99, or 33.3 per cent. Extending this reasoning to the 262 stars between 3d and 4th magnitudes, he finds 73 lying on the Milky Way, or 28.2 per cent. He estimates 164 stars in the northern hemisphere brighter than 4.0 magnitude, of which 52 lie on the Milky Way, or 31.7 per cent., while in the richer southern hemisphere he estimates 228 stars brighter than 4.0 magnitude, with 66 on the Milky Way, or 28.9 per cent. Out of the 5,356 lucid stars given in HEIS'

Atlas, GORE finds 1,186 on the Milky Way, or 22.1 per cent. It has been remarked that the percentage of bright stars is large, then a decrease takes place, and when we come to small magnitudes, there is again a large percentage. This can only mean that the body of the stars in the Milky Way are immensely distant, and that the abnormally bright stars are relatively so few in number, within moderate distances of the Sun, that the percentage of them in the Milky Way soon decreases, but then again increases when we deal with the average stars, of which the great star-clouds are composed.

As was long ago found by W. STRUVE, if we take only the stars plainly visible to the naked eye, there is a moderate increase in density towards the Milky Way, but the density rapidly increases as we descend to lower magnitudes, so that the faintest telescopic stars show a great inequality of distribution, a very large percentage of them lying in the Milky Way, which thus presents to the naked eye the aspect of a luminous zone. In certain portions of this belt the star-clouds present the appearance of a blaze of light so intense as to modify sensibly the illumination of the sky at night. In speaking of the intensity of the light of the Milky Way in the vicinity of the Southern Cross, the English astronomer, CAPTAIN JACOB, of the Bombay Engineers, justly remarked that: "Such is the general blaze of starlight near the Cross from that part of the sky, that a person is immediately made aware of its having risen above the horizon, though he should not be at the time looking at the heavens, by the increase of general illumination of the atmosphere, resembling the effect of the young Moon" (cf. HUMBOLDT'S *Cosmos*, Vol. III., p. 198).

PROCTOR was inclined to contest both the assumptions of HERSCHEL as to average equality of distribution, and his conclusions regarding the extent of the universe, because of the hypothesis of equal intrinsic brightness. But while these assumptions, as HERSCHEL points out, are not entirely rigorous, they will yet lead to general truth; and anyone who has noticed the close approach to equality in the brightness of millions of stars in the star-clouds along the Milky Way, will find in it a powerful support for the general validity of HERSCHEL'S argument.

§ 284. *Decrease in the Light of Stars Owing to Distance Alone.*

A star of the first magnitude gives 2.512 times as much light as a star of the second magnitude; and as light decreases inversely as the square of the distance, the light of the fainter star would equal that of the brighter star at a distance 1.585 times its present distance. Consequently, if two stars differ by

n magnitudes, their light would be equalized by placing the brighter star at a distance $\Delta = \sqrt{(2.512)^n} = (1.585)^n$. The following table gives the value of Δ from $n = 1$, to $n = 20$, which is about the difference of magnitude between the brightest star and the faintest visible in our most powerful telescopes.

Difference in Magnitude n	Distance to which the Brighter Star would have to be Removed to Equalize the Starlight Δ	Difference in Magnitude n	Distance to which the Brighter Star would have to be Removed to Equalize the Starlight Δ
1	1.585	11	158.49
2	2.512	12	251.19
3	3.981	13	398.11
4	6.310	14	630.96
5	10.000	15	1000.00
6	15.849	16	1584.9
7	25.119	17	2511.9
8	39.811	18	3981.1
9	63.096	19	6309.6
10	100.000	20	10000.0

Accordingly it appears that if distance alone were the cause of faintness, we could explain a difference of 20 magnitudes by a 10,000-fold increase of distance; 15 magnitudes by a 1,000-fold increase of distance; 10 magnitudes by 100-fold increase of distance; 5 magnitudes by a 10-fold increase of distance. Thus, to take a concrete case for illustration, the companion of *Sirius* is about 10 magnitudes fainter than *Sirius*; and if the two stars were intrinsically of equal lustre, with the fainter star merely behind the brighter, the distance of the fainter would be 100 times that of the brighter. And if we take the distance of *α Centauri* to be 4.5 light-years, the remotest stars 20 magnitudes fainter than *Sirius* would be at a distance of 45,000 light-years. The fact, however, that the universe lies mainly in the plane of the Milky Way and is enormously extended in this direction, would lead one to suppose that, of the more distant stars, we see chiefly the brightest ones; and of the most distant stars, only the few that are very large and brilliant. From the effect of distance alone the fainter stars become invisible, and we should see only the largest stars in the remoter regions of the Galaxy. If they be 10,000 times as bright as our Sun, or *α Centauri*, the distance may be 100 times 45,000 light-years, or 4,500,000 light-years. This surpasses the dimensions of the universe calculated by HERSCHEL, which never exceeded about 2,000,000 light-years (*Phil. Trans.*, 1802, p. 498).

§ 285. HERSCHEL'S *Calculation of the Distances of the Stars from Their Brightness.*

We have already discussed HERSCHEL'S method of gauging the depths of the starry stratum by counting the stars visible in his telescope. It now becomes

advisable to explain his method of estimating distances, not from the number of stars seen in telescopes, but from their brightness. This method was applied by HERSCHEL for determining the distances of clusters and other masses of stars, in his celebrated papers entitled "Astronomical Observations and Experiments Tending to Investigate the Local Arrangement of the Celestial Bodies in Space, and to Determine the Extent and Condition of the Milky Way," published in the *Philosophical Transactions* for 1817.

To explain HERSCHEL'S method most easily, it is sufficient to remark that, if all the stars were of the same intrinsic brightness, so that the difference in lustre depended wholly on the distance from us, then the observed magnitudes, varying inversely as the square of the distance, would enable us to fix the distances in space. HERSCHEL remarks that the method cannot be safely applied to individual stars, owing to their differences in intrinsic brightness, but he held that, as applied to whole classes or groups of stars, it would give their relative distances with essential accuracy. For it was pointed out that, although a single star of the 4th magnitude may be nearer to us than another of the 3d magnitude, yet it is impossible to doubt that the average distance of the 4th magnitude star exceeds that of the 3d, and this difference becomes still more certain as the fainter stars are approached, because the number of them is very large. Accordingly, this difference of distance depending on difference in brightness becomes a means of estimating distances of masses of stars admitting of considerable accuracy. The necessity of this method HERSCHEL explains as follows:

"With regard to objects comparatively very near to us, astronomers have completely succeeded by the method of parallaxes. The distance of the Sun; the dimensions of the orbits of the planets and of their satellites; the diameters of the Sun, the Moon, and the rest of the bodies belonging to the solar system, as well as the distances of comets, have all been successfully ascertained. The parallax of the fixed stars has also been an object of attention; and although we have hitherto had no satisfactory result from the investigation, the attempt has at least so far succeeded as to give us a most magnificent idea of the vast expansion of the sidereal heavens, by showing that probably the whole diameter of the Earth's orbit, at the distance of a star of the first magnitude, does not subtend an angle of more than a single second of a degree, if indeed it should amount to so much; with regard to more remote objects, however, such as the stars of smaller size, highly compressed clusters of stars and nebulae, the parallactic method can give us no assistance" (*Phil. Trans.*, 1817, pp. 302-303).

§ 286. HERSCHEL'S *Method of Converting Order of Magnitudes Into Order of Distances.*

To find the distances of the stars from their brightness it is necessary to convert the order of magnitudes into order of distances, and the following is the method given by HERSCHEL who was then in the 80th year of his age. W. STRUVE has remarked (*Études d'Astronomie Stellaire*, p. 44) on the wonderfully youthful penetration of spirit and clearness of judgment which enabled HERSCHEL to enjoy at this great age the composition of such a sublime and profound speculation. HERSCHEL'S discussion is as follows (*Phil. Trans.*, 1817, pp. 307-310):

"Of a Standard by Which the Relative Arrangement of the Stars May Be Examined.

"It is evident that when we propose to examine how the stars of the heavens are arranged, we ought to have a certain standard reference; and this I believe may be had by comparing their distribution to a certain properly modified equality of scattering. Now, the equality I shall here propose does not require that the stars should be at equal distances from each other; nor is it necessary that all those of the same nominal magnitude should be equally distant from us. It consists in allotting a certain equal portion of space to every star, in consequence of which we may calculate how many stars any given extent of space should contain. This definition of equal scattering agrees so far with observation that it admits, for instance, *Sirius*, *Arcturus*, and *Aldebaran* to be put into the same class, notwithstanding their very different lustre will not allow us to suppose them to be at equal distances from us; but its chief advantage will be that instead of the order of magnitudes into which our catalogues have arranged the stars, it will give us an order of distances, which may be used for ascertaining the local distribution of the heavenly bodies in space.

"To explain this arrangement, let a circle be drawn with any given radius about the point S, Fig. 1, Plate XV, and with 3, 5, 7, 9, etc. times the same radius drawn circles, or circular arcs, about the same centre. Then if a portion of space equal to the solid contents of a sphere, represented by the circle S, be allotted to each star, the circles, or circular arcs drawn about it will denote spheres containing the stars of their own order, and of all the orders belonging to the included spheres, and on the supposition of an equality of scattering, the number of stars of any given order may be had by inspection of the figure, which contains all the numbers that are required for the purpose; for those in front of the diagram express the diameters of spherical figures. The first row of numbers enclosed between the successive arcs are the cubes of the diameters; the next column expresses the

order of the central distances; and the last gives the difference between the cube numbers of any order and the cube of the next enclosed order.

"The use to be made of these columns of numbers is by inspection to determine how many stars of any particular order there ought to be if the stars were equally scattered. For instance, let it be required how many stars there should be of the 4th order. Then No. 4, in the column of the orders, points out a sphere of nine times the diameter of the central one, and shows that it would contain 729 stars, but as this sphere includes all the stars of the 3d, 2d, and 1st order as well as the Sun, their number will be the sum of all the stars contained in the next inferior sphere amounting to 343; which being taken from 729 leaves 386 for the space allotted to those of the 4th order of distances.

"*Comparison of the Order of Magnitudes with the Order of Distances.*

"With a view to throw some light upon the question, in what manner the stars are scattered in space, we may now compare their magnitudes, as we find

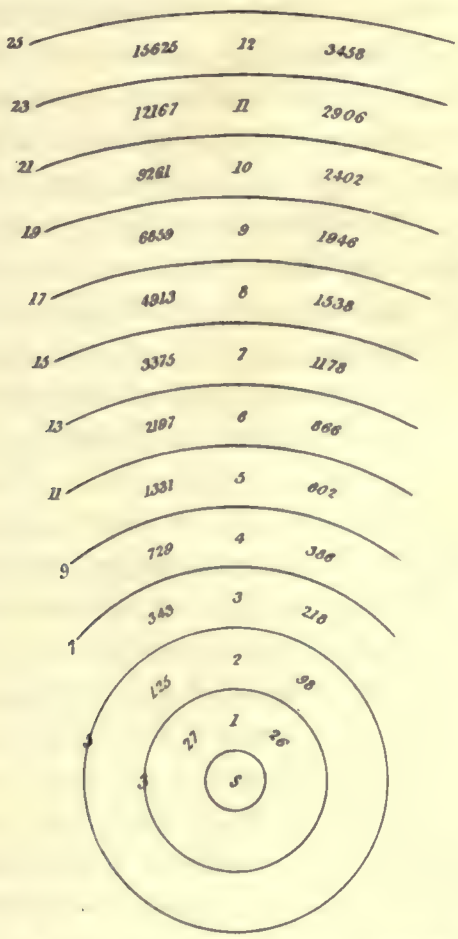


FIG. 43. HERSCHEL'S FIG. 1, PLATE XV, *Phil. Trans.*, 1817, SHOWING THE NUMBER OF STARS OF ANY PARTICULAR ORDER ON THE HYPOTHESIS OF EQUAL SCATTERING.

them assigned in MR. BODE's extensive catalogue of stars, with the order of their distances which has been explained.

"The catalogue I have mentioned contains 17 stars of the 1st magnitude; but in my figure of the order of the distances we find their number to be 26.

"The same catalogue has 57 stars of the 2d magnitude; but the order of distances admits 98.

"Of the third magnitude the catalogue has 206, and the order of distances will admit 218.

"The number of the stars of the 4th magnitude is by the catalogue 454, and by the order of distances 386.

"Before I proceed it may be proper to remark that, by these four classifications of the stars into magnitudes, it appears already that on account of the great difference in the lustre of the brightest stars many of them have been put back into the second class; and that the same visible excess of light has also occasioned many of the stars of the next degree of brightness to be put into the third class; but the principle of the visibility of the difference in brightness would have less influence with the gradually diminishing lustre of the stars, so that the number of those of the third magnitude would come nearly up to those of the third distance. And as the difference in the light of small stars is less visible than in the large ones, we find that the catalogue has admitted a greater number of stars of the 4th magnitude than the fourth order of distances points out; this may, however, be owing to taking in the stars that were thrown back from the preceding orders; and a remarkable coincidence of numbers seems to confirm this account of the arrangement of the stars into magnitudes. For the total number of the catalogued stars of the 1st, 2d, 3d, and 4th magnitudes, with the addition of the Sun, is 735; and the number contained in the whole sphere of the 4th distance is 729.

"Now the distinguishable difference of brightness becoming gradually less as the stars are smaller, the effect of the principle of classification will be, as indeed we find it in the 5th, 6th, and 7th classes, that fainter stars must be admitted into them than the order of distances points out.

"The catalogue contains 1,161 stars of the 5th magnitude, whereas the 5th order of distances has only room for 602.

"Of the 6th magnitude the catalogue contains not less than 6,103 stars, but the 6th order of distances will admit only 866.

"And lastly, the same catalogue points out 6,146 stars of the 7th magnitude, while the number of stars that can be taken into the 7th order of distances is only 1,178.

"The result of this comparison therefore is that if the order of magnitudes could indicate the distance of the stars, it would denote at first a gradual, and afterwards a very abrupt condensation of them; but that, considering the principle on which the stars are classed, their arrangement into magnitudes can only apply to certain relative distances, and show that taking the stars of each class one with another, those of the succeeding magnitudes are farther from us than the stars of the preceding order.

"Of a Criterion for Ascertaining the Profundity, or Local Situation of Celestial Objects in Space.

"It has been shown that the presumptive distances of the stars pointed out by their magnitudes can give us no information of their real situation in space. The statement, however, that one with another the faintest stars are at the greatest distance from us, seems to me so forcible that I believe it may serve for the foundation of an experimental investigation.

"It will be admitted that the light of a star is inversely as the square of its distance; if therefore we can find a method by which the degree of light of any given star may be ascertained, its distance will become a subject of calculation. But in order to draw valid consequences from experiments made upon the brightness of different stars, we shall be obliged to admit that one with another the stars are of a certain physical generic size and brightness, still allowing that all such deviations may exist, as generally take place among the individuals belonging to the same species.

"There may be some difference in the intrinsic brightness of starlight; that of highly colored stars may differ from the light of the bluish white ones; but in remarkable cases allowances may be made.

"With regard to size, or diameter, we are perhaps more liable to error; but the extensive catalogue which has already been consulted, contains not less than 14,144 stars of the seven magnitudes that have been adverted to; it may therefore be presumed that any star promiscuously chosen for an experiment, out of such a number, is not likely to differ much from a certain mean size of them all.

"At all events it will be certain that those stars of the light of which we can experimentally prove to be $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, $\frac{1}{36}$, and $\frac{1}{49}$ of the light of any certain star of the 1st magnitude, must be 2, 3, 4, 5, 6, and 7 times as far from us as the standard star, provided the condition of the stars should come up to the supposed mean state of diameter and lustre of the standard star, and of this, when many equalizations are made, there is at least a great probability in favor.

"*Of the Equalization of Starlight.*

"In my sweeps of the heavens, the idea of ascertaining the profundity of space to which our telescopes might reach, gave rise to an investigation of their space-penetrating power; and finding that this might be calculated with reference to the extent of the same power of which the unassisted eye is capable, there always remained a desideratum of some sure method by which this might be ascertained. . . . .

"The equalization of starlight, when carried to a proper degree of accuracy, will do away with the cause of the error to which the telescopic extent of vision has been unavoidably subject, we may therefore safely apply this vision to measure the Profundity of sidereal objects that are far beyond the reach of the natural eye; but for this purpose the powers of penetrating into space of the telescopes that are to be used must be reduced to what may be called gaging powers; and as the formula $\frac{\sqrt{x(A^2 - b^2)}}{a}$* gives the whole quantity of the space-penetrating power, a reduction to any inferior power p, may be made by the expression $\sqrt{\frac{p^2 a^2}{x} + b^2} = A$; when the aperture is then limited to the calculated value of A, the telescopes will have the required gaging power. Or, we may prepare a regular set of apertures to serve for trial, and find the gaging powers they give to the telescopes by the original formula" (pp. 318-319).

§ 287. HERSCHEL'S *Views on the Construction and Extent of the Milky Way.*

In the paper already cited in the *Philosophical Transactions* for 1817, HERSCHEL expresses himself regarding the Milky Way as follows:

"*Of the Construction and Extent of the Milky Way.*

"Of all the celestial objects consisting of stars not visible to the eye, the Milky Way is the most striking; its general appearance, without applying a telescope to it, is that of a zone surrounding our situation in the solar system, in the shape of a succession of differently condensed patches of brightness, intermixed with others of a fainter tinge.

"To enumerate a partial series of them, we have a very bright patch under the arrow of *Sagittarius;* another in the *Scutum Sobiescii;* between these two there are three unequally bright places; north preceding $\alpha\beta$ and γ *Aquilae* is a bright patch; between *Aquila* and the *Scutum* are two very faint places; a long faint

* "See *Phil. Trans.* for 1800, page 66."

place follows the shoulder of *Ophiucus*; near β *Cygni* is a bright place; near γ is another, and a third near α. A smaller brightish place follows in the succession of the Milky Way, and a large one towards *Cassiopea*. A faint place is on one side; a second towards *Cassiopea*, and a third is within that constellation; a very bright place is in the sword handle of *Perseus*; and α and γ *Cassiopea* inclose a dark spot.

"The breadth of the Milky Way appears to be very unequal. In a few places it does not exceed five degrees; but in several constellations it is extended from ten to sixteen. In its course it runs nearly 120 degrees in a divided clustering stream, of which the two branches between *Serpentarius* and *Antinous* are expanded over more than 22 degrees.

"That the Sun is within its plane may be seen by an observer in the latitude of about 60 degrees; for when at 100 degrees of right ascension the Milky Way is in the east, it will at the same time be in the west at 280; while in its meridianal situation it will pass through *Cassiopea* in the *Zenith*, and through the constellation of the Cross in the *Nadir*." . . . .

"Concluding Remarks.

"What has been said of the extent and condition of the Milky Way in several of my papers on the construction of the heavens, with the addition of the observations contained in this attempt to give a more correct idea of its profundity in space, will nearly contain all the general knowledge we can ever have of this magnificent collection of stars. To enter upon the subject of the contents of the heavens in the two comparatively vacant spaces on each side adjoining the Milky Way, the situation of globular clusters of planetary nebulae, and of far extended nebulosities, would greatly exceed the compass of this paper; I shall therefore only add one remarkable conclusion that may be drawn from the experiments which have been made with the gaging powers" (pp. 320-331).

§ 288. *Explorations of* Sir John Herschel, W. Struve *and* Boeddicker.

After the memorable explorations of the northern heavens by Sir Wm. Herschel, the necessity of a corresponding gauging of the southern hemisphere was fully appreciated by Sir John Herschel; and finally carried out at the Cape of Good Hope between 1834 and 1838, with a telescope similar to that used by his father in the gauging of the northern heavens. Sir John Herschel's studies at the Cape confirmed and completed the survey of the heavens begun by his father, and gave us a singularly symmetrical and complete view of the sidereal universe.

The original data of SIR WM. HERSCHEL were considerably improved by the discussion of W. STRUVE, embodied in his *Études d'Astronomie Stellaire*, Petersbourg, 1847. What SIR WM. HERSCHEL had left in the form of tables of gauges, STRUVE reduced to order by general mathematical formulae, admitting of exact calculation or graphical illustration.

Besides rediscussing the work of SIR WM. HERSCHEL and reducing it to approximate geometrical form, W. STRUVE extended the Herschelian theory to the data given in the catalogues of BESSEL and ARGELANDER. His conclusions essentially confirmed the views of SIR WM. HERSCHEL. The only important departure from HERSCHEL'S theory consisted in the conclusion that if the original principle of star-gauging was sound, the method of measuring distances from the order of brightness proposed by HERSCHEL in his 80th year, would give greater uniformity of distribution of the brighter stars over the heavens than they are observed to have. This difference, however, was not very pronounced, and in part the actual state of the heavens confirms both HERSCHEL and STRUVE in their inferences. STRUVE also concluded that the Sun occupies a position near the centre of a cluster of disc-like shape, which does not accurately follow a great circle in its course around the heavens, but presents some inflections or irregularities. From these studies W. STRUVE concluded that the Sun is a little north of the central plane of the cluster and that the distance from the centre corresponded to about the average distance of a star of the second magnitude.

Among the later workers in the study of the Milky Way, DR. BOEDDICKER, of LORD ROSS'S Observatory, deserves especial mention for having given us a faithful representation of the naked-eye aspects of the Galaxy. He reached the conclusion that a good naked-eye map was the first prerequisite to the study of the Milky Way, and devoted to it the best efforts of five years, 1884-1889. We have made much use of DR. BOEDDICKER'S maps, and cannot commend them too highly to the student of the Milky Way.

§ 289. DR. GOULD'S *Theory of the Solar Cluster and Milky Way.*

In the course of his study of the stars of the southern hemisphere, DR. GOULD reached the conclusion that the Milky Way as we see it probably is the result of the superposition of two or more Galaxies, or immense streams of stars (cf. *Uranometria Argentina*, p. 381). This question is one to which SIR JOHN HERSCHEL had given some attention during his memorable survey of the southern hemisphere made at the Cape of Good Hope from 1834 to 1838. It was then remarked by SIR JOHN that the Milky Way is crossed by a zone of large stars which traverses

the brilliant constellations of *Orion, Canis Major, Argo, Crux, Centaurus, Lupus* and *Scorpius*. "A great circle passing through ϵ *Orionis* and α *Crucis* will mark out the axis of the zone in question, whose inclination to the Galactic circle is therefore about 20°, and whose appearance would lead us to suspect that our nearest neighbors in the sidereal system (if really such) form part of a subordinate sheet or stratum deviating to that extent from parallelism to the general mass, which, seen projected on the heavens, forms the Milky Way" (*Results at the Cape*, p. 385).

This idea of a cluster about the Sun, which dates back to SIR WM. HERSCHEL, was further developed by DR. GOULD, in 1874 (*Proc. of Am. Assoc.*, 1874, p. 113, and in the *Uranometria Argentina*, pp. 348-370). GOULD bases his argument principally on the excessive number of stars brighter than 4th magnitude, above the number naturally to be expected from the numbers of 5th, 6th, 7th magnitude stars, on the assumption that the stars of all magnitudes are uniformly distributed in space. If only the stars brighter than 4th magnitude are considered, he concluded that there is evidence of the bifurcation of the belt of bright stars pointed out by SIR JOHN HERSCHEL.

Undoubtedly this belt of bright stars culminating in *Argo* and *Centaurus* is the most remarkable phenomenon presented to our contemplation by the sidereal heavens. The bright stars are so conspicuous in the constellations *Argo, Centaurus, Lupus*, etc., that when examining this part of the heavens from the Lowell Observatory in the City of Mexico, 1896-7, the author of this work used frequently to remark that the sky presented *a spotted appearance*, so densely were the large stars spangled over the region just north of the southernmost portion of the Milky Way.

As the result of his studies of this belt of bright stars, DR. GOULD reached the following conclusions:

1. "There is in the sky a girdle of bright stars, the medial line of which differs but little from a circle, inclined to the Galactic circle by a little less than 20°."

2. "The grouping of the fixed stars brighter than 4.1 magnitude is more symmetric, relatively to that medial line, than to the Galactic circle; and the abundance of bright stars in any region of the sky is greater as its distance therefrom is less." In other words, the circle is the one which practically makes the sum of the squares of the distances a minimum, so that it is the most probable medial line to which they can be referred.

3. "The known tendency to aggregation of faint stars towards the Milky Way is according to a ratio which increases rapidly as their magnitudes decrease, the law of which is such that the corresponding aggregation would be scarcely if at all perceptible for the bright stars."

4. "These facts, together with others, indicate the existence of a small cluster within which our system is eccentrically situated, but which is itself not far from the middle plane of the Galaxy. This cluster appears to be of a flattened shape, somewhat bifid, and to consist of somewhat more than 400 stars of magnitudes from the 1st to the 7th, their average magnitude being about 3.6 or 3.7."

The rest of DR. GOULD's discussion is of less interest and need not be given here. The considerations here adduced show that our Sun is really a star in a cluster of large extent; and yet it may be that some of our brighter stars are not permanent members of the solar cluster, but only passing through our portion of the Galaxy on a longer journey.

§ 290. *Photographic Exploration of the Milky Way by* BARNARD, RUSSELL, *and* MAX WOLF.

Of all the photographic explorers of the Milky Way, BARNARD is easily first. In fact, his photographic study of the Galaxy marks a distinct epoch in the subject scarcely less important than that of HERSCHEL a century ago. The photographic method is the method of the future, and it is impossible to estimate the value of the light which may thus be thrown upon the structure of the heavens. BARNARD began his great photographic survey of the Milky Way with a simple portrait lens at Lick Observatory, July 28, August 1 and 2, 1889. During the past twenty years he has extended the work over the entire Milky Way of the Northern Hemisphere, and as far south as —30°. The plates of the Milky Way given in this work are by BARNARD, and tell their own story more eloquently than any language could possibly do.

Let the reader study these pictures, and marvel at the wonders of the heavens! He will then understand why the Milky Way had such a charm for the HERSCHELS, while at the same time he will be surprised that the most beautiful part of the visible creation should be so little known to many contemporary men of science.

Soon after the beginning of his brilliant work, made possible by using a wide field and a strong light-grasp resulting from a six-inch lens and an exposure of several hours, PROFESSOR BARNARD wrote: "In the photographs made with the six-inch portrait-lens, besides myriads of stars, there are shown, for the first time, the vast and wonderful cloud forms, with all their remarkable structure of lanes, holes, and black gaps, and sprays of stars. They present to us these forms in all their delicacy and beauty, as no eye or telescope can ever hope to see them." (*Publications, Astron. Soc. of Pacific*, Vol. II, p. 242).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

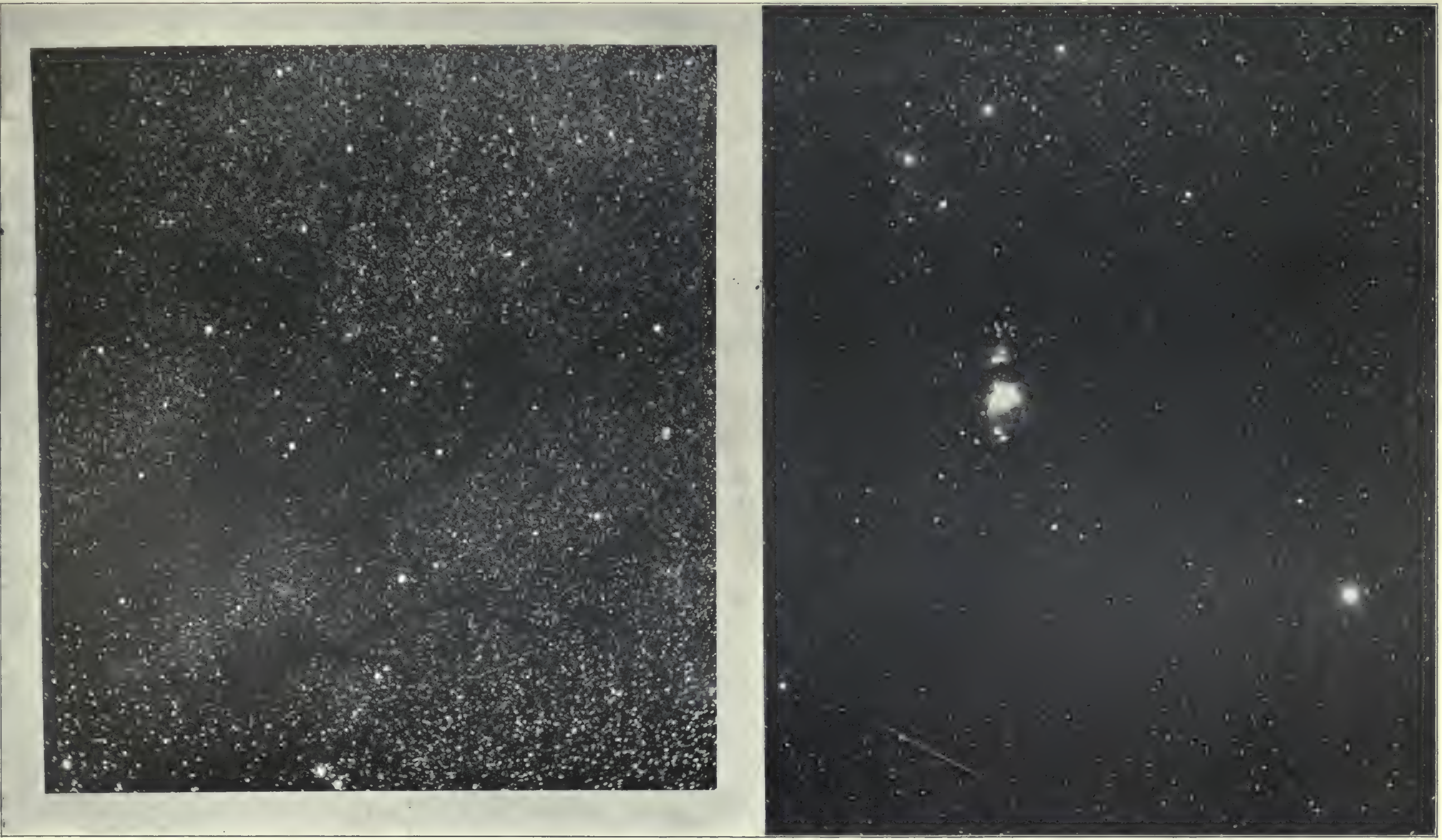


Plate α. Region of Taurus, Showing Dark Lanes with Nebulous Ground Work (Left), and the Orion Nebula (Right). Photographed by Barnard.

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

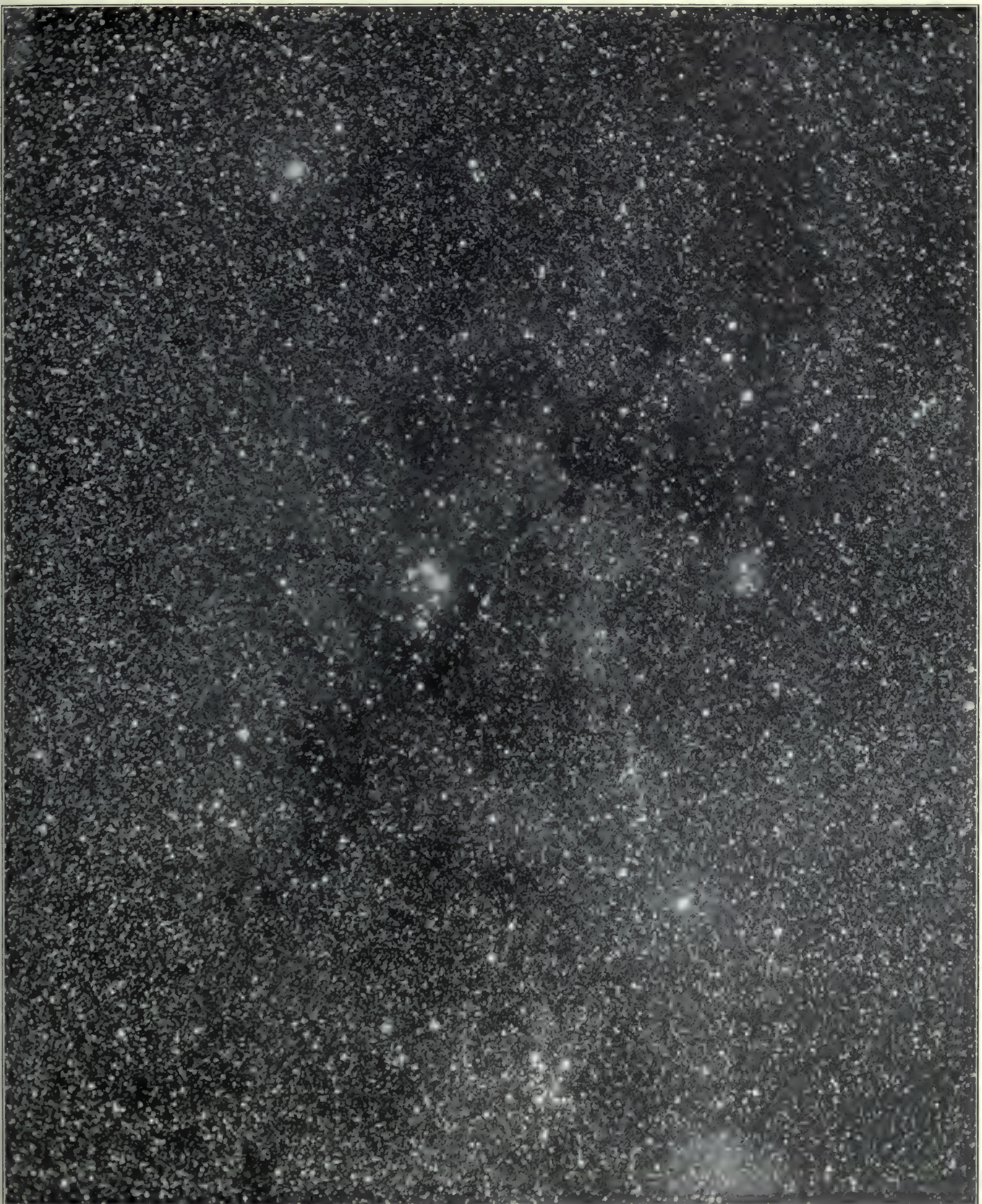


Plate β. Region of 15 Monocerotis. Photographed by Barnard, 1894, February 1 (Exposure $3^h\ 0^m$). This Plate Reveals the Wide Diffusion of Nebulosity over the Background of Certain Portions of the Heavens.

STAR CLOUDS DUE TO THE BREAKING UP OF THE MILKY WAY UNDER THE CLUSTERING POWER OF UNIVERSAL GRAVITATION, AS FIRST REMARKED BY HERSCHEL, AND ILLUSTRATING THE CAPTURE THEORY ON THE MOST STUPENDOUS SCALE.

PLATE γ. REGION OF RHO OPHIUCHI, SHOWING GREAT NEBULA AND VACANT REGIONS. PHOTOGRAPHED BY BARNARD, 1895, JANUARY 21–22 (EXPOSURE 7^h 30^m). BARNARD CONCLUDES THAT THE PHENOMENA HERE DISCLOSED GIVE EVIDENCE OF THE ABSORPTION OR EXTINCTION OF LIGHT IN TRAVERSING THE MASSES OF NEBULOSITY PERVADING CERTAIN PORTIONS OF SPACE.

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

Plate δ. Region of 58 Ophiuchi, Showing the Clustering Power at Work Over a Large Area of the Milky Way. Photographed by Barnard, 1895, June 26 (Exposure $4^h\ 5^m$).

STAR CLOUDS DUE TO THE BREAKING UP OF THE MILKY WAY UNDER THE CLUSTERING POWER OF UNIVERSAL GRAVITATION, AS FIRST REMARKED BY HERSCHEL, AND ILLUSTRATING THE CAPTURE THEORY ON THE MOST STUPENDOUS SCALE.

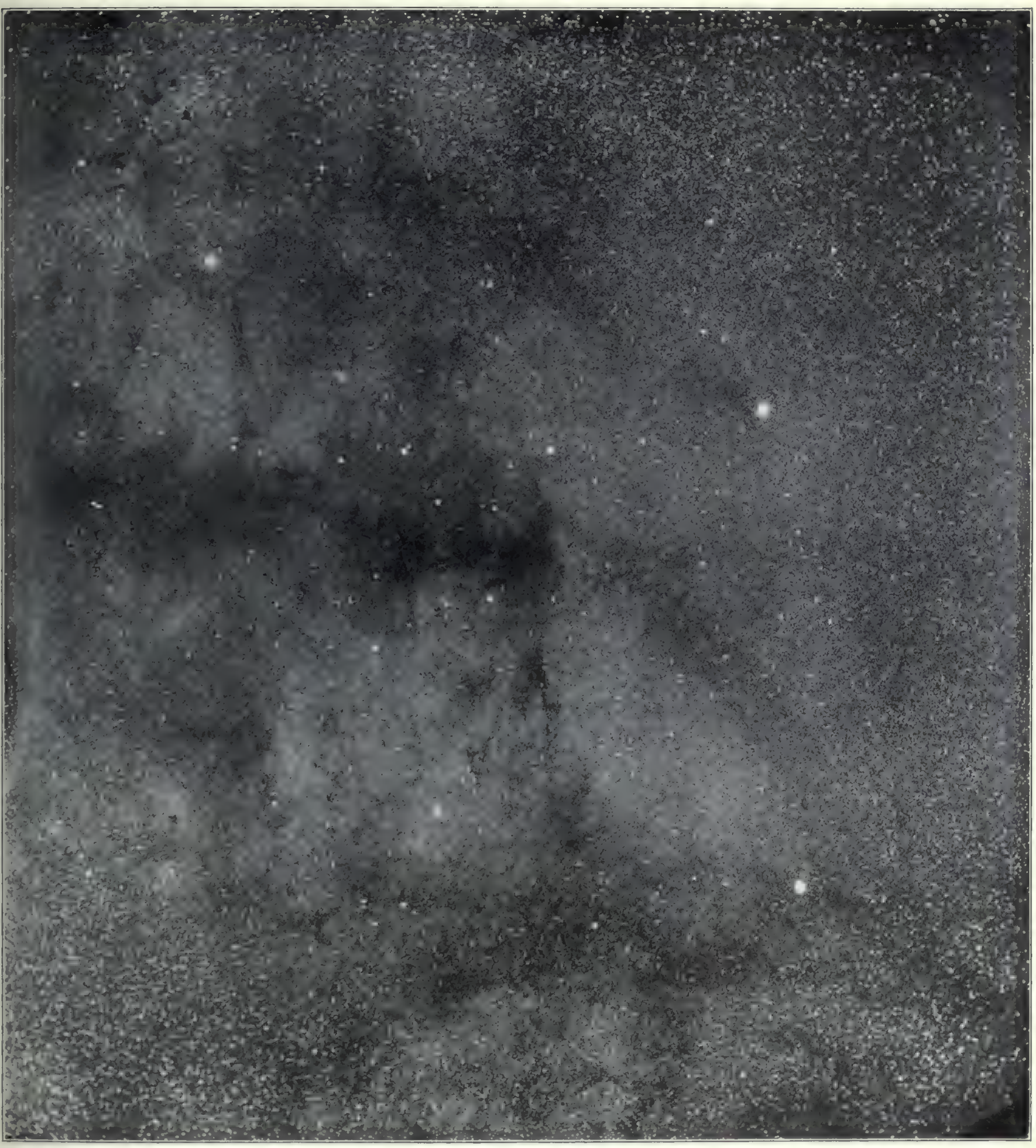


PLATE 6. REGION OF THETA OPHIUCHI, SHOWING VACANT REGIONS IN THE MILKY WAY. PHOTOGRAPHED BY BARNARD, 1894, JULY 6 (EXPOSURE, 3^h 36^m).

STAR CLOUDS DUE TO THE BREAKING UP OF THE MILKY WAY UNDER THE CLUSTERING POWER OF UNIVERSAL GRAVITATION, AS FIRST REMARKED BY HERSCHEL, AND ILLUSTRATING THE CAPTURE THEORY ON THE MOST STUPENDOUS SCALE.

PLATE ζ. VACANT REGION EAST OF THETA OPHIUCHI, SHOWING NEBULOSITY AND THE CLUSTERING TENDENCY. PHOTOGRAPHED BY BARNARD, 1905, JUNE 30 (EXPOSURE 3^h 45^m).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

Plate η. Region of Messier 8 and the Trifid Nebula in Sagittarius. Photographed by Barnard, 1894, July 5 (Exposure, $4^h\ 0^m$). In This Region the Background of the Sky is Largely Covered with Faint Nebulosity.

STAR CLOUDS DUE TO THE BREAKING UP OF THE MILKY WAY UNDER THE CLUSTERING POWER OF UNIVERSAL GRAVITATION, AS FIRST REMARKED BY HERSCHEL, AND ILLUSTRATING THE CAPTURE THEORY ON THE MOST STUPENDOUS SCALE.

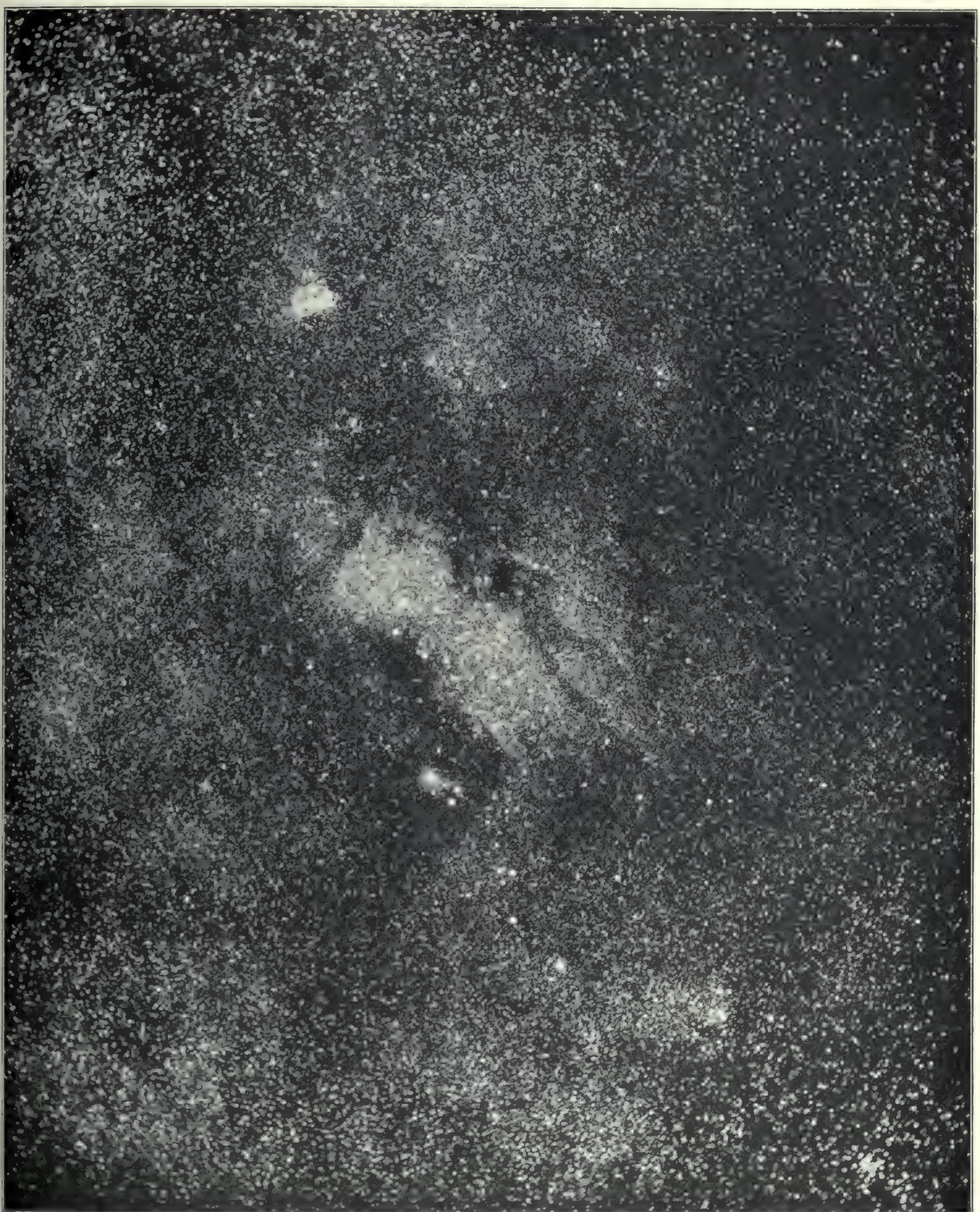
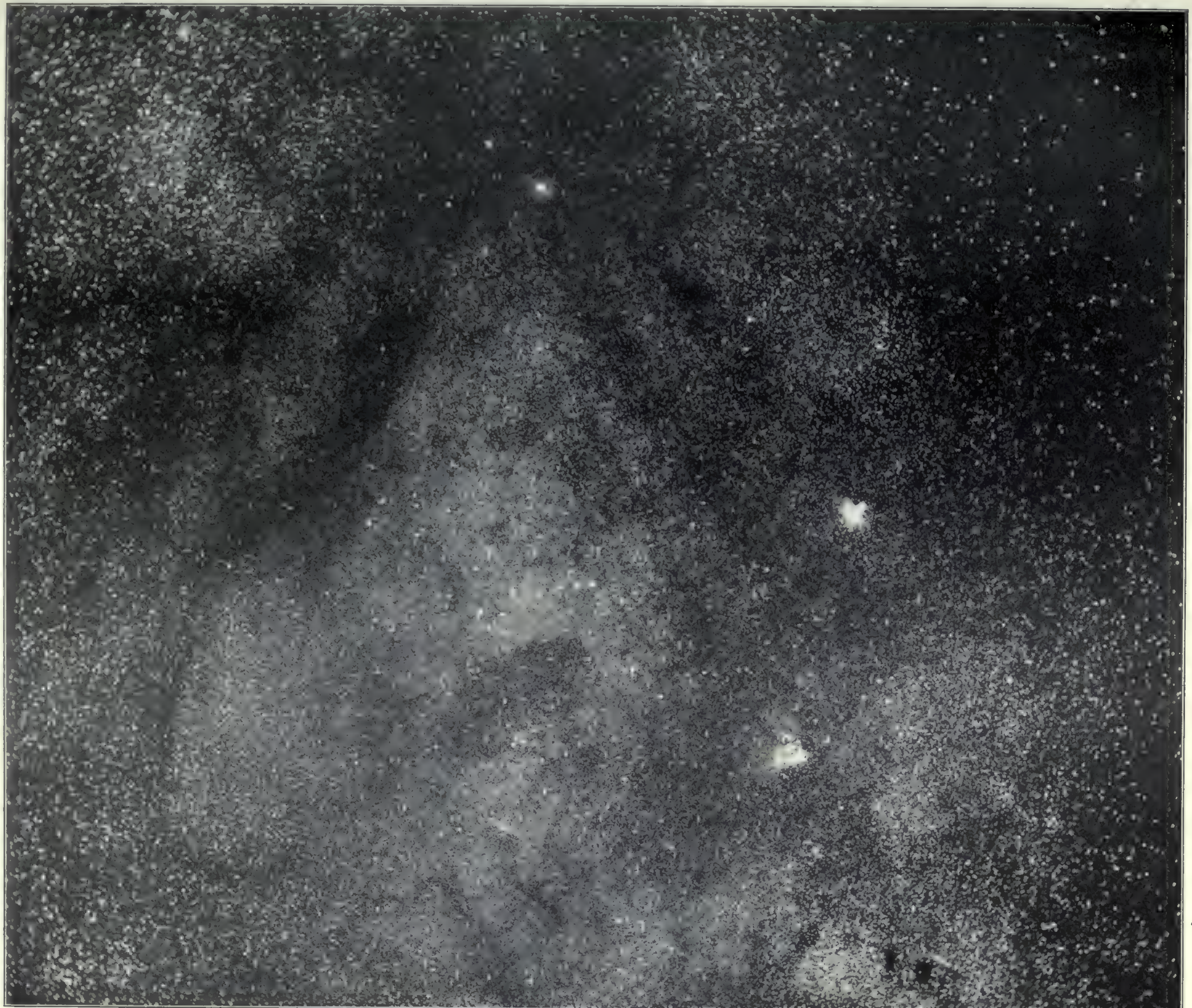


PLATE θ. REGION OF SMALL STAR CLOUD IN SAGITTARIUS. PHOTOGRAPHED BY BARNARD, 1892, JUNE 20 (EXPOSURE 4^h 15^m). THE RIFTS AND VACANCIES IN THE UPPER PART OF THIS STAR CLOUD ARE VERY REMARKABLE.

Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.



Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

Plate κ. Region of Great Star Cloud in Sagittarius, Showing Masses and Streams of Stars in the Process of Development. Photographed by Barnard, 1895, August 13 (Exposure, $3^h\ 5^m$).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

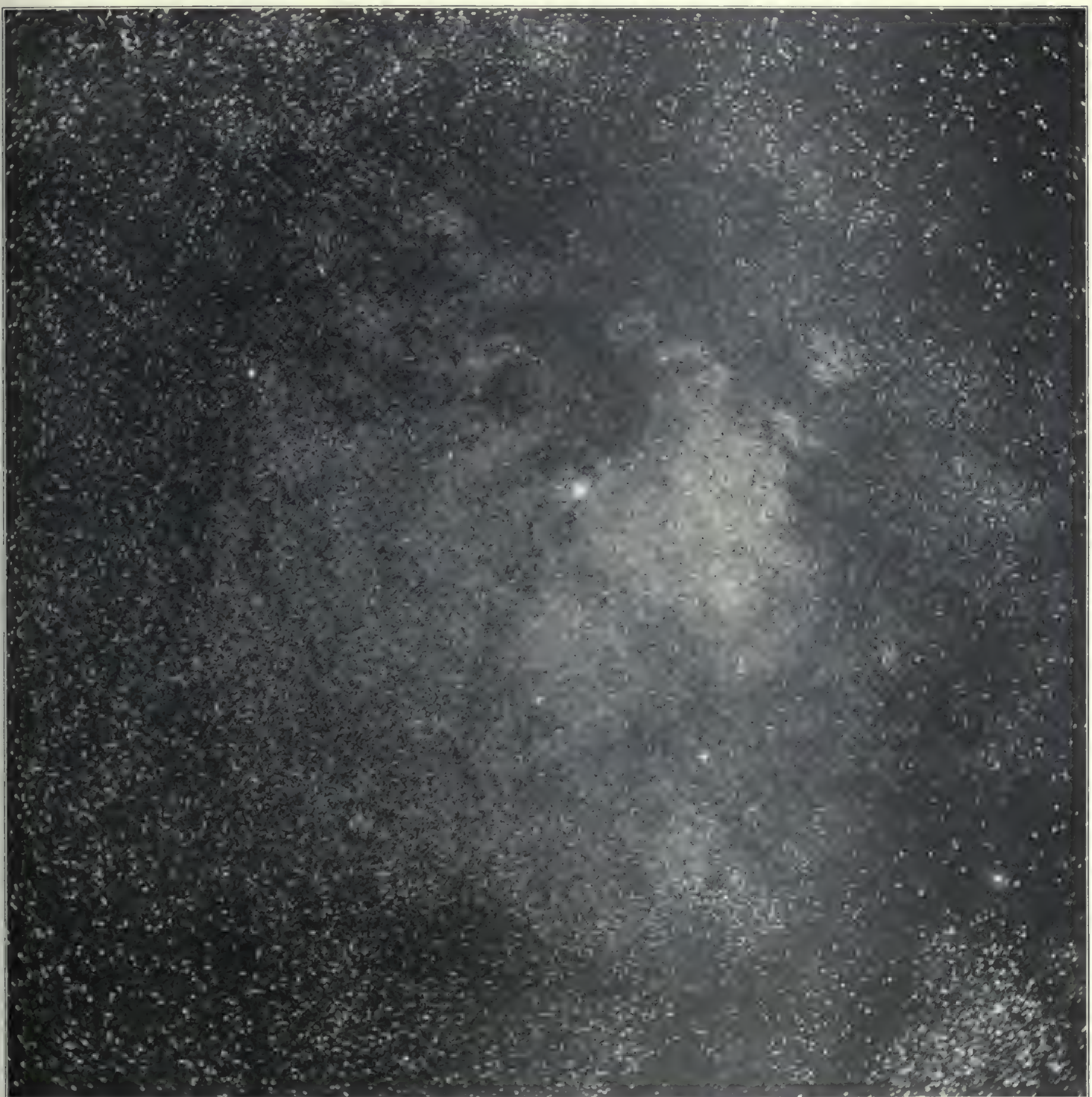


Plate λ. Region of Messier 11, with the Star Clouds in Antinous. Photographed by Barnard, 1892, June 29 (Exposure 3^h 25^m).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

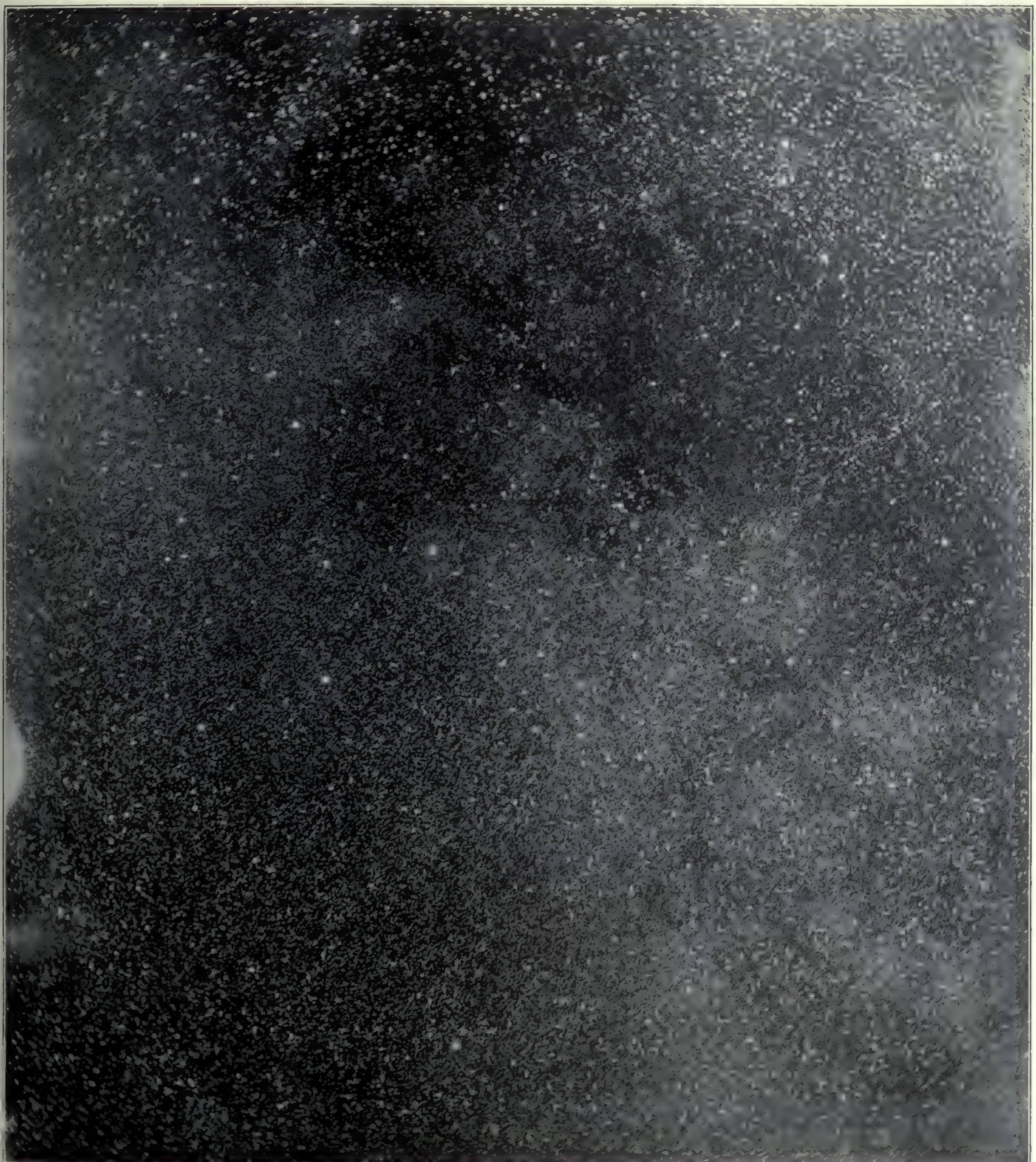


Plate μ. Region of Beta Cygni, in a Rich Portion of the Milky Way. Photographed by Barnard, 1893, October 12 (Exposure 5^h 17^m).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

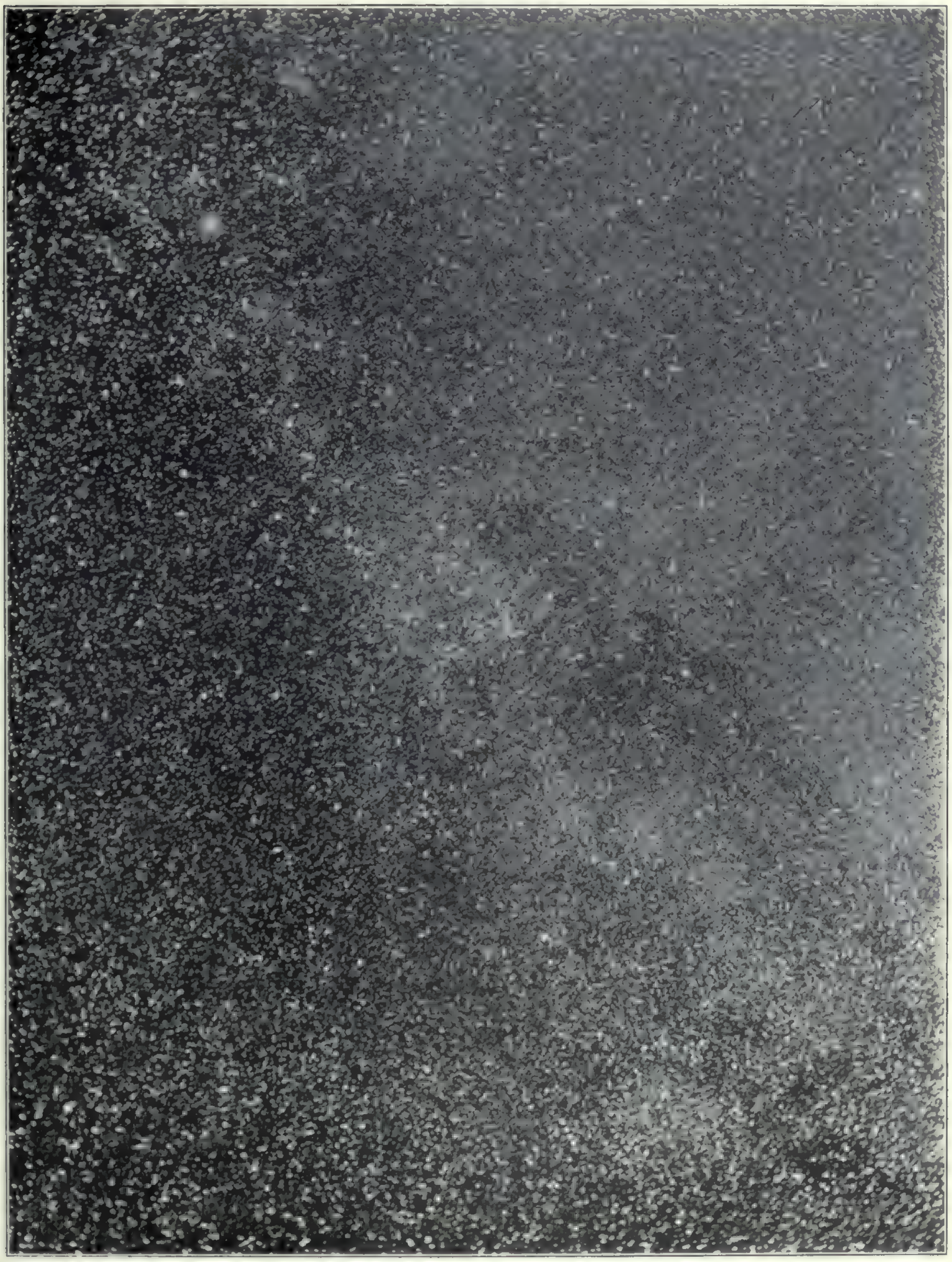


Plate v. Region of Chi Cygni, Showing One of the Densest Clouds of Stars Known in Any Part of the Heavens. Photographed by Barnard, 1892, October 20 (Exposure, $5^h\ 0^m$).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.

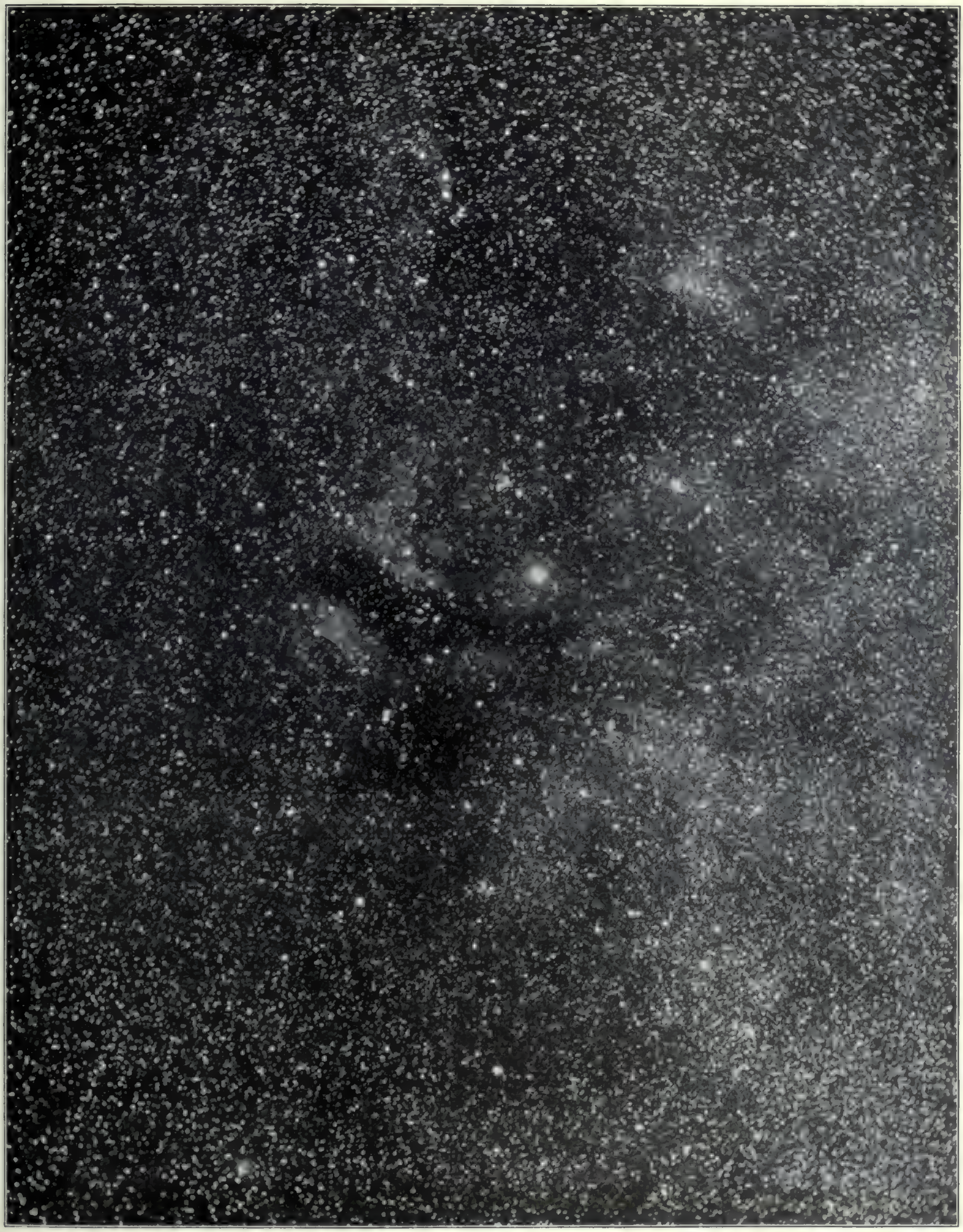
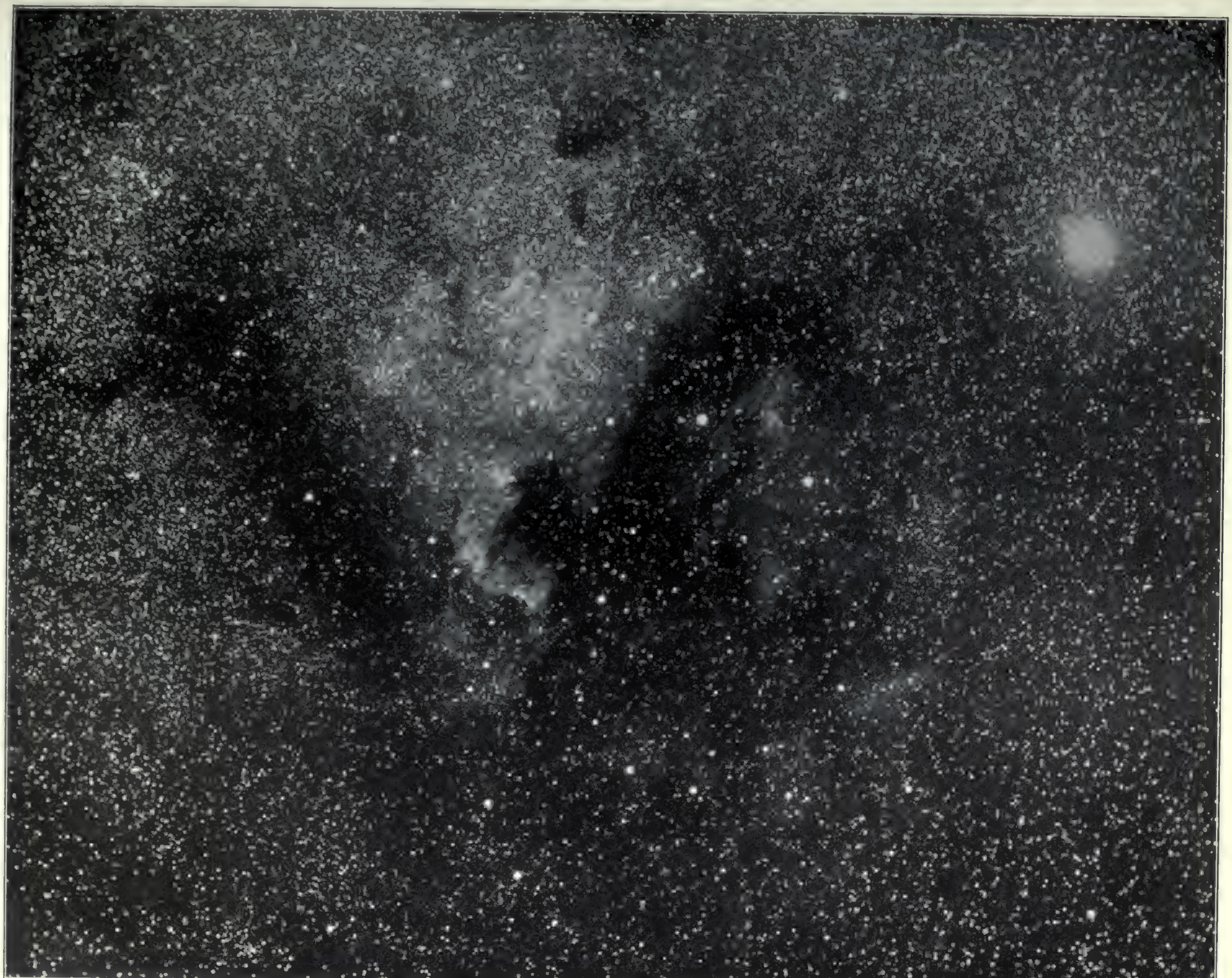


Plate ξ. Region of Gamma Cygni, Showing Nebulosity and Clustering Masses of Stars. Photographed by Barnard, 1905, August 28 (Exposure, 6^{h} 30^{m}).

Star Clouds Due to the Breaking Up of the Milky Way Under the Clustering Power of Universal Gravitation, as First Remarked by Herschel, and Illustrating the Capture Theory on the Most Stupendous Scale.



STAR CLOUDS DUE TO THE BREAKING UP OF THE MILKY WAY UNDER THE CLUSTERING POWER OF UNIVERSAL GRAVITATION, AS FIRST REMARKED BY HERSCHEL, AND ILLUSTRATING THE CAPTURE THEORY ON THE MOST STUPENDOUS SCALE.

PLATE π. REGION OF THE CONSTELLATION CEPHEUS, SHOWING NEBULOSITY AND DARK RIFTS IN THE MILKY WAY. PHOTOGRAPHED BY BARNARD, 1893, OCTOBER 13 (EXPOSURE, $7^h\ 5^m$).

BARNARD's complete photographic *Atlas of the Milky Way* is soon to be published by the Carnegie Institution of Washington. It is not too much to say that it will be one of the most splendid works of any age, and constitute a veritable *momentum aere perennius;* for it will place before every one the deepest and most penetrating view of the universe yet revealed to mortal sight, and in a form that will be almost ultimate. The structure of the Milky Way will no doubt change slightly, with the lapse of ages, but there will be no sensible change in the aspects of these gigantic structures for many thousands of years. Thus BARNARD's contribution to this branch of astronomical science will constitute a precious legacy to future generations which will be gratefully remembered in the remotest ages.

In 1890, H. C. RUSSELL, of Sydney, imitated BARNARD's example by applying photography to the southernmost portion of the Milky Way. His pictures showed the Coal Sack to be black only in the northern portion (cf. *Knowledge*, Vol. XIX, p. 301); while the other regions near the Southern Cross that show great rifts to the naked eye, were found by the more sensitive eye of the camera to be densely spangled with stars. RUSSELL's photographs of the Nubecula Major, resulting from an exposure of four and one-half hours, showed the structure to be a "complex spiral with two centres." A similar cluster was disclosed in the lesser Nubecula by an exposure of eight hours. One result of the explorations of BARNARD and RUSSELL was the realization that the structure of the heavens is everywhere much more complex than we had previously supposed.

While the spiral nebulae remain as before principally in the poles of the Galaxy, there are shown to be many irregular nebulae of diffuse character connected with the stars of the Milky Way. And the structure of the Galaxy itself is shown to be tenfold more varied than any one had previously imagined.

One of the earliest students of the Milky Way, by means of photographic methods, was DR. MAX WOLF, of Heidelberg, who holds a high and almost unique place in the history of astronomical photography. Along with the pioneer labors of BARNARD, of whose work we have treated above, DR. WOLF brought out the distinguishing features of nebulosity and other complexities in the structure of the Milky Way in *Cygnus*, *Perseus*, and other regions of the sky. What was thus recorded on photographic plates after hours of exposure, became permanently visible to every one; whereas in the early days of SIR WM. HERSCHEL the structure could be known only to the observer; and no language or description was adequate to convey to the mind a correct impression of the wonders of the heavens. The advantages thus gained by photography are great beyond calculation, and we owe much of the initiative in the photography of diffuse nebulae to DR. MAX WOLF, whose labors in so many lines are very justly celebrated.

CHAPTER XXIII.

Spiral Theory of the Milky Way, and of the Two Star Streams; Theory of Temporary and of Variable Stars, and of the Extinction of Light in Space.*

§ 291. Easton's *Spiral Theory of the Milky Way.*

Among the several investigators who have attempted to explain the phenomena of the Milky Way, Easton has best succeeded in outlining a satisfactory working theory, and he adopts the spiral hypothesis (*Astrophysical Journal,* Vol. XII, No. 2, September, 1900). We shall now give a brief account of Easton's theory, and of the observational foundation on which it rests. In his earlier work of 1895 (*Astrophysical Journal,* March, 1895) Easton concluded that the Milky Way might be of roughly annular structure, but he pointed out that the parts of the ring, even if closed, might be at very different distances from our Sun. Subsequently Seeliger studied by exact methods the problem of the distribution of the stars in space (cf. *Betrachtungen über räumliche Verteilung der Fixsterne,* 1898), and reached the conclusion that the stellar accumulations of the Milky Way are probably at different distances, which accords in fact with the result already reached by the elder Herschel. On the basis of this modern confirmation of Herschel's conclusions, Easton proceeds to examine the annular theory by subjecting it to successive tests, under the several conceivable hypotheses, and thus finds that it is inadmissible and must be wholly given up.

Easton lays much stress on the great superiority in brightness of the Milky Way near *Aquila* as compared to that near *Monoceros;* and says that this indicates that the stars are more numerous near the XVIIIth hour than near the VIth hour of right-ascension, a fact which has long been recognized by observers. He then adds: "The unequal distribution of the stars of the Milky Way, not only in detail, but also for the two halves of the zone as compared with each other, when

* It had been planned to devote this chapter to the "History of the Theories of Cosmogony," but on more careful consideration it seemed better to reserve this subject for subsequent treatment, while the foregoing discussion is here extended to include an outline of several subjects without which this volume would be very incomplete.

it is represented as divided along a line through *Crux* and *Cassiopeia*, is still more striking in the result of stellar gauges and enumerations. The mean result of WILLIAM HERSCHEL'S gauges in the region of *Aquila* is 161.5 stars; in that of *Monoceros*, 82.5 stars. Similarly, CELORIA, systematically counting the stars to about the eleventh magnitude in an equatorial band six degrees wide, has found 58,883 stars in the half of this band which is traversed by the Milky Way near XVIII^{h} and only 43,822 in the opposite half" (cf. STRUVE'S *Études d'Astronomie Stellaire*, 1847, Note 75; CELORIA "*Sopra alcuni Scandagli del Cielo*," *publ. del R. Osserv. di Brera*, Tome 13, p. 18).

ENCKE'S criticism of STRUVE'S theory that the Sun occupies an eccentric position (*A.N.*, Vol. 26, No. 622) made it clear that the stellar density in different directions should vary continuously from the maximum to the minimum. EASTON points out that local condensations of structure and brightness stand in the way of applying this principle; but as an approximation he considers the *Uranometrie Générale* of HOUZEAU, in which thirty-three bright spots and regions of the Milky Way are enumerated. Dividing the Milky Way into halves by a line through *Crux* and *Cassiopeia*, he finds that the half containing *Monoceros* includes four or five fairly bright spots, and not a single bright spot; while the half containing *Aquila* includes seven or eight fairly bright spots and seven bright spots—showing the same remarkable preponderance of light on the *Aquila* side as was indicated by HERSCHEL'S gauges. EASTON continues as follows:

"It follows that these apparent accumulations are comparatively most numerous in the region of *Aquila*, between $-45°$ and $+45°$, and that they are least numerous in the opposite zone near *Monoceros*. From this point of view these two zones, each embracing a quarter of the circumference, are in the ratio of 5.5 to 1, while for the corresponding *halves* of the Milky Way, the ratio is 2.8 to 1. On my chart of the Milky Way it may be seen that the general brightness diminishes pretty gradually from *Cygnus* to *Cassiopeia*; the same thing occurs between *Ara* and *Navis* in the Southern Hemisphere. But the gradation is very incomplete: between α *Persei* and α *Aurigae*, for example, the brightness of the Milky Way is much less marked than between α and θ *Aurigae*."

"GOULD remarks (*Uranometria Argentina*, p. 370) in speaking of the Milky Way in the Southern Hemisphere: 'Its brightest portion is unquestionably in *Sagittarius*, that in *Carina* being slightly inferior to this as regards intrinsic brilliancy, although far more magnificent and impressive on account of the great number of bright stars with which it is spangled.'"

After examining the various data bearing on the question, EASTON rightly concludes "that *for the fainter stars, taken as a whole, the Milky Way is widest in*

its brightest part, and at least for HERSCHEL's gauges, this certainly cannot be explained by local causes." Finally EASTON gives the accompanying figure to explain his spiral theory of the Milky Way, and concludes his discussion thus:

"It would be easy to push the comparison further and to find in it a plausible explanation of many features of the Galaxy. But I confine myself here to pointing out how easily this theory explains the luminous streams between the two branches of the Milky Way, in *Sagittarius* and *Cassiopeia;* the anomalous brightness of the secondary branch near *Cygnus;* the dark spaces surrounded by luminous streams between α *Cygni* and β *Cassiopeiae*, etc.; the 'lateral offsets' of the Milky Way; the connection of the clusters and the bright stars in *Taurus* and *Orion* with the nebulosities related to the Milky Way; the very faint region in

FIG. 6.

FIG. 44. EASTON'S SUGGESTED SPIRAL THEORY OF THE MILKY WAY.

Perseus, etc. — while retaining the advantages offered by the annular segments. I wish to insist upon the fact that Fig. 6 *does not pretend to give an even approximate representation of the Milky Way*, seen from a point in space situated on its axis. It only indicates in a general way how the stellar accumulations of the Milky Way might be distributed so as to produce the Galactic phenomenon, in its general structure and its principal details, as we observe it."

§ 292. *Theory of the Two Star Streams.*

It was first pointed out by Sir William Herschel that the solar system as a whole is moving towards the constellation *Hercules*, and that there results therefrom a systematic tendency in the apparent motions of the stars. Those in front of our Sun should gradually appear to separate, owing to our approach, and those behind should apparently draw together, owing to our recession to greater and greater distance. This was clearly foreseen by Herschel, and the principle thus outlined enabled him to locate the apex of the Sun's way with considerable precision. Herschel's conclusions were for a time disputed by Bessel, but they were later confirmed by Argelander, and their correctness has since been generally recognized by astronomers. But various investigators have obtained different values for the apex of the Sun's way, according to the stars chosen and the method employed. Here are a few of the results (cf. Kobold's *Bau des Fixstern systems*, p. 101).

	A_0	D_0	Number of Stars	Mean Proper Motion ″
Airy	261.5	24.7	113	. .
Dunkin	263.7	25.0	1167	. .
L. Struve	273.3	27.3	2509	. .
Stumpe	287.4	42.0	551	0.23
	279.7	40.5	340	0.43
	287.9	32.1	105	0.85
	285.2	30.4	58	2.39
Porter	281.9	53.7	576	0.23
	280.7	40.1	533	0.45
	285.2	34.0	143	0.90
	277.0	34.9	70	>1.20

These results depend on Argelander's method, while by Bessel's method the corresponding place of the solar apex would be in south declination near $-5°$. In 1895 Dr. Kobold directed attention to these discrepancies depending on method, the underlying hypothesis being that the "peculiar" motions are haphazard, or devoid of preference for any particular direction. Of course this hypothesis could be admitted only as a first approximation to the order of nature.

After a prolonged study of the proper motions of 2,400 stars included in the AUWERS-BRADLEY Catalogue, extending from the North Pole to 30° South, PROFESSOR J. C. KAPTEYN announced to the British Association in 1905 that there were two principal streams of stars (cf. *Report* for 1905, p. 257). He divided the area of the sky into twenty-eight regions, and found the directions of the apparent motions in each region; and when these directions were plotted on a sphere they were seen to converge to two points, showing that the apparent linear motions are parallel, as in the case of a shower of meteors which converge to a radiant point in the sky. PROFESSOR F. W. DYSON, Director of the Royal Observatory of Edinburgh, has discussed these problems in the *Proc. Roy. Soc.* of Edinburgh (1908, p. 231, and 1909, p. 376) and in *Nature*, of Nov. 4, 1909; and we have availed ourselves of PROFESSOR DYSON'S analysis in making up the present account of the two streams of stars.

It is concluded, therefore, that, relatively to the Sun, the stars are moving in two streams, which are inclined at a considerable angle to one another. When the solar motion is subtracted, so as to give the absolute motion in space, it is found that the two streams are moving in diametrically opposite directions, relatively to the centre of gravity of all the stars. KAPTEYN concluded that the motion was in the plane of the Milky Way, and towards the star *ξ Orionis*, $\alpha = 91°$ $\delta = +13°$; and towards the opposite point of the celestial sphere in *Sagittarius*. The apparent motions of the stars were thus resolved into: (1) a haphazard motion, without preference for direction; (2) the reversed solar motion; (3) the streaming movement towards *ξ Orionis* and the opposite point.

These investigations have been carried on by KAPTEYN, EDDINGTON, DYSON, SCHWARTZSCHILD, and others, with slightly varied results, but all agreeing in the existence of the two streams of stars mentioned. The convergent point varies a little according to the material used. SCHWARTZSCHILD has made a considerable improvement in the method of analysis by adopting a spheroidal instead of a spherical distribution of the velocities of the stars. In view of HERSCHEL'S proof that the Galaxy is of small thickness compared to its breadth, this is equivalent to the assumption that the more dominant velocities are in the plane of the Galaxy. The existence of two streams of stars moving in opposite directions with spheroidal distribution of velocity, is a necessary result of the construction of the heavens.

If we define the "apex" as the direction of the Sun's motion relative to the centre of gravity of the stars, and the "vertex" as the direction of motion of one stream relatively to the other (KAPTEYN), or the major axis of SCHWARTZSCHILD'S spheroid, the different investigators give the following results:

	Apex α	Apex δ	Vertex α	Vertex δ
	°	°	°	°
KAPTEYN–BRADLEY Stars	..	..	91	+13
EDDINGTON–GROOMBRIDGE Stars	266	+31	95	+3
SCHWARTZSCHILD–GROOMBRIDGE Stars	266	+33	93	+6
DYSON-Stars of large proper motion	281	+42	88	+24
BELJAWSKY–PORTER'S Stars	281	+36	86	+24
EDDINGTON–Zodiacal Stars	..	..	109	+6

These data are taken from DYSON'S paper in *Nature*, of November 4th, 1909. In the *Monthly Notices* of the Royal Astronomical Society for November, 1909, MR. S. S. HOUGH, Director of the Cape Observatory, gives results of the study of the radial velocities of 318 stars brighter than 4.5 magnitude within 120° of the South Pole. The radial velocities may also be explained on the two-drift hypothesis; for HOUGH'S results confirm DYSON'S conclusions based on the transverse motions, so that both motions point to one and the same phenomenon.

§ 293. *The System of Ursa Major.*

Of late years a great deal of attention has come to be given to the detection of particular systems among the fixed stars, and especially among the brighter stars which compose the conspicuous constellations. Already in the 18th century MICHELL had concluded from the theory of probability that a physical relationship existed between the stars of the *Pleiades;* and on similar grounds SIR WILLIAM HERSCHEL announced that *Coma Berenices* was a cluster comparatively near our Sun. Boss' investigation of the *Hyades* cluster has already been mentioned in § 266. It was pointed out by PROCTOR over thirty years ago that most of the stars of the *Dipper* have a common proper motion, and are drifting together in space. He applied the new theory of *star drifts*, giving systems of stars with parallel motion, to other constellations, with a number of suggestive results. But the recent investigations with the spectrograph have greatly augmented the material at the disposal of the astronomer, because by combining the motion in the line of sight with the motion perpendicular to the visual ray and the parallax, it becomes possible to determine the absolute direction of the motions in space. We shall not go into the details of these calculations, but merely content ourselves with a few of the results that have been obtained. The principal centres of the spectrographic work on the motion in the line of sight, are the Lick, Yerkes, and Potsdam Observatories. And PROFESSOR LUDENDORFF, of Potsdam, has extended and greatly improved our knowledge of the *Ursa Major* system of bright stars. Recently PROFESSOR EBERHARD, DR. MUND, DR. HERTZSPRUNG, and others

have been occupied with these researches. HERTZSPRUNG has shown that *Sirius* is a member of the *Ursa Major* system; and in *A.N.*, 4376, LUDENDORFF shows that β *Aurigae* and α *Coronae Borealis* belong to the same group. LUDENDORFF's list of stars includes the following objects, but many more will doubtless be added when our knowledge of the heavens becomes more complete:

β *Aurigae*	γ *Ursae Majoris*
α *Canis Majoris*	δ *Ursae Majoris*
37 *Ursae Majoris*	ϵ *Ursae Majoris*
β *Ursae Majoris*	ζ *Ursae Majoris*
δ *Leonis*	α *Coronae Borealis*

The agreement among the motions of the stars is quite satisfactory, and there is no doubt that they form parts of a cluster drifting together in space. In fact these stars not only drift together, but lie approximately in a plane, and according to PROFESSOR LUDENDORFF, nearly in a right line. The known convergent point of the system was used by DR. HERTZSPRUNG to detect new and remote members of the system, such as *Sirius*, δ *Leonis*, β *Aurigae*, etc. (cf. *Astrophysical Journal*, Sept., 1909); and he gives the convergent point as $\alpha = 127°.8$, $\delta = +40°.2$. The velocity of the system referred to our Sun is found to be 18.4 km. per second.

§ 294. *Theory of the Milky Way Here Adopted.*

The accompanying sketch gives what seems to be the most probable arrangement of the Milky Way. It is a modification and extension of EASTON's spiral theory, so designed as to account for the two streams recently recognized to exist among the stars near our Sun, which alone show proper motion. It is impossible as yet to work out the details of the spiral constituting the Milky Way; but it is clear that the general arrangement here outlined enables us to account for the two streams. There is a secular whirling motion in progress by which our Sun is carried along nearly in the plane of the Milky Way, towards *Hercules* or *Lyra*, as HERSCHEL first pointed out. The centre of the vortex is towards *Sagittarius*, *Aquila* and *Cygnus*, and that is why the stars are so dense and bright in that direction. The streams near our Sun interpenetrate, some going faster and some slower than our Sun; and the result is that those which are moderately remote appear to be left behind, by the greater motion of our Sun; while those outstripping our Sun in the whirlpool of stars rush ahead towards *Cygnus*, *Aquila* or *Sagittarius*. This last point is the one towards which the movement is most concentrated, and from our point of view seems to be the centre of the Milky Way.

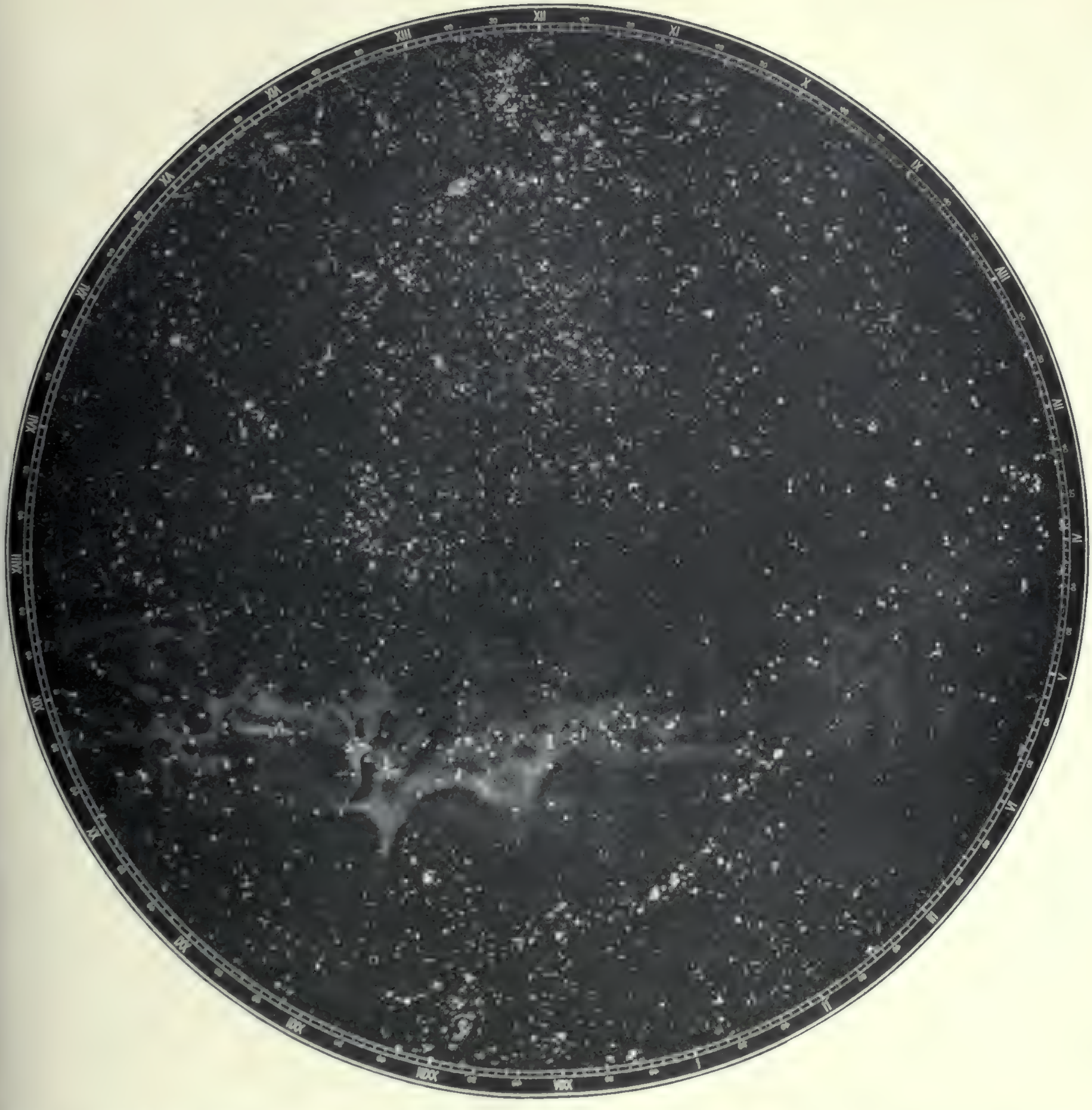


Plate ρ. Isographic Projection of the Northern Celestial Hemisphere upon the Plane of the Equator, Viewed from Above, Showing the Breaking Up of the Milky Way into Vast Clouds and Streams of Stars, with Clusters Forming Along the Galactic Circle, and Nebulae in the Regions of the Poles. The Clusters are Represented by Crosses, the Nebulae by Dots. The Data for This Projection of the Naked Eye Milky Way is Supplied by the Maps of Boeddicker, and Dreyer's New Catalogue of Nebulae and Clusters.

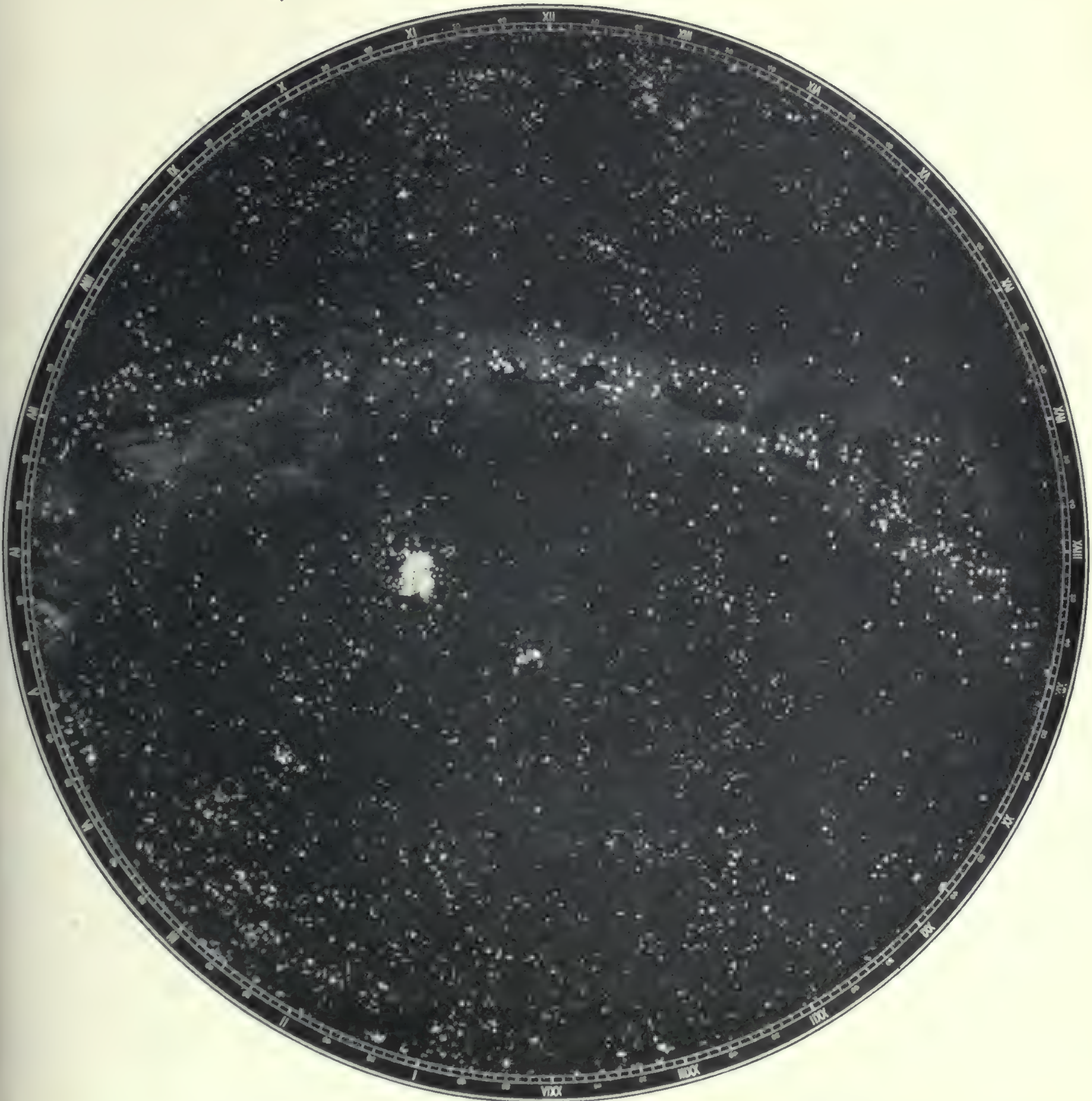


Plate σ. Isographic Projection of the Southern Celestial Hemisphere upon the Plane of the Equator, Viewed from Above, Showing the Breaking Up of the Milky Way into Vast Clouds and Streams of Stars, with Clusters Forming Along the Galactic Circle, and Nebulae in the Regions of the Poles. The Clusters are Represented by Crosses, the Nebulae by Dots. The Data for This Projection of the Naked Eye Milky Way is Supplied by the Maps of Gould, and Dreyer's New Catalogue of Nebulae and Clusters.

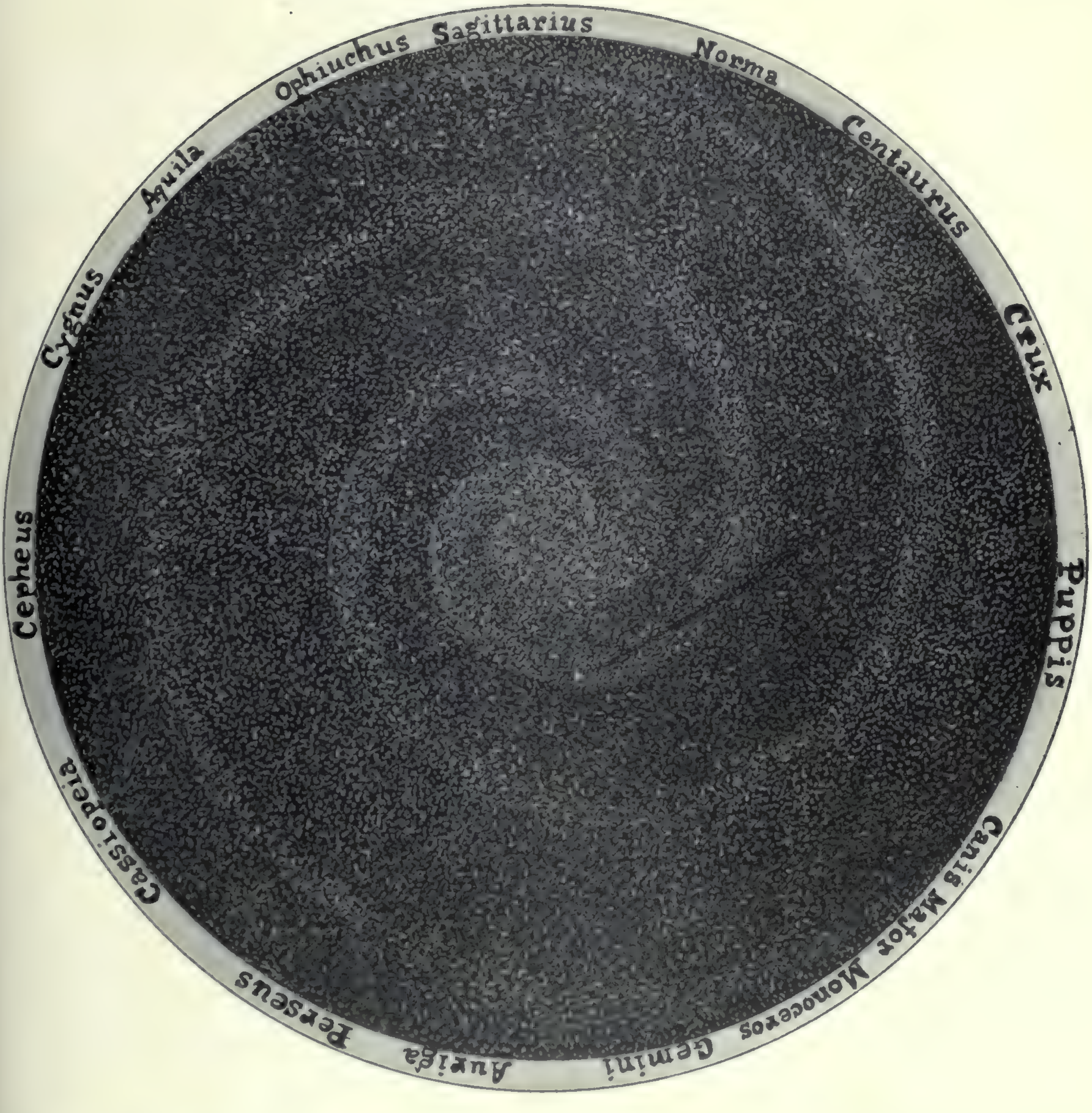


Plate 7. Spiral Theory of the Sidereal Universe, Devised by T. J. J. See, for Explaining the Two Star Streams, the Secular Motion of the Solar System Towards the Constellation Cygnus, the Asymmetry and Bifurcation of the Milky Way, and Such Phenomena as the Coal Sack, Near the Southern Cross, Here Represented by a Vacant Lane Along the Stream of Stars. The Sun is Supposed to be a Member of a Cluster Represented by the Bright Point Below the Centre of the Spiral, so that as Viewed from our Unsymmetrical Situation the Milky Way Appears Widest and Brightest Towards the Constellation Sagittarius, and Narrowest and Faintest in Monoceros.

At any rate it is the centre of the portions of the Milky Way nearest the Sun. This simple figure enables us to give a physical interpretation to the researches of astronomers. The result, however, is a mere outline, and does not yet enable us to indicate to what extent the spiral branches of the Milky Way interpenetrate. It may be long ages before we can make out the details of the star streams, and the exact way in which they are arranged about our Sun. But it seems fairly certain that this rough sketch gives us the fundamental basis of the Galactic vortex in which the Sun is immersed. The study of this gigantic sidereal movement will doubtless engage the attention of astronomers for many centuries.

§ 295. *List of the Principal Novae Which Have Appeared Since the New Star of* HIPPARCHUS, 134 *B.C.*

1. *Nova Scorpii,* 134 *B.C.;* between β and ρ *Scorpii.* Observed by HIPPARCHUS, and by the Chinese.
2. *Nova Ophiuchi,* 123 *A.D.;* between α *Herculis* and α *Ophiuchi.*
3. *Nova Centauri, Dec.* 10, 173, *A.D.;* between α and β *Centauri;* very brilliant, changed color, visible eight months.
4. *Nova Sagittarii, April to July,* 386; between λ and ϕ *Sagittarii.*
5. *Nova Aquila,* 389; near α *Aquilae,* visible three weeks, and rivalled *Venus* in brightness, according to CUSPINIANUS, who had himself seen it.
6. *Nova Scorpii,* 393, *March;* in the tail of *Scorpion,* observed by the Chinese.
7. *Nova Scorpii,* 827 (?). Visible four months; observed at Babylon in reign of CALIPH AL MAMOUN, by Arabian astronomers HALY and GIAFAR BEN MOHAMMED ALBUMAZAR; brightness "equalled that of the Moon in her quarters" [This might be a comet!].
8. *Nova Cephei-Cassiopeiae,* 945; recorded by CYPRIANUS LEOVITIUS, Bohemian astronomer, from manuscript chronicle, credited by TYCHO.
9. *Nova Arietis,* 1012, *May to August;* variable in brightness and dazzling to the eyes; chronicled in Annals (709-1044) of HEPIDANNUS, Monk of St. Gall, who died in the year 1088.
10. *Nova Scorpii,* 1203, *July;* in tail of *Scorpion;* recorded by the Chinese as of "bluish white color, without luminous vapor, and resembling *Saturn.*"
11. *Nova Ophiuchi,* 1230, *middle of December to end of March,* 1231; between *Ophiuchus* and the *Serpent;* from the *Ma-tuan-lin,* which contains an accurate account of comets and fixed stars back to 613 B.C., in the time of THALES.
12. *Nova Cephei-Cassiopeiae,* 1264; mentioned by CYPRIANUS LEOVITIUS, the Bohemian astronomer, and credited by TYCHO.

13. *Nova Cassiopeiae, Nov.* 11, 1572; TYCHO's celebrated star. Equaled *Jupiter* and rivaled *Venus;* visible till March, 1574. Changed color from dazzling white to ruddy, and then appeared pale. Visible in daylight when brightest.
14. *Nova Cygni,* 1600; JANSEN's star, and according to BAYER, now known as 34 *Cygni,* 3d to 6th magnitude; vanished in 1621, but found by CASSINI in 1655 to be again of 3d magnitude; 1677-1682 again discovered to be of the 6th magnitude. SIR JOHN HERSCHEL classed it as a variable, but ARGELANDER considered it a *Nova.*
15. *Nova Ophiuchi, Oct.* 10, 1604; KEPLER's celebrated star; brighter than *Jupiter* and *Saturn,* but fainter than *Venus;* white but scintillating. Vanished in March, 1606.
16. *Nova Vulpeculae,* 1670, *June* 20, near *β Cygni;* discovered by Carthusian Monk ANTHELMUS; 3d magnitude, variable; vanished in March, 1672.
17. *Nova Ophiuchi,* 1848, *April* 28; discovered by HIND; 5th magnitude, reddish yellow, afterwards died down to 13th magnitude, 1874-5.
18. *Nova Scorpii, May* 18, 1860, in a nebula, *Messier* 80; 7th magnitude, rapidly declined. Observed by AUWERS and POGSON.
19. *Nova Coronae Borealis, May* 12, 1866; rose to 2d magnitude; shown by HUGGINS to give bright hydrogen lines; changed color from white to orange; faded to 10th magnitude.
20. *Nova Cygni, Nov.* 24, 1876; rose to 3d magnitude, rapidly declined to 6th magnitude Dec. 15th; color changed from yellow to bluish, thus probably becoming a planetary nebula.
21. *Nova Andromedae,* 1885, *August* 16, in the Great Nebula; rose to 7th magnitude, and rapidly declined; color changed from yellow to nebular tint.
22. *Nova Persei,* 1887; ninth magnitude; hydrogen lines and $\lambda = 4060$ bright; discovered at Harvard College Observatory.
23. *Nova Aurigae,* 1892, *Feb.* 1; rose from below 10th to 4th magnitude, and after fluctuations became a planetary nebula in August, 1892.
24. *Nova Normae,* 1893; 7th magnitude; twelve bright lines, nebular spectrum 13th Feb., 1894; Harvard College Observatory.
25. *Nova Carinae,* 1895. Bright hydrogen lines; faded from 8th to 11th magnitude between April and July. Harvard College Observatory.
26. *Nova Centauri,* 1895; 7.2 magnitude; spectrum July 18, resembled that of 30 *Doradûs*; in a nebula; Harvard College Observatory.
27. *Nova Sagittarii, March* 8, 1898, 4.7 magnitude; bright hydrogen lines; spectrum nebular, 13 March, 1899; Harvard College Observatory.

28. *Nova Aquilae, April* 21, 1899; 7th magnitude; bright line spectrum, July 3, 1899; nebular spectrum October, 1898.
29. *Nova Persei, No.* 2, *Feb.* 22, 1901; the brightest temporary since KEPLER'S celebrated star of 1604; rose to exceed *Capella;* bluish white; showed great changes of color and spectrum; at first of *Orion* type, then of hydrogen type; color changed from white to red; faded and spectrum became that of planetary nebula in July, 1901.
30. *Nova Geminorum,* 1903, *March* 6th; observed by TURNER and BELLAMY at Oxford.
31. *Nova* (RS) *Velorum,* 1905, *Dec.* 5th. Discovered by MISS LEAVITT, at Harvard College Observatory.
32. *Nova Aquilae,* 1905, *Aug.* 18. Discovered by MRS. FLEMING, at Harvard College Observatory.
33. *Nova Circini,* 1906, *Feb.* 14. Discovered by MISS LEAVITT, at Harvard College Observatory.
34. *Nova Scorpii, No.* 2, 1906, *June* 14. Discovered by MISS CANNON, at Harvard College Observatory.

Many of the notes about the earlier stars are taken from HUMBOLDT'S *Cosmos,* Vol. III, and of the latest stars from data kindly supplied by PROFESSOR E. C. PICKERING, Director of the Harvard College Observatory, where so many Novae have been discovered in recent years.

§ 296. *On the Physical Cause of the Appearance of New Stars Near the Path of the Milky Way.*

It was remarked by TYCHO and KEPLER, over three centuries ago, that nearly all of the new stars which have suddenly blazed forth and afterwards died down to comparative obscurity, have appeared near the course of the Milky Way, while the regions of the heavens remote from the Galaxy have seldom or never been illuminated by these temporary stars. And although this tendency of the Novae to follow the Milky Way has attracted more and more attention from astronomers of late years, during which the list of these objects has been so greatly increased, especially by the search for stars having peculiar spectra, carried on at Harvard College Observatory, there has been, so far as I know, no satisfactory explanation of this fact.

Even if the cause producing Novae should remain more or less obscure, it would still be very desirable to know where they appear and why, because if this were well established, it might guide us to a large extent in the discovery of

Novae, and perhaps ultimately throw light on the cause which occasions these sudden outbursts of very obscure objects. We propose, therefore, to examine the relationship of the Novae to the Milky Way, in the hope of ascertaining why they are essentially confined to the Galactic region of the heavens.

If we were required to calculate the total number of stars included within a sphere of radius r, we should have to evaluate the triple integral

$$N_t = \int_0^r \int_0^\pi \int_0^{2\pi} \sigma \,.\, dr \,.\, r d\phi \,.\, r \cos \phi d\psi \,, \tag{547}$$

where σ is the density of the stars per unit volume, and ϕ and ψ the Galactic latitude and longitude, respectively. As the stars are irregularly scattered and gathered into clusters σ is very complex and can only be defined by the general relation

$$\sigma = F\{C \,,\, R(r) \,,\, \Phi(\phi) \,,\, \Psi(\psi)\} \,. \tag{548}$$

Here C is some initial constant applying to a particular portion of space, while the functions $R(r)$, $\Phi(\phi)$, $\Psi(\psi)$ are complex and at present quite unknown.

In default of actual knowledge of the details of the distribution of the stars in space, we are obliged to fall back on W. STRUVE's discussion of the gauges of SIR WM. HERSCHEL, as affording the best available approximation to the actual state of the sidereal universe. In these discussions r will be taken as the distance to which HERSCHEL's telescope could penetrate, assumed to be uniform in all directions and the above triple integral thus reduces to a double integral, relative to the Galactic latitude and longitude, respectively.

But although there are some known irregularities in the density depending on the Galactic longitude, they are not yet accurately determinable and so much less important than those depending on the Galactic latitude, that we shall here content ourselves with the latter only, and thus but a single integral is required to express the approximate number of stars scattered over the face of the heavens.

In his *Études d'Astronomie Stellaire*, Petersbourg, 1847, W. STRUVE has shown that the average density of the stars per field of HERSCHEL's telescope in different Galactic latitudes is as follows:

$$z = \frac{6.5713 - 5.03 \cos 2\phi - 1.39 \cos 4\phi}{1 - 1.23088 \cos 2\phi + 0.23212 \cos 4\phi} \,.$$

0°	$z = 122.0$	by	151	multiple	gauges
15	30.30	"	56	"	"
30	17.68	"	34	"	"
45	10.36	"	48	"	"
60	6.52	"	18	"	"

And on page 34 of the Notes on the *Études*, STRUVE calculates a table giving the value of z for each region of Galactic latitude, as follows:

$\phi =$	$z =$	Diff.
0°	122.0	11.3
1	110.7	21.2
2	89.5	17.6
3	71.9	11.8
4	60.1	7.7
5	52.4	5.1
6	47.3	3.7
7	43.6	2.8
8	40.8	2.1
9	38.7	1.8
10	36.9	1.6
11	35.3	1.3
12	34.0	1.4
13	32.6	1.1
14	31.5	1.2
15	30.3	1.0
16	29.3	1.1
17	28.2	0.9
18	27.3	0.9
19	26.4	

$\phi =$	$z =$	Diff.
20°	25.4	4.2
25	21.2	3.5
30	17.7	2.9
35	14.8	2.5
40	12.3	1.9
45	10.4	1.6
50	8.8	1.3
55	7.5	1.0
60	6.5	0.8
65	5.7	0.6
70	5.1	0.4
75	4.7	0.3
80	4.4	0.2
85	4.2	0.1
90	4.1	

He remarks that the most rapid change of z is near $\phi = 2°$, and adds that this circumstance explains why the Galaxy is visible to the naked eye only over a comparatively small width of about 4°.

Taking STRUVE's discussion of HERSCHEL's data as the basis of our calculations, we are required to integrate certain zones of the Galactic hemisphere, to ascertain if there is any physical reason why the new stars should follow the path of the Milky Way. The values of z being given, we obtain the total number of stars ζ included between two semicircles which cut each other at the pole of the Galaxy at an angle of 904″, corresponding to the field of view of HERSCHEL's telescope, in the plane of the Milky Way, by the expression

$$\zeta = 4 \cdot \frac{2}{\pi} \int_0^{\frac{\pi}{2}} z \cos \phi \, d\phi \, . \tag{549}$$

This integral gives the number of stars included within a half lune of angle 904″, and to get the contents of the entire hemisphere we must multiply by $\mu = \frac{1296000''}{904''} = 1433.6$. The total number of stars in a Galactic hemisphere thus becomes

$$Z = 1433.6 \, \frac{8}{\pi} \int_0^{\frac{\pi}{2}} z \cos \phi \, d\phi \, , \tag{550}$$

= 10,187,017 stars, according to the calculations of STRUVE, based on the data of HERSCHEL. This gives 20,374,034 stars in the entire celestial sphere—an estimate which faithfully represents the gauges of HERSCHEL, who avoided the regions of clusters and other groups of stars of abnormal density.

According to this mode of integration the numbers of the stars in the eighteen zones of 5° width extending from the Milky Way to the poles, may be found by the formula

$$N_i = \frac{8}{\pi}\frac{9000''}{904''}\int_a^b \phi(s)\,ds\ ; \tag{551}$$

and the results obtained by neglecting the second differences in STRUVE's table of $z = \phi(s)$ are as follows:

ϕ_i	N_i	ϕ_i	N_i
0° – 5°	3161500	45° – 50°	350470
5 – 10	1630000	50 – 55	297610
10 – 15	1226600	55 – 60	255550
15 – 20	1016700	60 – 65	222690
20 – 25	850610	65 – 70	197140
25 – 30	710600	70 – 75	178880
30 – 35	593240	75 – 80	166110
35 – 40	494670	80 – 85	156980
40 – 45	414350	85 – 90	151500
	$\sum_{i=1}^{i=9} N_i = 10098270$		$\sum_{i=1}^{i=18} N_i = 12075270$

$$\sum_{i=1}^{i=18} N_i - \sum_{i=1}^{i=9} N_i = 1976930$$

STRUVE found for the number of the HERSCHEL stars in a hemisphere 10,187,017, while SEELIGER obtained 13,500,000. The values here found thus lie between these extremes.

Making use of STRUVE's data and integrating the Galactic hemisphere from the pole to 45°, $\phi = 90°$ to $\phi = 45°$, I find in this zone with about 0.3 of the area of a hemisphere only 1,976,930 stars, almost exactly one-sixth of the stars of the hemisphere, according to the author's integration given above. Accordingly, we find that the zone within 45° of the Milky Way contains five-sixths of all the stars counted by HERSCHEL; and it is a fact that all the Novae which have been recorded since the earliest ages have appeared within less than 45° of the Milky Way.

Now, our modern telescopes and photographic plates show many more stars than HERSCHEL's gauges. Perhaps instead of 20,374,034 stars we should have

ten times that many, or about 200,000,000. Moreover, as the universe is known to extend enormously in the direction of the Galaxy, it is certain that more than the average proportion of the additional stars now visible, but unseen or uncounted by HERSCHEL (who avoided the clusters in which the regions of the Milky Way are so rich), would be near the plane of the Galaxy. Thus to represent our present knowledge, our multiplier near the Galaxy would considerably exceed 10, and might reach 20, while near the poles it would be less than 10, and might be no larger than 5 or even 3. Accordingly, to our modern investigation the stars are much thicker, and the universe is relatively more extended, in the direction of the Galaxy than HERSCHEL estimated, while they are relatively rarer near the poles of the Galaxy. Thus the above ratio of 5 to 1 may be increased to something like 33 to 1, or even a higher value.

If this view be admissible, the belt within 45° on either side of the Milky Way may include not only five-sixths of the stars, as in HERSCHEL's time, but perhaps thirty-two thirty-thirds, or even a higher proportion of all the stars. For some of the Galactic stars are rendered very faint by great distance and extinction of their light; and even if below telescopic vision in our most powerful instruments, might suddenly rise into view and even prominence by the outbursts which augment their light by so many magnitudes.

The fact that out of the total number of Novae recognized in recent years, a considerable part are faint stars which have been detected by their peculiar spectra, chiefly at Harvard College Observatory, and in former times would have passed unnoticed, while only a few Novae attain conspicuous naked-eye brightness, lends support to the view that the Milky Way utilizes its almost unlimited supply of very faint stars in the production of Novae, as well as those brighter than say 14th magnitude.

Accordingly it seems practically certain that the preference of Novae for the plane of the Milky Way rests on no other physical cause than the great accumulation of stars in that part of the heavens. And in the effort to discover these objects, the Galaxy is the region, above all others, in which search should be made, while search near the poles of the Galaxy would be almost if not quite hopeless.

As the stars are centres of cosmical systems made up of planets, satellites and comets, Novae obviously are due to collisions of suns with smaller masses, such as planets, satellites, comets, or small nebulae. Spectroscopic evidence indicates that Novae sometimes pass into planetary nebulae, as in the case of *Nova Persei*, No. 2, 1901; *Nova Aurigae*, 1892, etc.

Owing to the great extension of the universe in the direction of the Galaxy,

and the effect of perspective from distance alone, all the observed phenomena* find an easy and simple explanation on this hypothesis. Incidentally the decided preference of Novae and of the WOLF-RAYET stars for the plane of the Milky Way lends some support to the view that the sidereal universe is of unlimited extent, and that the more distant portions of the Galaxy remain invisible from mere faintness and extinction of the starlight.

§ 297. *General Theory of Variable Stars.*

The subject of variable stars is a very large one, and the limits of this volume will not permit a detailed treatment of it; but if any light can be thrown upon the underlying cause producing variable star phenomena, even a very brief treatment will not be wholly without value. In spite of the variety of phenomena exhibited by variable stars, it is easy to divide them into about six principal classes:

1. *Algol Variables*, with regular extinction of part of the light by a dark companion moving in a plane passing nearly through the Sun.
2. *β Lyrae Variables* of double maxima and minima due to two bright stars, revolving in close proximity, with alternate eclipses of the component stars.
3. *The Cepheid Variables*, with rapid increase in the light and a more gradual decrease, accompanied by irregularities resembling secondary maxima in the descending branch of the light curve. Type of *δ Cephei*.
4. *The Geminid Variables*, with light curves of nearly symmetrical shape, the increase and decrease following essentially the same law of change.
5. *The Cluster Variables*, usually of short regular period, and characterized by rapid increase and somewhat more gradual decrease in the light, with prolonged dead level minima, the reverse of the *Algol Variables*, and therefore called by HARTWIG *Ant-Algol Variables*.
6. *Long-Period Variables*, of which *o Ceti* may be taken as a type.

The eclipsing stars, comprised here in the first three classes may be represented by such types as *Algol*, *β Lyrae*, and *δ Cephei*. The first is due to an ordinary eclipse of a bright by a dark companion; the second, to a bright star circulating about a larger but less lustrous globe, and thus exhibiting an intensified totality, with alternate eclipses of the stars; the third, to two stars undergoing

* We may estimate that at least 20 Novae have blazed forth with dazzling splendor in 2000 years, or one in a century. These are the extraordinary results of millions of collisions, with minor bodies, among a total assemblage of perhaps a billion stars. Thus in the long run any one star might suffer such a great collision once in 100,000,000,000 years. There need therefore be no fear that a body large enough to produce a great conflagration will strike our Sun, though in the course of immense ages it may suffer slight conflagrations from the impacts of comets, such as those of 1680, 1843, and 1882, a little change in the perihelion distances of which would have brought about actual collisions.

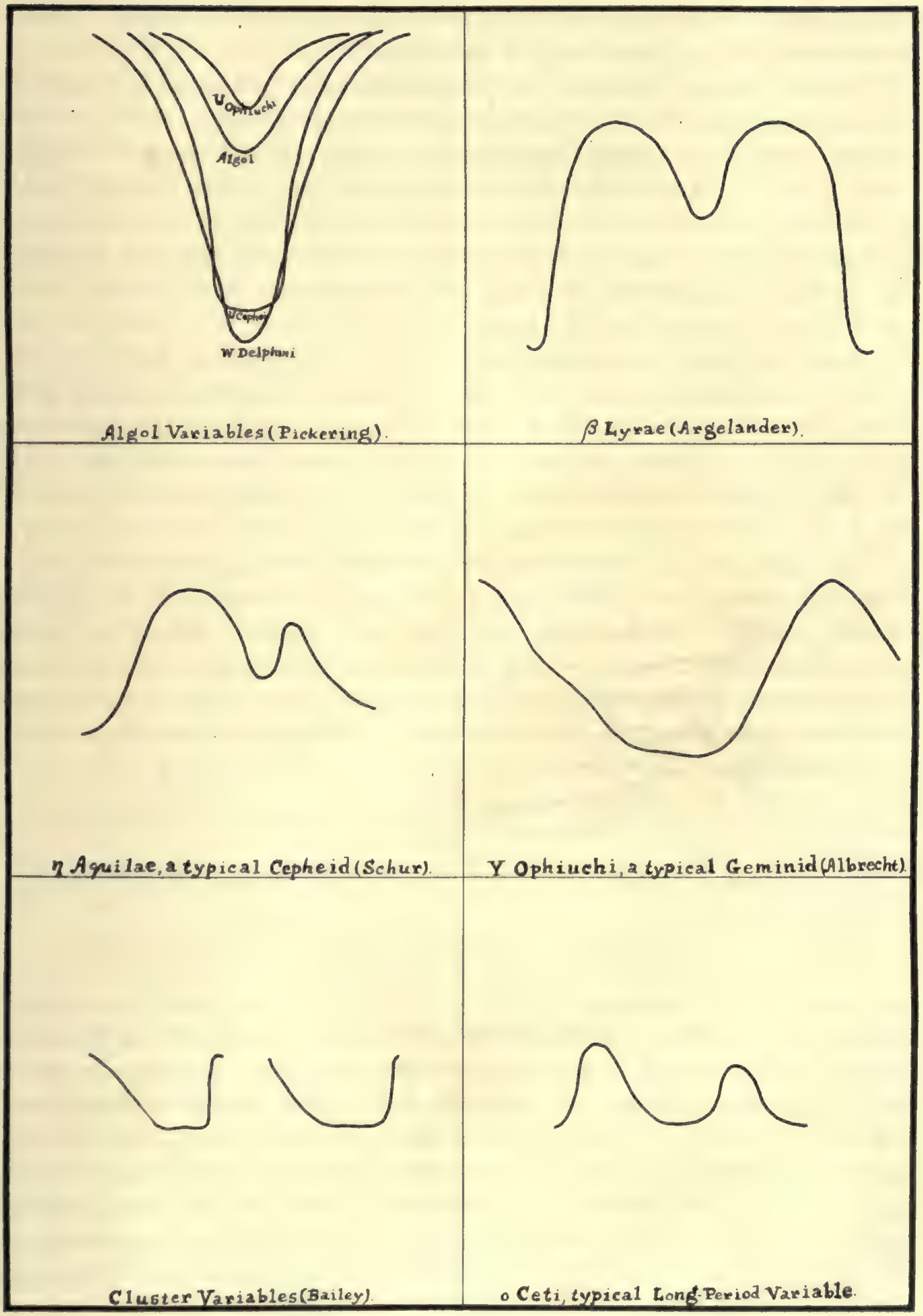


FIG. 45. LIGHT CURVES OF TYPICAL VARIABLE STARS.

unequal double eclipses with periastral resistance, and having a period of revolution corresponding to two periods of variation.

All these types of eclipsing stars have been studied by various investigators, and their phenomena admit of rational explanation by known causes.

The exact details of the classification of eclipse variables adopted is unimportant, since the general character of the theory is not thereby altered; but it is natural to adhere to that earliest developed, if it proves adequate to explain the observed phenomena, which is believed to be the case. We need not therefore dwell on eclipsing stars, but may pass at once to the *Geminids* and *Cluster Variables*, and variables of long period.

Already in 1904 (*Astrophysical Journal*, Vol. 20, p. 186 and *L.O.B.* 62) Dr. R. H. Curtiss remarked that "It is easy to construct a plausible explanation for the light and velocity curves of *W Sagittarii* on the assumption that the system is pervaded by a resisting medium which enhances the brightness of the side of the star which faces the direction of motion. . . . . Until more data are available, it would be premature to follow out such theories."

The subject has since been further investigated by Dr. Sebastian Albrecht of the Lick Observatory (*L.O.B.*, 118, p. 138) and by Professor F. H. Loud of Colorado College (*Astrophysical Journal*, Dec. 1907), and by others; and it has come to be believed that the resisting medium is one of the most important causes operative in the development of variable star phenomena. For those who wish to consult the latest researches attention should be called to the work of the following investigators:

1. Dr. Ralph H. Curtiss, in *L.O.B.*, 62, July, 1904, with full discussion of the orbit of *W Sagittarii*.
2. Dr. Sebastian Albrecht, in *L.O.B.*, 118, May, 1907, with orbits of *Y Ophiuchi* and *T Vulpeculae*.
3. Dr. J. C. Duncan, in *L.O.B.*, 157, April 19, 1909, with orbit of *X Sagittarii*.

In these papers an observational basis is laid which justifies the conclusion that the resisting medium is the most important cause affecting the light of variable stars. It accounts for the important fact of the synchronism of maximum light and minimum positive velocity in line of sight which appears to be characteristic of variables of the *Cepheid* type. These and other phenomena have been shown to be consistent with the effects of a resisting medium, and these conclusions were recorded by the foregoing investigators independently of the present author's researches in Cosmical Evolution. The discovery of independent lines of evidence pointing in the same direction is always interesting, and greatly strengthens the theory thus suggested.

It is not necessary to go into a detailed discussion of the phenomena which may be thus explained, but it is sufficient to remark that by varying the character of the medium, and the law of its density, the eccentricity and size of the orbit, the relative mass and relative brightness of the components, and the inclination of the system with respect to the visual ray, the position of the periastron, etc., we may account for all the known phenomena of variable stars.

Even the *Ant-Algol Variables* in clusters find a simple and natural explanation in good accord with their short periods and regular light-curves. It is scarcely to be doubted that the sudden rise in their light is due to bodies moving in highly eccentric orbits, and encountering resistance chiefly near periastron; while the more prolonged dead level tracts of their light-curves correspond to the longer sweep of the companion over the region near apastron. The rapid rise and more gradual decline in light may be explained by the direct effect of resistance, which would produce a rapid blazing up and a more gradual cooling down. The great regularity in the periods of these stars shows that orbital motion* must necessarily be the controlling principle.

In the case of long-period variables such as *Omicron Ceti,* the same explanation will hold by introducing disturbing bodies which we have every reason to suppose must often be present. Moreover, the red color of variables points also to a resisting medium, since such a medium would imply that selective absorption is at work and the principal star surrounded by a widely extended atmosphere of fine cosmical dust, or nebulosity, too close and too faint to be visible in our telescopes. For as we recognize in the telescope a good many *nebulous stars,* there must be a considerable number of such objects also quite beyond the limits of our vision; and when the nebulosity is dense enough and in a state of fine particles it may give the star the reddish aspect so characteristic of variables. This explains a difficulty long recognized in variable star astronomy, by the simplest and most general cause known to be at work everywhere in space, and shown to have exerted an enormous influence on the development of the solar system.

In view of the great part played by the nebular resisting medium in the past history of the universe, as shown by the known process involved in the evolution of the solar system, and the ease with which the theory of the resisting medium adapts itself to the phenomena of variable stars, it is impossible to doubt that it is the principal physical cause of stellar variability.

It must be left to future research to determine the exact mode of operation involved in the different types of variables. But as it is proved that nearly all

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* In *Astron. Nachr.*, No. 4409, BARNARD discusses the use which might be made of certain variables of this type for the measurement of time.

the fixed stars have companions of planetary or stellar character revolving about them, and that some nebulosity still survives about nearly all the heavenly bodies, it will be apparent that as a whole the stars are more or less variable, and that fluctuation in brightness is one of the most general of all the laws of the stars. Since cosmical systems are everywhere in the process of formation, it should occasion us no surprise that the luminous bodies which light up the depths of space should exhibit fluctuations in their starlight.* Our wonder should rather be that the changes are no greater than they are. The smallness of the changes, in most cases, would seem to imply that, as a rule, cosmical transformations come about gradually, and extend over vast intervals of time; which shows that systems of worlds usually develop under conditions of great stability, just as in the known history of our solar system. Notwithstanding very gradual changes this result bears impressive testimony to the general stability of the order of Nature.

§ 298. *The Extinction of Light in Space as Inferred from the Researches of* CHESEAUX, OLBERS, STRUVE, BRACE, KAPTEYN, SEELIGER *and* BARNARD.†

The problem of the extinction of light in space dates back to 1744, when CHESEAUX, of Geneva, published a treatise on the great comet of 1743 and 1744, with an addition containing diverse observations and astronomical dissertations, of which the second is entitled *Sur la force de la Lumière et la propagation dans l'Ether, et sur la distance des Étoiles Fixes.* In 1823 this subject was independently treated by DR. OLBERS, of Bremen, in a memoir entitled *Ueber die Durchsichtigkeit des Weltraums,* published in the *Berliner Jahrbuch* for 1826, pp. 100-121, and reprinted in OLBERS' Werke.

Basing their work on observed data respecting the stellar distribution, both CHESEAUX and OLBERS concluded that, if the number of stars is infinite and distributed with anything like uniformity in space, and there be no extinction or absorption of light, the sky would appear all over of a brightness approaching that of the Sun, since at any point the brightness depends on the depth of the luminous layer and the solid angle which it subtends at that point. If, therefore, the layer of stars is indefinitely thick, and no light is lost by extinction or absorption, the luminous points would finally tend to cover the background of the heavens as the stars do in a dense cluster, and the whole sky would become luminous like the Sun's disc.

In his *Études d'Astronomie Stellaire,* 1847, W. STRUVE attempted to deduce the law of the extinction of light in space. He used the data of HERSCHEL's gauges

* The Novae appear to be extreme cases of fluctuation, due to actual collisions of stars with bodies large enough to produce violent but temporary conflagrations, and therefore with masses probably corresponding to our planets, satellites and comets.

† cf. *Astrophysical Journal,* January, 1910.

and assumed an average uniformity of stellar distribution in layers parallel to the Milky Way; and on the hypothesis that the brightness is a measure of the relative distance, deduced tables to show that extinction takes place. If the distribution were uniform, the number of calculated stars of different magnitude should vary inversely as their brightness; but as the number of calculated stars of any magnitude exceeds slightly the number observed, the excess being greater with the diminution of the magnitude, he concluded that extinction should be invoked to explain the discrepancy.

STRUVE also concluded that the space-penetrating power of a telescope was less than HERSCHEL had concluded from the natural enfeeblement of light incident to the inverse square of the distance, and that light is lost in its passage through celestial space.

Using ARGELANDER'S estimates of magnitudes as the basis of calculation, STRUVE found that the apparent brightness of a star ξ is a function of its distance and of the extinction, of the form

$$\xi = \frac{\lambda^{x-1}}{x^2} = \frac{1}{x^2}0.990651^{x-1}, \tag{552}$$

where x is the distance of the star, and $\lambda = 0.990651$, the coefficient of extinction.

The following is STRUVE'S table:

Distance $x =$	Coefficient of Extinction $\lambda =$	Relative Brilliancy: Without Regard to Extinction $(\xi) =$	Relative Brilliancy: With Regard to Extinction $\xi =$	No. of Collected Stars which would Produce the Same Brilliancy	
1,0000	0,99065	1,0000	1,0000	1,00	Mean Dist. of Stars 1A
1,2638	0,99821	0,6261	0,6246	1,60	Radius " " 1A
1,8031	0,98300	0,3076	0,3053	3,28	Mean Dist. of Stars 2A
2,1408	0,98009	0,2182	0,2159	4,63	Radius " " 2A
2,7639	0,97437	0,1309	0,1287	7,77	Mean Dist. of Stars 3A
3,1961	0,97043	0,0979	0,0959	10,43	Radius " " 3A
3,9057	0,96398	0,0656	0,0638	15,68	Mean Dist. of Stars 4A
4,4374	0,95918	0,0508	0,0492	20,34	Radius " " 4A
5,4545	0,95006	0,0336	0,0322	31,02	Mean Dist. of Stars 5A
6,2093	0,94334	0,0259	0,0247	40,49	Radius " " 5A
7,7258	0,93000	0,0168	0,0157	63,58	Mean Dist. of Stars 6A
8,8726	0,92003	0,01270	0,01180	84,76	Radius " " 6A
14,4365	0,87319	0,00480	0,00423	236,44	Radius " " 7B
24,8445	0,79186	0,00162	0,00129	772,20	" " " 8B
37,7364	0,70154	0,000702	0,000497	2010,9	" " " 9B
227,782	0,11770	0,00001928	0,00000229	436696,0	Radius " " H

The second column shows the effect of extinction. Only one per cent. of the light is lost for a star of the first magnitude, but for a star of ARGELANDER'S sixth magnitude, the loss mounts up to eight per cent., while for stars of BESSEL'S ninth magnitude, thirty per cent. is lost. Finally, for the smallest stars at the greatest depth to which HERSCHEL'S telescope could penetrate, the total loss is 88 per cent. If λ be the coefficient or function to which the intensity of the light is reduced by extinction, the loss is $1-\lambda$; and we have for great distances:

	Extinction	Distance
$\lambda = 0.01$	0.99	$x = 490.26$
$\lambda = 0.001$	0.999	$x = 735.40$
$\lambda = 0.0001$	0.9999	$x = 980.53$
$\lambda = 0.00001$	0.99999	$x = 1470.80$

This enfeeblement of the light of the more distant stars and the corresponding limitation of the penetrating power of telescopes is the direct effect of the extinction of light in its passage through space.

STRUVE has examined also the fractional part of the light of the Galaxy given by the stars of different classes. Taking a differential cone with vertex in the eye of the observer, and having an element of the base of thickness dx, the differential brilliancy is

$$dL = C \cdot \lambda^x \cdot z dx \,, \tag{553}$$

where C is a constant, λ the coefficient of extinction for unit of distance, and z the density of the stars in the differential layer. The total light of the cone from the eye to the distance δ is

$$L = C \int_{x=0}^{x=\delta} \lambda^x z dx \,. \tag{554}$$

In the direction of the middle of the Galaxy $z = 1$, and we get

$$\left.\begin{aligned} L' &= C \int_{x=0}^{x=\delta} \lambda^x \,.\, dx = C \frac{1-\lambda^\delta}{-\log_e \lambda}; \\ \text{and if} \qquad \delta &= \infty \quad , \quad L' = \frac{-C}{\log_e \lambda}. \end{aligned}\right\} \tag{555}$$

But as the light of the middle of the Milky Way is taken to be unity, this gives $C = -\log_e \lambda$. And thus for the distance δ we have

$$L' = 1 - \lambda^\delta \,. \tag{556}$$

And

$$L = -\log_e \lambda \int_{0=0}^{x=\delta} \lambda^x dx \tag{557}$$

gives the brilliancy in any direction from the Milky Way to its poles. But as the analytical expression of z is very complicated, this is best evaluated by quadrature, which is made very simple by taking the distance of the remotest HERSCHEL stars as the unit of distance, and the corresponding coefficient of extinction $\lambda = 0.11770$. Thus STRUVE finds the following results:

Stars of Classes	Portion of the Light of the Middle of the Galaxy	Stars of Classes	Portion of the Light of the Middle of the Galaxy
1 to 6 A	0.07993	1 to 6 A	0.07993
1 to 7 B	0.12683	7 B	0.04690
1 to 8 B	0.20814	8 B	0.08131
1 to 9 B	0.29845	9 B	0.09031
1 to H	0.88230	9 B to H	0.58385
1 to ∞	1.00000	H to ∞	0.11770
			1.00000

This table gives a very useful summary of the total light of the stars of the different ARGELANDER, BESSEL and HERSCHEL classes. It shows that most of the light comes from the stars between BESSEL's ninth magnitude and the faintest HERSCHEL stars, while the stars beyond HERSCHEL's vision give less than 12 per cent. of the total light of the Galaxy. Unless we suppose the sidereal universe to extend but little beyond the depths to which HERSCHEL's telescope could penetrate, we must recognize the rapid increase of extinction at great depths.

That the universe should have been accidentally so constituted as practically to cease at the limits of HERSCHEL's vision, or that HERSCHEL would have just happened to make telescopes which nearly penetrated to the borders of the universe, no one will believe who is acquainted with the theory of probability. The chances are almost infinity to one against the coincidence. And we are obliged to recognize, therefore, that the remoter stars are finally rendered invisible by the extinction of light in space, and that the penetrating power of telescopes is less than its theoretical amount by a proportion which increases with the distance and finally cuts off the light of the more distant stars entirely. Thus the most distant regions of the universe will always be veiled in the blackness of everlasting night, and no increase in telescopic power can ever overcome this difficulty, by which nature limits our explorations of the sidereal universe.

Since the appearance of the great work of STRUVE, over sixty years ago, on the *extinction* of light in space, the subject of the *absorption* of light has been considered by many other writers, but with less significant results. We shall only refer to a few of these investigations.

In the *Studies* of the University of Nebraska for July, 1888, the late PROFESSOR D. B. BRACE has an important paper " On the Transparency of the Ether," in

which he has considered the effect of imperfect elasticity of the ether, and shown by a careful discussion of the differential equations for the motion of a viscous fluid that the principal effect of imperfect elasticity or friction in the ether would be to give *increasing coloration to the remoter stars*, by cutting down the shorter vibrations, and thus rendering them somewhat reddish. But he remarks that the observed whitish color of the remotest stars is strong evidence that there is little or no absorption of light depending on the friction of the ether; and he thus concludes that the apparent finiteness of the universe, as shown by STRUVE's researches, cannot be due to *absorption*.

BRACE, however, takes no account of *extinction* as distinguished from *absorption*, the former effect depending on the pressure of cosmical dust, the latter on the imperfect elasticity of the ether. We may therefore dismiss the supposition that the ether is imperfectly elastic, and consider only the effects which will arise from cosmical dust in space. If the particles of the dust be appreciably larger than the wave lengths of light, the effect will be extinction without sensible selective absorption. Now this is exactly what appears to take place. Nebulosity pervades the celestial spaces, and this star dust is made up of solid particles of all sizes. Like very coarse dust floating in the air it cuts down the depth through which the starlight can penetrate; so that the remoter stars become invisible but do not show sensible coloration.

As *extinction* of light is an unquestioned fact, this result would seem to indicate that most of the cosmical dust is made up of grains larger in diameter than the wave lengths of light; which accords with our knowledge of the nature of meteoric dust falling upon the Earth and shown to exist in considerable abundance throughout the celestial spaces.

Within the last few years the subject of the *extinction* and *absorption* of light in space has been further considered by TURNER, KAPTEYN, and SEELIGER.* In *A.N.*, 4359, p. 246, SEELIGER remarks that the recent results tending to show a *selective absorption* are far from conclusive, and that the subject must be followed much further before any proof can be obtained. TURNER's results are published in the *Monthly Notices* of the Royal Astronomical Society, while KAPTEYN's more elaborate discussions will be found in the *Astrophysical Journal* for January, November, and December, 1909.

KAPTEYN finds from his careful discussion of the observations of several hundred stars that the difference d representing the amount of change *Phot.—Vis.* for unit distance is $d = +0.0031 \pm 0.0006$ mag. (*Aph. J.*, Dec., 1909, p. 399). This

* Also by BARNARD in the *Astrophysical Journal* for January, 1910, and by SCHWARZSCHILD and HARTZSPRUNG, in *Astron. Nachr.*, No. 4422, June 28, 1910.

value of the *absorption* is excessively minute, and whilst it may be real, it cannot yet be regarded as established. The data on photographic and visual magnitudes of the stars are not yet accurate enough to permit of the certain recognition of such minute quantities. The difficulty is like that of fixing the ten-thousandth of a second of arc in the solar parallax, when the thousandth is still uncertain, and the hundredth barely established.

Accordingly the indications of absorption found in the spectra of certain stars by KAPTEYN and SLIPHER must be accepted with extreme reserve. Many other explanations might be found for the same phenomena, but the causes involved might prove to be wholly different. In conclusion, we may therefore say that a small *absorption* probably exists, but that it is not yet certainly detected. It cannot be large, even for the remotest stars, as we infer from the observed absence of sensible coloration; while *extinction* is everywhere at work and known to be so large for the most distant portions of the Galaxy as to limit the depths to which our telescopes can penetrate, as has been shown by STRUVE from HERSCHEL'S gauges.

§ 299. SEELIGER'S *Investigation of the Spacial Distribution of the Fixed Stars.*

In the *Abhandlungen* of the Munich Academy of Sciences for 1898 (II Class, XIX Band, III Abtheilung) PROFESSOR H. VON SEELIGER has treated with entire mathematical rigor of the problem of the spacial distribution of the fixed stars. We can here give only the briefest summary of his method, but we may remark that he takes account of the following causes: (1) The unequal intrinsic lustre of the stars; (2) The unknown law of distribution; (3) The extinction of light in space. And when the resulting equations cannot be integrated, owing to the presence of unknown factors under the integral signs, he proceeds to determine the unknown functions in such a way as to satisfy observations; and the data of observation are thus harmonized with rigorous theory.

Put A_m for the number of stars from the brightest to the magnitude m, which lie in a given direction on an apparent surface area ω; then if the stars which lie in a portion of space $d\tau = \omega r^2 dr$, at the distance r were transferred to the distance 1, the apparent brightness would be i, which denotes the absolute intensity; and the value of i may be defined by the frequency function $\phi(i)$. And if D be the number of the stars in a unit volume of space, their intensity i will be between i and $i+di$, and $D\phi(i)\,di\,d\tau$ will be the number of the stars which are in the surface ω; so that we have the equation

$$A(d\tau) = D \,.\, \phi(i)\,di\,d\tau \,. \qquad [\alpha]\ (558)$$

The intensity i is to be taken so that it varies between $i = 0$ and $i = H$, and $\phi\,(i)$ is to be determined so that

$$\int_0^H \phi(i)\, di = 1\ . \qquad [\beta]\ (559)$$

The apparent brightness h of a star at the distance r will not be assumed to be $\frac{i}{r^2}$ but may be reduced by extinction to

$$h = \frac{i\psi(r)}{r^2}, \qquad [\gamma]\ (560)$$

where $\psi\,(r)$ is the extinction function, and depends on the direction of the star. Taking as the element of space $d\tau = \omega r^2 dr$, and substituting for di its value $di = \frac{r^2}{(\psi)\, r} dh$, we therefore get

$$A\,(d\tau) = \omega \frac{D\,.\,r^4 dr}{\psi(r)}\, \phi \left(\frac{h\, r^2}{\psi(r)}\right) dh\ . \qquad [\delta]\ (561)$$

This gives the number of stars in the area ω, at distance between r and $r + dr$, and brightness between h and $h + dh$. Integrating this expression between the limits of the brightest stars to those of brightness h_m, we get

$$A_m\,(d\tau) = \omega \frac{D(r)\, r^4 dr}{\psi(r)} \int_{h_m}^{H \frac{\psi(r)}{r^2}} \phi \left(\frac{h\, r^2}{\psi(r)}\right) dh\ . \qquad [\epsilon]\ (562)$$

If we integrate from $r = 0$ to $r = \sigma$, where there are stars of brightness $\geqq h$, we get

$$A_m = \omega \int_0^\sigma \frac{D(r)\, r^4}{\psi(r)}\, dr \int_{h_m}^{H \frac{\psi(r)}{r^2}} \phi \left(\frac{h r^2}{\psi(r)}\right) dh\ . \qquad [\zeta]\ (563)$$

The value of σ may be found from the equation

$$\frac{\sigma^2}{\psi(\sigma)} = \frac{H}{h_m}, \qquad [\eta]\ (564)$$

so long as $\sigma < r_1$, the upper limit of r; and in the other case $\sigma = r_1$. Therefore, on putting $x = \frac{h\, r^2}{\psi(r)}$, and $dh = \frac{\psi(r)}{r^2} dx$, we have the two equations

$$\left.\begin{aligned} A_m &= \omega \int_0^{\sigma} D(r)\, r^2\, dr \int_{h_m \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx \qquad r_1 > \sigma\,; \\ A_m &= \omega \int_0^{r_1} D(r)\, r^2\, dr \int_{h_m \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx \qquad r_1 < \sigma\,. \end{aligned}\right\} \qquad [\theta]\ (565)$$

In the same way the mean distance ϱ_{mm_1} of the stars between the magnitude m and m_1 becomes

$$\varrho_{mm_1}\left(A_m - A_{m_1}\right) = \omega \int_0^{\sigma} D(r)\, r^3\, dr \int_{h_m \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx - \omega \int_0^{\sigma_1} D(r)\, r^3\, dr \int_{h_{m_1} \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx\,. \qquad [\iota]\ (566)$$

Here σ_1 is to be determined from equation [η] when m_1 is put in place of m. If we prefer mean parallaxes π_{mm_1} in seconds instead of ϱ_{mm_1} we have

$$\frac{\pi_{mm_1}}{0''.2}\left(A_m - A_{m_1}\right) = \omega \int_0^{\sigma} D(r)\, r\, dr \int_{h_m \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx - \omega \int_0^{\sigma_1} D(r)\, r\, dr \int_{h_{m_1} \frac{r^2}{\psi(r)}}^{H} \phi(x)\, dx\,. \qquad [\kappa]\ (567)$$

These formulae (ι, κ) hold so long as $r_1 > \sigma > \sigma_1$. In other cases we have $\sigma_1 < r_1 < \sigma$, and in the first integral $\sigma = r_1$, and σ_1 remains unchanged. Finally if $r_1 < \sigma_1$ then upper limit of the first integral are both $= r_1$. If we introduce a new variable ϱ defined by the relations

$$\varrho^2 = \frac{r^2}{\psi(r)}\,, \quad r = f(\varrho)\,, \qquad [\lambda]\ (568)$$

and call

$$\Delta(\varrho) = D\{f(\varrho)\}\left(\frac{f(\varrho)}{\varrho}\right)^2 f'(\varrho)\,, \qquad [\mu]\ (569)$$

we get

$$\left.\begin{aligned} A_m &= \omega \int_0^{\sqrt{\frac{H}{h_n}}} \Delta(\varrho)\, \varrho^2\, d\varrho \int_{h_m \varrho^2}^{H} \phi(x)\, dx\,, \quad m < n\,, \\ A_m &= \omega \int_0^{\sqrt{\frac{H}{h_n}}} \Delta(\varrho)\, \varrho^2\, d\varrho \int_{h_m \varrho^2}^{H} \phi(x)\, dx\,, \quad m > n\,. \end{aligned}\right\} \qquad [\nu]\ (570)$$

Herein h_n represents the apparent brightness given by the brightest stars at the border $(r = r_1)$ of the sidereal system. In the same way $[\kappa]$ becomes

$$\frac{\pi_{mm_1}}{0''.2}\left(A_m - A_{m_1}\right) = \omega\int_0^{\sqrt{\frac{H}{h_m}}} \varDelta(\varrho)\,\frac{\varrho^2\,d\varrho}{f(\varrho)}\int_{h_m\varrho^2}^{H}\phi(x)\,dx - \omega\int_0^{\sqrt{\frac{H}{h_{m_1}}}} \varDelta(\varrho)\,\frac{\varrho^2\,d\varrho}{f(\varrho)}\int_{h_{m_1}\varrho^2}^{H}\phi(x)\,dx\,. \quad [\xi]\ (571)$$

We may also include the special case in which $m - m_1$ is very small. Since $\psi(r) = \frac{r^2}{\varrho^2}$, we get the formula

$$\frac{\pi_m}{0''.2} = \frac{\displaystyle\int_0^{\sqrt{\frac{H}{h_m}}} \varDelta(\varrho)\,\frac{\varrho^4}{f(\varrho)}\,\phi(h_m\varrho^2)\,d\varrho}{\displaystyle\int_0^{\sqrt{\frac{H}{h_m}}} \varDelta(\varrho)\,\varrho^4\,\phi(h_m\varrho^2)\,d\varrho}, \quad [o]\ (572)$$

which in practice is generally sufficient for nearly all purposes. This fundamental formula is, in fact, included in $[\theta]$, but in a slightly different form. In case no absorption takes place, we have $\psi(r) = 1$, $r = \varrho$, $\varDelta = D$. If the result of enumeration A_m is known, and also the parallax π_m, then the functions $\varDelta$ and ϕ depend on the data of observation, by means of $[\nu]$ and $[\xi]$, which are so-called integral equations. Under certain circumstances, the two integral equations $[\nu]$ determine both functions ϕ and $\varDelta$.

To take account of absorption, we remark that if

$$\varDelta(\varrho) = c\,.\,\varrho^{-\lambda}\,; \quad [\pi]\ (573)$$

then it will follow for any frequency function $\phi(i)$

$$A_m = \gamma\,.\,h_m^{\frac{1}{2}(\lambda-3)} \quad [\varrho]\ (574)$$

Conversely if A_m is given by this formula, then it follows that $\varDelta(\varrho) = c\,.\,\varrho^{-\lambda}$; for if we overlook the absorption, we shall have for $\varDelta(\varrho) = c\,.\,\varrho^{-\lambda}$,

$$\frac{\pi_m}{0''.2} = \frac{\displaystyle\int_0^{\sqrt{\frac{H}{h_m}}} \varrho^{3-\lambda}\,\phi(h_m\varrho^2)\,d\varrho}{\displaystyle\int_0^{\sqrt{\frac{H}{h_m}}} \varrho^{4-\lambda}\,\phi(h_m\varrho^2)\,d\varrho} = \frac{h_m^{\frac{1}{2}}\displaystyle\int_0^{\sqrt{H}} x^{3-\lambda}\,\phi(x^2)\,dx}{\displaystyle\int_0^{\sqrt{H}} x^{4-\lambda}\,\phi(x^2)\,dx} = \Gamma.\ h_m^{\frac{1}{2}}. \quad [\sigma]\ (575)$$

That is, the mean parallaxes π_m for magnitudes $m < n$ are proportional to the square roots of the brightnesses, which was long ago remarked by HERSCHEL. Parallaxes which proceed according to equation [σ] are called "normal" parallaxes. Equation [ϱ] naturally carries with it equation [σ] as a direct consequence. Accordingly if the gauges of the stars confirm equation [ϱ], then the mean parallaxes must proceed according to the square root of the brightnesses. If this does not take place, there must be absorption, and it will be indicated by rigorous mathematical relations.

If A_m follows formula [ϱ] only approximately, it is not always necessary for the π_m to fulfill [σ] with equal accuracy. On the contrary, it may be anticipated that both the integral equations [ν] and [ξ] may be fulfilled by appropriate choice of the functions ϕ and $\varDelta$, when absorption is neglected; that is, a representation of π_m can be forced, but this can happen only in a certain range for A_m, for when [ϱ] is *accurately* fulfilled, [σ] must likewise hold.

It follows from [ν] that A_m is determined only through the function $\varDelta$ and not through D. Wherefore it is impossible, even with a knowledge of ϕ to separate the spacial density D from the absorption so that both may be determined from the apparent distribution of the stars. The attempts which have actually been made with this end in view are frail. Yet such a separation in a certain sense is possible by the use of [ν] and [ξ], since in [ξ] we do not have $\varDelta(\varrho)$ but $\varDelta(\varrho)\frac{1}{f(\varrho)}$. If for example $\varDelta(\varrho)$ is known, it follows from the definition equation for $\varDelta(\varrho)$ that

$$D(r) = \frac{\psi(r) - \frac{1}{2} r \cdot \psi'(r)}{(\psi(r))^{5/2}} \cdot \varDelta\left(\frac{r}{\sqrt{\psi(r)}}\right). \qquad [\tau] \ (576)$$

For $\varDelta\varrho = \varrho^{-\lambda}$, for example, we have

$$D(r) = r^{-\lambda} \cdot \frac{\psi(r) - \frac{1}{2} r \cdot \psi'(r)}{(\psi(r))^{1/2(5-\lambda)}}, \qquad [\upsilon] \ (577)$$

and when we postulate the simplest case of general absorption $\psi(r) = e^{-\nu r}$, it follows that

$$D(r) = r^{-\lambda}\left(1 + \frac{\nu}{2} r\right) e^{\frac{3-\lambda}{2}\nu r}. \qquad [\phi] \ (578)$$

This is an outline of SEELIGER's investigation of the spacial distribution of the fixed stars, substantially as given in *Astronomische Nachrichten*, No. 4359. For further details the reader must consult the original memoir of SEELIGER, *Betrachtungen über die räumliche Verteilung der Fixsterne*, in the *Abhandlungen* of the Munich Academy of Sciences for 1898; and a second paper with the same title in the Munich *Abhandlungen* for 1909.

§ 300. SEELIGER'S *Principal Conclusions.*

(I) "The number of the stars augments with the magnitude so much the more rapidly the nearer the contemplated region of the heavens approaches the Milky Way."

(II) "The number of the fainter stars increases in regions remote from the Milky Way very slowly and in an excessively slower ratio than is the case with the brighter stars."

(III) As for the HERSCHEL stars, SEELIGER shows that in the case of these faint stars, the number increases with decreasing brightness approximately according to the law holding for the brighter stars only in the Milky Way; while outside of this zone the number of the faint stars increases very slowly, and the phenomena of the Milky Way itself forces upon us the conclusion that in that remote depth of space, in which the stars producing this appearance exist, the rigorous law would hold true, and that the number of the stars in the Milky Way alone is subjected to further increase (KOBOLD, *Der Bau des Fixsterne systems*, p. 176).

This reasoning establishes the vast extension of the sidereal universe in the direction of the Milky Way, and its extreme thinness in the direction perpendicular to the Galaxy.

(IV) Taking the magnitude of the brightest stars as —2, SEELIGER concludes that the boundary of the sidereal system in the direction of the poles is, in round numbers, 500, and in the plane of the Milky Way 1100 Sirian units of distance, each corresponding to a parallax of $0''.2$, or about 16 light-years; and he calculates the total number of the HERSCHEL stars at 27,000,000.

(V) He concludes that the hypothesis of the extinction of light in space, advanced by OLBERS and STRUVE, is natural, plausible and well adapted to explaining the small surface luminosity of the background of the heavens, but not adequate to account for the distribution of stars of different brightness, unless we also assume a bounded system with finite density in a limited space.

(VI) In the denser portions of the Milky Way there is more, and in the vacant portions less, matter, than in the average of that zone, where attractive forces are at work; whether the clustering tendency is due to the cumulative effects of gravity or to super-position of chance phenomena, he does not decide; but, like HERSCHEL, he notices the approximate compensation between the bright and the vacant regions, which are often in comparative proximity.

(VII) In our immediate neighborhood the density of stars in the Milky Way decreases more rapidly than in the direction perpendicular to it; but at great distances this proportion becomes reversed, and the decrease becomes rapid

towards the poles and slow near the Milky Way. This is illustrated by the accompanying figure showing curves of equal stellar density, taken from PROFESSOR KOBOLD'S *Bau des Fixsterne systems*, p. 205. The small circle represents the region of the naked eye stars, while the larger circle represents the distance of the stars of ninth magnitude; and in general the curves of equal density run nearly parallel to the plane of the Milky Way, as was long ago imagined by STRUVE.

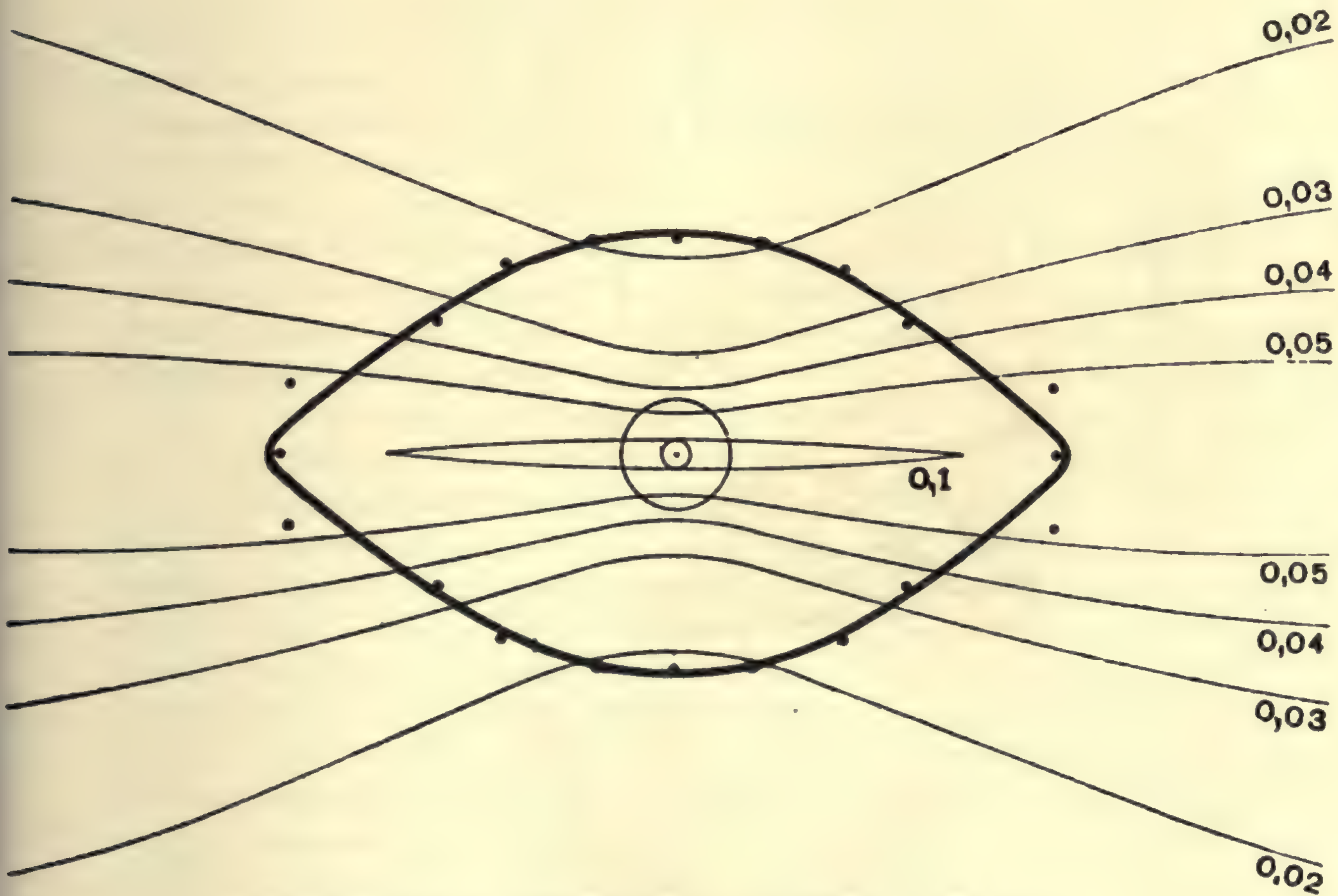

FIG. 46. KOBOLD'S INVESTIGATION OF THE DENSITY OF THE STARS.

(VIII) SEELIGER has shown also that the HERSCHEL stars follow a different law from the brighter stars to the 11.5 magnitude; but KOBOLD remarks that this fact is not yet satisfactorily explained. The following table shows how the ratio of the HERSCHEL to the Durchmusterung stars varies in the nine zones each 20 degrees in width between the Galactic poles.

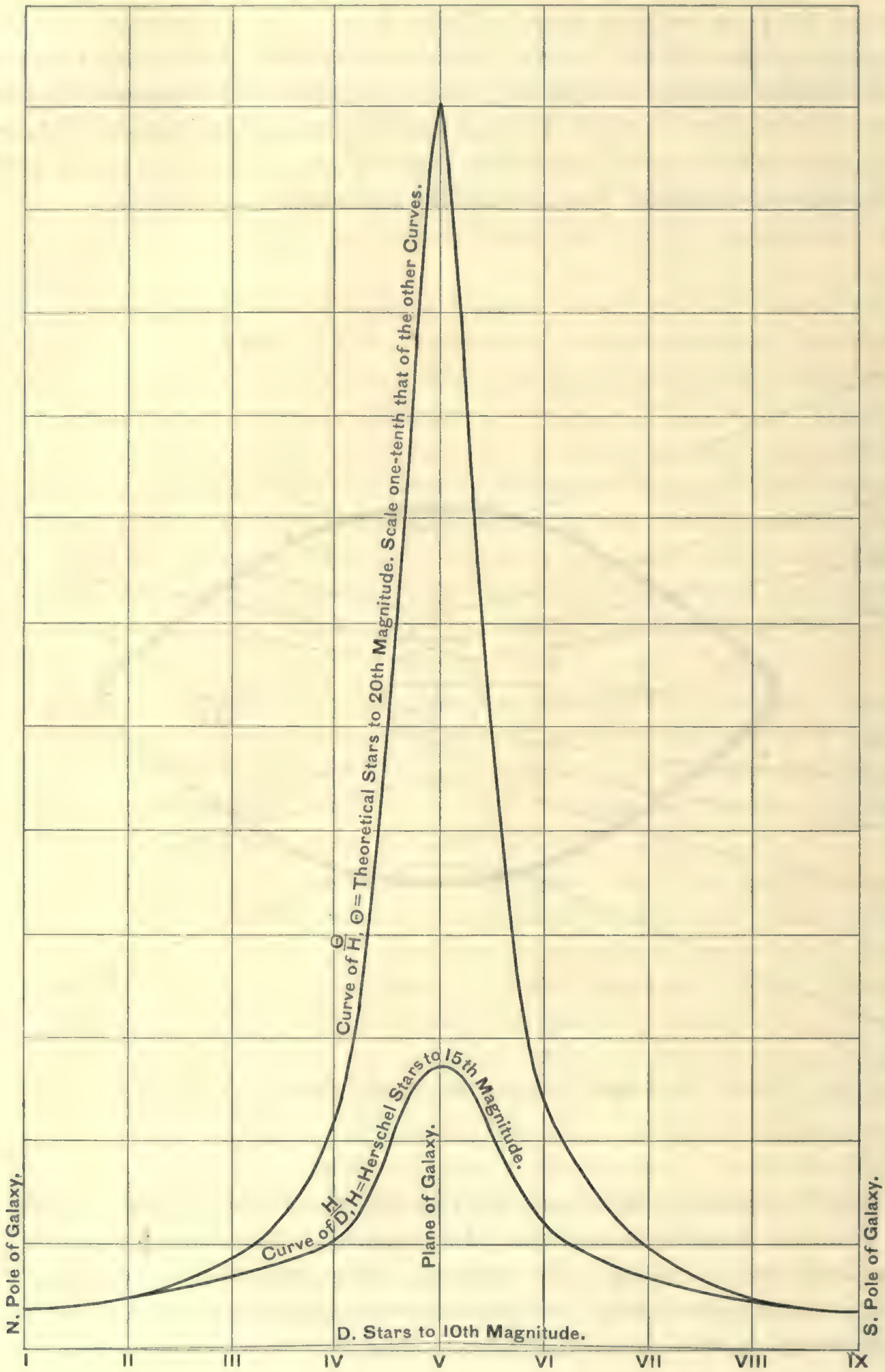

FIG. 47. RATIO OF THE HERSCHEL TO THE DURCHMUSTERUNG STARS, WITH THEORETICAL RATIO OF STARS TO 20TH MAGNITUDE TO THE HERSCHEL STARS, CALCULATED BY T. J. J. SEE.

Zone	H	D	$\frac{H}{D}$
I	107	3.06	35.0
II	154	3.24	47.5
III	281	3.80	73.9
IV	560	5.34	104.9
V	2019	7.36	274.3
VI	672	5.94	113.1
VII	261	3.99	65.4
VIII	154	3.56	43.3
IX	111	3.51	31.6

This law of the relative increase of the Herschel stars towards the plane of the Galaxy is so regular as to be very remarkable, and no cause for it has yet been assigned.

It would seem that the indefinite extension of the starry stratum in the direction of the Milky Way, thus giving a certain proportion of very large stars which might appear at very great distances as the small Herschel stars, owing to the effect of distance and the extinction of light in space, is the only reasonable explanation of this law of concentration of the Galactic light towards the medial plane of that luminous zone. If this explanation be admissible, it will afford a new argument for the indefinite extension of the starry stratum and also for the extinction of light in space, as was long ago inferred by Struve from the excess of the calculated stars of a given magnitude over those actually observed by Herschel in his gauges of the heavens.

The accompanying figure illustrates the tendency of distribution among the stars of the different classes to the 20th magnitude. This theoretical deduction from the laws found to hold true for the *Durchmusterung* and the Herschel stars constitutes what appears to be a legitimate and most powerful argument for the indefinite extension of the sidereal universe and for the unavoidable extinction of light in space. Thus a limit is fixed to the space-penetrating power of telescopes, and the most remote regions of the universe are seen to be veiled in the blackness of impenetrable night; but obviously the borders of the visible universe may be pushed back with each increase of telescopic power. To observe stars of the 20th magnitude, we should require a telescope of about 14 feet aperture. This is within the range of our modern powers of construction and as the limits of the universe would be expanded by 65 per cent., such an effort for nearly doubling the bounds fixed by Herschel cannot be too urgently recommended to the attention of astronomers.

CHAPTER XXIV.

The Laws of Cosmical Evolution.

§ 301. *The Inadequacy of the Detachment Theory as Developed by* Laplace *and his Successors.*

This concluding chapter will consist principally of a summary of the results at which we have arrived in the course of the present volume, and although such a review will be restricted mainly to general considerations, it will not be without value for our grasp of the work as a whole.

In Chapter XV we have considered especially the origin of the solar system, and have adduced various criteria which show the inadequacy of the detachment theory to explain the origin of the planets and satellites. Among the several independent proofs that the bodies of the solar system have not been detached from the central bodies which now govern their motions, by acceleration of rotation, as imagined by Laplace, that based on the mechanical principle of the conservation of areas, and known as Babinet's criterion, appears to be the most satisfactory, and least open to any valid objection. For the dynamical rigor of Babinet's criterion is universally recognized, and the numerical results of its application show that the planets never were detached from the Sun, and that the satellites never were detached from their several planets, as incorrectly held by Laplace and generally believed by astronomers since 1796.

This alone is sufficient to show that the theory of Laplace, which has dominated all our thought in Cosmogony for over a century, is quite devoid of real foundation. In fact, this fatal defect in Laplace's theory is conceded in *Nature* of July 29, 1909, by a distinguished reviewer of the paper on the cause of the circularity of the orbits of the planets and satellites (*A.N.*, 4308). It is true that a criticism is made by this writer in *Nature*, that, because detachment is disproved, capture is not necessarily established; but the arguments given in the present volume answer this objection completely. The results of several other writers who have followed the general conceptions of Laplace need not be considered, since the

work of their great master is shown to be entirely vitiated by a false premise. Thus we have passed over the writings of FAYE, LIGONDES, ANDRÉ, NÖLKE, and many others as not materially contributing to the subject, except in variations, which are not vital, when the foundation itself is unsound. DR. NÖLKE has discussed the capture theory in *A.N.*, 4374, and he agrees with the present author that the irregular satellites are captured, but thinks that the regular ones may have been formed in the atmospheres of the planets. The weakness of this position is apparent.

It will doubtless appear very remarkable to future investigators that we should not earlier have perceived that the roundness of the orbits of the planets and satellites was to be explained by the secular action of a resisting medium and by no other cause whatsoever. The force and accord of traditional opinion, however, was so great that for a long time no one recognized that a false premise had entirely vitiated LAPLACE's original reasoning. BABINET's criterion lay buried in the *Comptes Rendus*, and remained unknown to KELVIN, NEWCOMB, DARWIN, TISSERAND, POINCARÉ and other modern investigators.* And even after I had investigated this criterion and discovered the false premise in LAPLACE's argument, there were many who considered it rash to criticize the traditional theory, lest the author's views be rejected as heterodox and contrary to those held by the majority of astronomers. To the earnest student of nature such arguments plead for popularity rather than truth, and, when I was fully convinced of the erroneous premise, I did not hesitate to attempt the permanent overthrow of the Laplacian hypothesis. This effort has been so satisfactory that probably no one hereafter will ever again give serious consideration to a theory which is shown to be absolutely untenable. In the place of it the explanation that the bodies have been captured and have had their orbits reduced in size and rounded up under the secular action of a resisting medium, is established by rigorous mathematical criteria which leave little or nothing to be desired.

And just as the satellites are collected near the centres of the closed spaces about their several planets, indicating that with the lapse of ages the orbits have been reduced

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* As a specimen of the errors heretofore current we may cite the following passage from LOOMIS' *Treatise on Astronomy*, edition of 1880:

"*How this hypothesis (of* LAPLACE) *may be tested.* — It has been attempted to subject this hypothesis to a rigorous test in the following manner. The time of revolution of each of the planets ought to be equal to the time of rotation of the solar mass at the period when its surface extended to the given planet. It remains, then, to compute what should be the time of rotation of the solar mass when its surface extended to each of the planets. It has been found that if we suppose the sun's mass to be expanded until its surface extends to each of the planets in succession, its time of rotation at each of these instants would be very nearly equal to the actual time of revolution of the corresponding planet; and the time of rotation of each primary planet corresponds in like manner with the time of revolution of its different satellites.

"The nebular hypothesis must therefore be regarded as possessing considerable probability, since it accounts for a large number of circumstances which hitherto had remained unexplained" (p. 315).

in size and rounded up under the secular action of a resisting medium; so also our *planetary orbits were originally of much vaster dimensions and greater eccentricity than at present.* This indicates that with the lapse of ages the planetary system has contracted its dimensions enormously. In the nebular stage the original dimensions of the system may have been one hundred or one thousand times what they are now. The theory here developed is therefore profoundly different from that outlined by LAPLACE, in all its main features; and is believed to be firmly established by the introduction of the necessary and sufficient conditions usually required to establish the truth of theorems in the Science of Mathematics.

In this most exact of all sciences it is not sufficient to show that a given cause will explain a given phenomenon; it is necessary to prove also that it is the *only possible* cause which may be assigned to account for it. Then we have an argument which is entirely conclusive. And such an argument we have endeavored to develop respecting the origin of the planets and satellites.

§ 302. *The Capture Theory Confirmed by the Widest Inductions from Nature.*

Before proceeding with the details of the Capture Theory, it is desirable to consider how far it is supported by the general indications of Nature. It is sufficient to recall the following facts:

(1) That molecules, meteorites, satellites and all kinds of bodies are captured and gathered together in nebulae. For a nebula is formed by the agglomeration of fine dust expelled from the stars by the repulsion of their light and by the electric forces, and when this is collected together it begins to condense and develop into a cosmical system. And whilst the collection and condensation of cosmical dust forms stars, the expulsion of this dust from the stars again forms nebulae, and the capture of cosmical dust is the ordinary process in the formation of nebulae.

(2) In a nebula a vast number of meteorites are formed by the precipitation of ions, whence small globes develop, and as the mass is widely scattered, each small globe has about it a considerable sphere of influence within which its attraction is supreme. The result is that the small globes grow by accretion, and in time very much augment their masses. This again is a process of capture, and consists in augmenting the masses of the bodies forming in a nebula, and in decreasing their number correspondingly. And the process of sweeping up small masses thus suggested certainly is one of the most general of all the laws of nature. Every meteor consumed in our atmosphere is a visible illustration of this process of capturing small particles of cosmical dust, while the craters on the Moon survive

to tell of the capture of larger masses, and of their great abundance in our primordial nebula.

(3) The development of a system from a nebula is a process of gradual clearing up, by the capture and absorption of the smaller masses within it. The very constitution of the solar system thus bears impressive witness to nature's all-ensnaring process of capture by which cosmical systems are built up.

(4) Moreover, within historical times comets have been captured by the perturbations of the planets, after the system was practically free from the existence of a sensible resisting medium. This shows that even when no resistance is at work, a system may gather in and augment its mass by the capture of small bodies passing through it.

(5) In the same way a cluster captures stars, as HERSCHEL long ago remarked, and thus arise those glorious swarms of stars scattered so abundantly over the face of the sidereal heavens.

(6) The same process is exhibited on a stupendous scale by the formation of star-clouds in the Milky Way.

(7) And just as individual stars collect into clusters and star-clouds in the Milky Way, so also the mutual action of all these individually complex masses causes them to circulate as a Galaxy, which probably is the highest order of sidereal vortex.

(8) At the other extreme of the physical universe, we have atoms capturing electrons, and molecules capturing atoms; while vast systems of these small particles connected together give us all manner of physical bodies.

(9) The resisting medium aids in all processes of capture, under central forces, because these forces cause small bodies to drop down towards these centres of attraction.

(10) Under repulsive forces matter is dispersed instead of being captured, and in actual nature both processes are at work together.

In general, nature's process of capture is most effective on comparatively large masses, because attractive forces thus come to predominate over repulsive forces; while in the case of very small masses or particles moving under repulsive forces, diffusion easily predominates over centralization and attraction, and fine particles are dispersed even against the power of gravity.

§ 303. *The Planets Were Formed in the Outer Parts of the Solar Nebula.*

We have seen how a nebula grows by the gathering together of cosmical dust. Much of this is very fine, but some of it collects into larger nuclei, and thus arise bodies such as the satellites or planets. Now in an immense nebula the condensa-

tion goes on at an infinitely great number of centres, and the solid globes thus arising are analogous to our Moon. When the larger of these globes have gathered together a vast number of moons and united them into one mass, the result is a planet. But in the early history of a nebula, multitudes of these moons and planets go into the central body to form the governing sun.

On the one hand this builds up the Sun's mass, while on the other it clears up the inner parts of the nebula. The result is that only planets with orbits of large original perihelion distance survive, and even they have their mean distances and eccentricities greatly reduced. Accordingly we perceive that the surviving planets were formed at a great distance from the Sun, or in the outer parts of the solar nebula. This proposition is important, and follows at once from theorems regarding the decrease of the major axis and eccentricity under the secular action of a resisting medium, which is the true cause of the roundness of the planetary orbits. We have already remarked how untenable it is to consider the planets as any part of the Sun. But along with the fundamental change in our conceptions to the effect that the planets never were any part of the Sun, it is necessary to include another related principle, that when they were formed in the outer parts of the solar nebula, they moved in much larger and more elongated orbits than at present. It may be that some of the planets, if not all of them, had begun to form before they entered the solar nebula, but it is evident that the masses have since been enormously augmented by the capture of satellites.

§ 304. *The Capture of Satellites.*

In Chapters X and XI we have treated at some length of the capture of satellites, and have shown in what a variety of ways the small body may come to move about the two large bodies conjointly and become permanently attached to one of them. This is chiefly by dropping down nearer and nearer these centres of attraction, till the satellite passes within the closed surface about one of the larger masses. When the neck of the hour-glass space connecting the spheres of influence of the two large masses is narrow, the satellite may pass from the control of one mass to that of the other, and a fixed status is not yet established. But if resistance or disturbances* occur by which the velocity changes and the neck closes, the satellite may abide with the body around which it is then revolving. *It thus ceases to be a temporary and becomes a permanent satellite.*

*Throughout this work *disturbances by a fourth body, fifth body*, etc., are always tacitly grouped under the general effects of a resisting medium, since the resistance is supposed to be *nebular*, and nebulae are proved to contain an infinite number of small bodies, some of which attain the size of comets, moons, and planets. The roundness of the orbits is a direct witness to the action of a resisting medium, and this property is so fundamental that all actions which have taken place are naturally included under the corresponding cause.

If the neck of the hour-glass space is just closed by some disturbance of the satellite's relative velocity, it is clear that a subsequent contrary disturbance may cause the neck to open again, and the satellite might afterwards escape. Thus a satellite which was already captured might be again released, but the chance of this occurring is less than that of capture, because resistance tends to bring the satellite nearer and nearer the centres of attraction. And after the capture has been effected long enough to cause the satellite to acquire a constant of the Jacobian integral larger than that at the border of the closed HILL space, it is more difficult for such a satellite to become lost again.

From this line of thought it follows that the capture of satellites is easily possible and actually has taken place.* Under different conditions the satellite may pass very near the planet or the Sun, and by the shifting of the orbit the body come into collision with one of the larger masses. This leads to the absorption of satellites, which has so much augmented the masses of the planets, and also rendered the Sun's mass seven hundred and forty-six times that of all the attendant planets combined.

Accordingly it will be seen that the capture of satellites is a general process in nature, and as applicable to the systems of double and multiple stars as to the solar system. Cosmical systems exist everywhere among the fixed stars, and in all such systems, whether made up of planets or stars of binary, ternary, or higher order, or clusters, it may be assumed that small bodies are circulating and being captured and having their orbits transformed.

§ 305. *Retrograde Satellites.*

In discussing the capture of satellites we have pointed out that in passing from the Sun to *Jove*, it is possible for the satellite to cross the line *SJ* before coming completely under the control of *Jove*, and the result is a path resembling a figure-of-eight. In the *Monthly Notices* of the Royal Astronomical Society for December, 1909, PROFESSOR SIR G. H. DARWIN has treated more critically the problem of the transition from direct to retrograde motion. This is a very important problem, and will give us much new light on the exact dynamical conditions which lead to the origin of the retrograde satellites.

* In *A.N.*, 4408, MR. SELIG BRODETSKY, of Cambridge, England, has treated of the problem of a resisting medium and tried to make it appear improbable that the moon was captured; but as he evades nearly the whole of my argument his criticism has no weight. Moreover he concedes the possibility of capture under certain conditions. It may be a hundred years before mathematicians work out *all the conditions* under which capture may occur; but for reasons given in this volume we already know with certainty that the moon never was any part of the earth, and, capture being a possibility, it can therefore be nothing else than a planet from space. Any other origin of the moon is an absolute impossibility. MR. BRODETSKY is unfortunate in writing from Cambridge, where the terrestrial theory is taught.

In the case of actual satellites of the solar system, it may be difficult to restore the exact processes which took place. It is evident, however, that there are a number of possible disturbances which might give retrograde satellites. Out of all these satellites only a few would survive — a result which accords with the phenomena observed in the solar system. On the other hand, direct satellites are much more frequent, and consequently direct motions predominate about our actual planets, the direct satellites being something like nine-tenths of the whole, and not over one-tenth being retrograde.

These considerations are important, as throwing light upon the magnitude of the vortices about the planets, and the influence of these vortices in accelerating the axial rotations of the planets. This influence seems to predominate over that of tidal friction; and from the conditions furnished by the solar system, it is possible to estimate the tendency to accelerate the rotation of a planet with considerable accuracy.

It is not yet possible to determine the exact ratio of retrograde to direct satellites, but it seems likely that that already derived from the observed satellites is a fair indication of what takes place among the million of unseen particles circulating about the planets, but too small to become visible in our most powerful telescopes. Thus from the observed satellites, we may deduce the approximate ratio of the retrograde to the direct orbital motions among all satellites whatsoever.

§ 306. *The Rotations of the Planets on Their Axes.*

Among all the particles revolving about a planet, some direct and some retrograde, there is a preponderance of direct motion, as we may infer from the nature of the hour-glass surface about the planet, and from the motions of the satellites observed in the solar system. This gives a direct rotation of the planet on its axis. Moreover, this tendency to produce a vortex with direct rotation among the particles circulating as satellites, will exist also among the elements of cosmical dust coming from a great distance and passing near the planet. This follows at once from the nature of the restricted problem of three bodies, whether the particle be attached to and circulating about the two large bodies, or be in rapid flight about the Sun and accidentally passing near the planet, for the tendency to develop a vortex with direct rotation is always present. It follows from this that when such passing particles or meteoric swarms collide with the planet, they accelerate the rotation on the axis in the direction of the planet's orbital revolution. If the planet's rotation be not direct, there is a tendency to make it so; and the absorption

of satellites thus swept up by the planet tends to tilt the equator into coincidence with the plane of the planet's orbit, as in the typical case of the planet *Jupiter*.

In the course of recent researches on the motion of the Moon, NEWCOMB and others have suspected that some cause is at work counteracting the supposed retardation of the Earth's rotation by tidal friction; but it does not appear that any one has heretofore supposed that the cause at work was of such primordial character as it has now been shown to be. As the planetary system does not revolve in a perfect vacuum, but in a space still filled with a very rare medium of cosmical dust, which is being gradually swept up by the planets, as witnessed by the meteors consumed in the Earth's atmosphere, we cannot suppose that the cause which has established the rotations of the planets has yet ceased to act; but on the contrary, we must assume that it is still at work, though of less intensity than formerly. It is not therefore remarkable that there is no observable effect traceable to the secular effects of tidal friction.

§ 307. *The Equatorial Accelerations of the Sun, Jupiter, and Saturn.*

Observational evidence of the existence of vortices about the planets is furnished not only by the motions of the individual satellites which may be seen from the Earth, and by the motions of the particles of *Saturn's* rings shown by the spectrograph, but also by the recognized equatorial accelerations of the Sun, *Jupiter* and *Saturn*. These three globes have been found by observation to rotate most rapidly near their equators, and for a long time this fact seemed puzzling. Attempts were made by WILSING and SAMPSON to account for the equatorial acceleration of the Sun, by the shrinkage of the mass of this gaseous globe, due to secular cooling.

So long as no better explanation was available, this traditional view seemed plausible enough; but the theory of secular contraction due to loss of heat has recently fallen into some disfavor, by the discovery that other more dominant causes are at work. Thus the secular acceleration of the Sun's equator shows that considerable matter is falling into the Sun; and the constancy of the length of the terrestrial day shows that another cause is counteracting tidal friction, which is most likely the downfall of cosmical dust. Such a cause, and such a cause alone, enables us to account for the present rotation of the Earth, with a speed exceeding that of *Mars*, in spite of the fact that the tidal friction on the former is nearly a thousand times more powerful than on the latter. And the phenomena of the Sun, *Jupiter*, and *Saturn* concur in indicating that the vortices about these planets are of appreciable density. The descent of some of this matter against the surfaces of the globes is the easiest and most direct way of accounting for the equa-

torial accelerations observed. Near the surface of these immense masses, a particle would have a large velocity, and be nearly always so directed as to conform approximately to the rotation of the body on its axis. Thus a very moderate number of particles colliding with the planet would keep the equatorial region rotating more rapidly than the polar regions. As this simple explanation accords with the known processes of planetary growth, it must be held to be the true cause of the phenomena observed in nature, and all other explanations are superfluous and improbable.

§ 308. *Acceleration of the Earth's Rotation, Due to the Fall of Cosmical Dust.*

The equatorial accelerations observed on the Sun, *Jupiter*, and *Saturn* naturally represent a tendency also at work on the rotation of the Earth; and as the Earth is a solid body, the fall of meteorites into the atmosphere must augment, if ever so little, the eastward movement of the Earth about its axis. For the impulses communicated to the atmosphere are transmitted to the solid body of the globe by friction, and there thus arises a small secular acceleration of the Earth's rotation. It will not do to suppose that this tendency to acceleration is overcome by tidal friction; for we have shown that tidal friction is nearly a thousand times more powerful on the Earth than on *Mars*, and yet *Mars* rotates forty-one minutes slower than the Earth, in spite of the latter's tidal retardation throughout immeasurable ages. This seems to indicate that some cause opposite to tidal friction is at work accelerating the rotation of the Earth, and that tidal friction has left no appreciable trace of its action on the observed periods of the Earth and *Mars*. And the rotation of *Venus* likewise appears to be thirty-five minutes faster than the Earth, confirming the operation of a similar cause nearer the Sun; and this cause can be nothing else than the vortices of cosmical dust whirling about these planets, and by impact against their atmospheres accelerating the axial rotation.

The theory of the outstanding inequality in the Moon's secular acceleration points to a vortex of cosmical dust whirling about the Earth and shortening the day. And if tidal friction should now be partially counteracting this acceleration, yet as the observed period is less than that of *Mars*, it appears that throughout past time the accelerating tendency has been the most powerful. Should this be so, it is scarcely possible that it can as yet have ceased to act; on the contrary, we must suppose that the day is slowly getting shorter, but that at present the change is excessively small.

The question of the uniformity of the length of the day is one of the most important in the Science of Astronomy; and whilst some progress is being made in its solution, it is not likely that the motion of *Mercury* or the First Satellite of *Jupiter*, by which Professor Newcomb attempted to test the constancy of the

length of the day, will give us any very satisfactory criterion, because of the doubt as to the freedom of the motions of these bodies from disturbing influences. The vortices about the Sun and *Jupiter* might affect the motion of *Mercury* and Satellite I, as these influences might easily produce inequalities in the movement of these bodies of higher magnitude than the supposed changes in the rotatory motion of the Earth. In dealing with this question, therefore, we must simply wait for a measure of time more accurate than that of the Earth's rotation, and it is not yet clear what that will be, if it can be discovered at all.* It cannot fail to be very satisfactory to astronomers to find that the day, after all, is of very constant length.

§ 309. *The Capture of the Moon by the Earth.*

In Chapter XI we have given a somewhat full and critical discussion of the problem of the origin of the Moon, and have shown that it must be a captured planet, for the following reasons: (1) The other satellites were captured by their several planets, and there are no grounds upon which we could justify an exceptional mode of formation in the case of the satellite of the Earth. (2) There are no forces at work in the solar system which could give the Earth such a rapid axial rotation as to bring about a rupture of its figure of equilibrium, as was originally imagined by DARWIN. (3) If such a rupture took place, it is easily shown that the result would be a series of fragments or groups of small bodies, which could never unite into one mass. (4) Investigators have been unable to trace the Moon back to the Earth, on any admissible hypothesis, so that an unaccountable discontinuity in the history of the system opposes itself the moment we attempt to trace the Moon to a terrestrial origin. (5) These contradictions indicate that the old theories rest on false premises, which must be permanently given up; and that the Moon is a planet which came to us from the heavenly spaces. (6) As the capture of the Moon is shown to be possible, and similar captures occurred in the case of the other satellites, we must hold that it occurred also in the case of the satellite of the Earth.

The theory that the Moon is a captured planet explains all the phenomena of the Lunar-Terrestrial system so satisfactorily that it has been adopted by DR. NÖLKE in *A.N.*, 4374, and a similar conclusion seems to have been reached by all others who have studied the question closely. If one still clung to the terrestrial theory, it would be nearly if not quite impossible to account for the large moment of momentum of the Moon's orbital motion; whereas, on the capture theory, this necessarily would result, without postulating any material reduction in the Earth's

* In *Astron. Nachr.*, No. 4409, BARNARD shows that the cluster variable No. 33 of MESSIER 5 (*Libra*), with period of $12^h\ 2^m\ 7^s.30219 \pm 0^s.01781$, may be used as a time constant.

axial rotation. And as tidal friction is nearly 1,000 times more powerful on the Earth than on *Mars*, we cannot postulate such a reduction of the Earth's axial rotation without introducing an even greater anomaly into the rotation of *Mars*, the slowness of which could be assigned to no known cause.

A celestial origin of the Moon offers the only escape from these difficulties, and fortunately such an origin is indicated also by the craters and surface phenomena, which are shown to be due to collisions with asteroids, probably when the Moon was revolving as an independent planet. The capture of the Moon by the Earth may therefore be regarded as a demonstrated fact, which seems destined to become an accepted item of scientific philosophy.

§ 310. *Outstanding Inequality in the Secular Acceleration of the Moon's Mean Motion.*

It is found by the observations of eclipses extending over the past 3,000 years that the Moon has an actual secular acceleration of about 8″.2, while the amount of this movement which can be explained by gravitational theory is only about 6″.2, leaving an outstanding inequality of 2″ per century in a century. It was formerly supposed by FERREL, DELAUNAY, DARWIN and others, that this residual difference was to be accounted for by the retardation of the Earth's rotation due to tidal friction, which would give rise to an apparent acceleration of the motion of the Moon. But it was pointed out by OPPOLZER that it was very doubtful whether this effect would result from the friction in our actual oceans; and the alternate hypothesis of a secular acceleration, due to the downfall of meteoric dust, was suggested to account for the outstanding inequality. OPPOLZER accompanied his suggestion by calculations showing that the postulated effect could be produced by a very moderate downpour of cosmical dust.

In his latest researches on the motion of the Moon, PROFESSOR NEWCOMB reached the conclusion that some unknown cause is at work counteracting and perhaps exceeding in importance the effects of tidal friction; and he commended this subject to the attention of future investigators. The examination of the problem seems to show that both OPPOLZER and NEWCOMB are right; that the downfall of meteoric dust does actually take place and produce an acceleration of the rotation of the Earth about its axis, while at the same time the resistance to the Moon's orbital motion and the increase in the mass of the Earth and Moon produces a small outstanding inequality in the Moon's mean motion, which is shown to be of the order of 2″ per century in a century.

This explanation is confirmed by the evidence of deposits of cosmical dust on the older lunar craters far from the maria; for these craters show the clearest

and most unmistakable signs of ageing, their features becoming rounded and indistinct, as if they were suffering from erosion. But as erosion is not present, the only known cause which could give rise to a similar effect is the downpour of cosmical dust. If this ageing effect appeared only near the maria it might be explained by the dust scattered when these large areas were vaporized; but as it occurs in all parts of the Moon's surface, and is a sure sign that the formations are of great age, it must be that the cosmical dust obscuring the outlines of the craters comes in the main directly from the heavenly spaces. Moreover, if the *satellites* producing the craters come from the heavens, it would be strange if quantities of *dust* did not come also; and we know that this must be so from the meteors swept up by the Earth. Thus the secular acceleration of the Moon's mean motion is connected with the causes known to exist, and to produce the effects found by observation.

§ 311. *The Craters on the Moon Due to Impact of Satellites.*

One of the most remarkable facts established in this volume is the non-volcanic origin of the lunar craters. Not only is it shown that the craters are due to the impact of satellites against the lunar surface, but also that the maria have had a similar origin — in the melting of whole areas, under the heat of great conflagrations arising from the impact of large satellites. And we have shown that the maria would render the lustre of our Moon somewhat variable as seen from a distance; and thus we have connected the maria with the variability in the brightness of the satellites of *Jupiter* and *Saturn*, the cause of which had remained very obscure. The fact of the variation in the light of *Jupiter's* satellites was suspected by Galileo. It is fully established by modern research, and especially by the photometric measures of Guthnick, who has also determined the fluctuations in the brightness of the satellites of *Saturn*. To discover the cause of the maria on our own Moon, and thence deduce the cause of the variability of the satellites in general, is a material advance in the interpretation of the phenomena of the solar system. But if all the satellites are captured planets, and essentially of similar character, why is it not legitimate to assign to the other satellites surface conditions due to the same causes which have been at work on the Moon? It merely happens that our Moon is the only satellite near enough to enable us to understand the details of its surface; but when this is once made out, we may confidently ascribe to the same cause the fluctuations in the brightness of the satellites of *Jupiter* and *Saturn*.

The mystery so long hanging over the origin of the lunar craters is very remarkable, but in view of our difficulty in discovering the true cause of mountain formation on the Earth and the natural disposition to ascribe the mountains on the

Moon to the same cause, we readily see why it could not easily be overcome. It is not, therefore, strange that the fallacious volcanic theory should have persisted from the age of GALILEO to our own time. But if it continues to find adherents in the future, it will simply prove to us that the love of mystery is stronger than the love of truth, which is always true with a considerable number of persons. In his *History of Astronomy During the Middle Ages*, DELAMBRE remarks that the Arabian sceptics were accustomed to employ reasoning which was vague and chimerical. Some modern writers may have similar mental tastes, but they seldom attain to much real grasp of the physical sciences which correctly represent the order of nature.

§ 312. *Craters Similar to Those on the Moon Once Existed on the Earth, but Have Since Been Obliterated by the Oceans and the Atmosphere.*

It is very remarkable that terrestrial mountain formation was long ascribed to the shrinkage of the Earth, whereas it is really due, as was first shown by the present author in 1906, to the secular leakage of the oceans. Owing to our deception in regard to the origin of terrestrial mountains, our failure to understand the origin of the lunar craters was natural; for analogy of process was sought, on the premise that the Moon was of terrestrial origin. And as terrestrial volcanoes were explained by eruptions, the same hypothesis was applied to the lunar surface, in spite of the fact that terrestrial and lunar craters were strikingly dissimilar. But this error is now corrected in a way which ought to be lasting and satisfactory.

It only remains then to speak of the craters analogous to those on the Moon which once existed on the Earth. For since the lunar craters are due to impact, similar impacts of satellites in remote ages must have indented the surface of the Earth also. Of this there can be no possible doubt. But in the case of the Earth, the globe is enveloped by an atmosphere and largely covered by oceans, so that the effect of the impact of a satellite against our planet would be somewhat different from what it was in the case of a barren, airless globe like the Moon. Moreover, on the terrestrial globe, the greatest changes would in time be wrought by the erosive effects of the air and water. Thus craters and maria which might be preserved almost indefinitely, on the Moon, would here be washed away and wholly obliterated. Accordingly if the impacts occurred a long time ago, we should not expect any trace of them to survive on the Earth, but they would still remain comparatively fresh and distinct on the Moon. Now the continents of the Earth show that the mountains and volcanoes are traceable to the sea, while very few are

to be ascribed to satellite impacts;* which seems to show that our Earth has suffered few collisions since geological history began. The phenomena on the Earth and the Moon are therefore in entire accord, when each group is traced to its true cause.

§ 313. *Obliquities of the Major Planets Modified by Capture of Satellites.*

The craters on the Moon illustrate in a clear and unmistakable manner what results take place when a small satellite comes into collision with a solid globe. And as the Moon is literally covered with craters, one would be justified in concluding that the absorption of satellites by collision is a common process among the planets. The large planets, like *Jupiter* and *Saturn*, however, are too far away to show satellite indentations, even if they were solid globes; and as they are gaseous to a great depth, it is impossible to discover from their surface phenomena any evidence of the capture of satellites.

Is there no other evidence of satellite capture which may be recognized at that great distance? Let us recall the obliquities of *Jupiter* and *Saturn*, and the rigorous investigation which we have made of the causes there disclosed. We have remarked that on the average, satellites moving near the plane of *Jupiter's* orbit and colliding with that planet will tend to decrease the planet's obliquity; for the rotations will be compounded according to the principle of the parallelogram of forces. The planet will tend to rotate about an axis becoming more and more nearly perpendicular to the plane of his orbit, and the obliquity will tend to vanish. And whatever may have been the original obliquity, the growth of the planet by the absorption of satellites will tend to give zero obliquity, as is observed in the case of *Jupiter*. Considering the immense influence exerted by *Jupiter* in the capture and transformation of the orbits of comets, it is easy to see that his capture and absorption of satellites in the past has been enormous.

In order to leave no doubt of the numerical sufficiency of this cause to tilt a planet over towards a position of zero obliquity, we considered what would happen to *Saturn* if that planet should have his mass made equal to that of *Jupiter* by the capture of satellites; and we found by rigorous calculation that the present obliquity of *Saturn* would vanish like that of *Jupiter*. Accordingly it was inferred that at one time the obliquity of *Jupiter* may have been as large as that of *Saturn*, or even larger, but has been made to disappear by the capture and absorption of satellites. Therefore, whilst we cannot observe satellite indentations on the outer planets, owing to their gaseous constitution and great distance, yet in the modified

* Such as Coon Butte and Meteor Crater in Arizona.

obliquities we find visible traces of the capture of satellites which are quite as conspicuous as the craters on the Moon.

§ 314. *Modification of the Obliquities of the Other Planets.*

Passing beyond *Saturn* to the planet *Uranus*, we find an obliquity of about 97°, and in the case of *Neptune*, 145°. Now the mass of *Saturn* surpasses those of *Uranus* and *Neptune* in the ratio of about 6.3 and 5.2 to 1, and if their masses were correspondingly increased by the capture of satellites moving near the planes of the orbits of these outer planets, it is certain that the obliquities would be decreased to correspond closely to that of *Saturn*. Hence it has been inferred that the surviving high obliquity of the two outer planets is due to the failure of those two planets to capture an adequate share of satellites.

In fact, it is clear that *Jupiter* robbed *Saturn* of building material, and *Saturn* joined *Jupiter* in robbing *Uranus*, while all three conspired in robbing *Neptune;* and the result was that the outer planets had their supply of satellites so far cut off that the obliquities remained large. Accordingly it appears that *Jupiter* and *Saturn* exhibit most clearly the normal type of development of planetary obliquities; while the development of *Uranus* and *Neptune* was so early arrested that their phenomena can be interpreted only by means of the other phenomena of the solar system. But as *Jupiter* and *Saturn* were able to cut off the supply of building material for the outer planets, we thus obtain an indication that the satellites traversing the solar system moved in orbits of considerable eccentricity; otherwise they could not thus be intercepted by the neighboring planets.

As regards the terrestrial planets, *Mars* has the highest obliquity, and the Earth next; while that of *Venus* is no doubt the smallest of these three planets. This is what ought to result from theory, and it seems to be confirmed by observation; for SCHIAPARELLI and others have inferred that the obliquity of *Venus* does not exceed 10° or 12°. It seems safe to predict that it does not exceed 20°. Accordingly it appears that there are many phenomena which point to the capture of satellites, both in the outer and inner parts of our system: and in the observed obliquities we have a series of phenomena which bear witness to this process as clearly as the craters and the maria on the Moon. These indentations alone exhibit imprints indicating the diameter of the small bodies which were once so numerous in the solar system; and the swarm of small bodies thus disclosed gives us a new and impressive light on the constitution of a nebula, which probably could not be obtained from any other existing source. Thus although our solar nebula has long since vanished, we still have traces of its constitution in the dents preserved

in the face of the Moon, and in the obliquities of the planets, which can supply us with knowledge that never could be obtained from the existing nebulae, owing to their immense distances. The study of the survivals in the solar system is therefore of the utmost value in developing a true theory of cosmical evolution.

§ 315. *The Observed Rotation of Venus, and the Theoretical Reason Why This Planet Should Rotate Faster Than the Earth.*

We have discussed the older observations of *Venus*, and the resulting periods of rotation thus deduced; and have shown that the period of $23^h\ 21^m$ was deduced by careful observers who were free from any theoretical bias. From an observational point of view, this rotation period must be considered the most probable. Some such short period is distinctly indicated by the observations made by the author with the 26-inch refractor at Washington in 1900. And now it has been found that there is a theoretical cause which would make an inner planet like *Venus* rotate somewhat more rapidly than the Earth, while a remoter planet like *Mars* should rotate more slowly. This brings theory and observation into striking accord, and leaves very little doubt that we have at last found the true laws of nature. It is found that satellites moving in eccentric orbits and passing *Venus* would have greater velocity than when passing the Earth; and therefore when they collide with *Venus* they tend to give that planet a greater impetus about its axis of rotation. The result would be that if we integrate the effects arising from an equal mass of such satellites, colliding with two such planets in the course of ages, the cumulative rotation given to *Venus* would be greater than that given to the Earth; and in the same way the rotation of Mars would be still slower. Accordingly if we admit this cause of planetary rotation, it follows that *Venus*, the Earth and *Mars* ought to rotate in periods of increasing length, such as we find by observation. On the other hand, if the rotations be as here indicated, this fact becomes a powerful argument in support of the capture of satellites as the dominant process in the formation of the solar system.

It is shown that the solar tidal friction on *Venus* slightly exceeds that due to the Sun and Moon combined on the Earth, while the solar tidal friction on *Mars* is nearly a thousand times smaller. The present slow rotation of *Mars*, the more rapid rotation of the Earth, and the still more rapid rotation of *Venus* is therefore strongly adverse to the view that tidal friction has exerted a sensible influence on the past history of the solar system. If the facts be as here assumed, we shall have to give up the view that tidal friction exerts a sensible effect

on the rotations of the planets, and recognize in the capture of satellites a much more powerful cause, the influence of which has everywhere proved to be paramount.

§ 316. *Extension of the Solar System Beyond Neptune.*

We have seen that the extreme circularity of the orbit of *Neptune* is a distinct indication that the solar system does not terminate at the present known boundary, but extends much further out. The fact that this orbit is so round shows that the resisting medium was of considerable density at that great distance, and thus indicates the existence of several planets beyond. The orbit of *Neptune* bears a striking parallel to that of *Jupiter's* Fourth Satellite, which was long supposed to be the outer limit of the Jovian system, whereas three remoter satellites have been discovered since 1905. We can therefore no more believe that *Neptune* marks the outer limit of the solar system than that *Jupiter's* Satellite IV indicates the termination of the Jovian system, or that *Saturn's* system terminates with *Titan*. It is impossible to say how many more planets may exist beyond *Neptune*, but there are not likely to be less than three, and there may be more.

And it is practically certain that the system extends to a distance of at least one hundred astronomical units, or over three times the present known dimensions. And just as the satellite systems have contracted their dimensions under the secular effects of a resisting medium, so also the solar system itself must originally have been of much vaster dimensions than at present.

In fact, it is highly probable that our system once extended to a distance of not less than 1,000 astronomical units, and has since contracted its dimensions. We conclude this not only from the phenomena now observed in the solar system, but also from the comparison with the angular dimensions of other nebulae, all of which must be absolutely of immense size. And whilst our nebula evidently was not of such vast extent and mass as some of the larger nebulae which we see in the sky, yet it must have been enormously larger than the dimensions of the developed planetary system as we now behold it. We may therefore fix 1,000 astronomical units as defining the *order of distance* of the outer border of the primordial solar nebula, though the comets extend still further out.

The present vast dimensions of certain spiral nebulae and their conspicuous transparancy indicate that they may hereafter likewise undergo enormous shrinkage. Thus the shrinkage postulated for the solar nebula is but a special case of the general process of nature, as shown by the appearances of the nebulae throughout the sidereal heavens; and this again confirms the theory that the solar system

extends much beyond *Neptune*, and that the primitive boundaries were much more distant still.

§ 317. *The Capture of the Asteroids and Periodic Comets by the Perturbations of the Planets — Views of* PROFESSOR H. A. NEWTON.

The subject of the capture of the periodic comets by the perturbing action of the planets is very familiar to astronomers, and thus requires no further treatment here. But the capture of the asteroids by *Jupiter* has been much less discussed. It is obvious that just as the planet *Jupiter* may capture a comet by transforming its orbit so as to lie wholly within his own orbit, so also this great planet may exert a like influence on the orbits of the asteroids which extend beyond the Jovian orbit. For however these two classes of small bodies are classified as to name, the dynamical effect is the same; and moreover it is extremely probable that the asteroids once were rather large comets which have since lost or would have had *tails*, had their perihelion distances been smaller.

In addition to the conclusion already cited (§ 94, pp. 190-192, and § 177, pp. 376-378) regarding the origin of the Asteroids and Periodic Comets, that these small bodies were thrown into their present positions by the perturbing action of *Jupiter*, it should be noted that a similar result was reached by PROFESSOR H. A. NEWTON, of Yale University, at least sixteen years ago, although the writer did not recall the circumstance till two years after he had independently obtained the same result, and just as the last chapter of this volume is going through the press. Early in March, 1894, PROFESSOR F. L. CHASE and the writer called to see this eminent mathematician, and found him quietly working in his study, while the snow was drifting in high banks about his house. He conversed with us for some time on the perturbations of the asteroids and periodic comets, on which he was so great an authority; and then, pointing out of the window to the way in which the drifting snowflakes were being thrown in behind a heavy bank, he remarked that in a similar manner the small planets and comets were gathered by *Jupiter* within his own orbit, the region beyond being one in which they could not maintain stable movement. The researches of PROFESSOR NEWTON thus confirmed the theory early entertained by STEPHEN ALEXANDER, and more recently verified by CALLANDREAU, and uniformly adopted in this work. As the asteroids are spread over a wide zone, those most remote from *Jupiter* show the cumulative effects of a resisting medium most clearly — the decrease in the mean distance being relatively rapid for a small mass, but very slow indeed for a large mass such as *Jupiter*, as remarked on pp. 235, 236. Carried still further back in time this

reasoning indicates that the nuclei of the terrestrial planets, *Mercury*, *Venus*, Earth, Moon, and *Mars*, once were thrown within the orbit of *Jupiter*, as pointed out on page 399.

Origin of the Body of the Comets to be Sought in the Outer Parts of the Primordial Solar Nebula.

If it be asked what is the most probable origin of the body of the comets, our answer would be that they come, in all probability, mainly from the outer parts of our primordial solar nebula; and as this had immense dimensions, it is not strange that the body of the comets move in elongated ellipses, which closely resemble parabolas in the parts of the orbits covered by observations.

For we have seen that ring nebulae and spiral nebulae are but different phases of nebular vortices whirling and condensing towards the centre. And we have shown that most of the matter in such a vortex goes into the central star, while small masses form the planets revolving around it. In the course of time these planets come to move in orbits which are much reduced in size and nearly circular. But comets receding to the original border of the solar nebula would revolve in very long periods, and therefore some of them might escape capture by the planets and still revolve in orbits with very large major axes.

Accordingly it looks as if the comets come from an outer shell about our Sun, which corresponds to the original border of the primordial solar nebula. This is the so-called "Home" of the comets which PROFESSOR PEIRCE and others have imagined, but never fully understood.

If such a connection be admissible at all, the explanation here given has great inherent probability, owing to the analogy with other nebulae. Moreover, it accounts for nearly all known phenomena in a very satisfactory manner. The distribution of cometary orbits about our Sun is not yet fully investigated; but it is generally believed to be comparatively equable in the various directions, with about as many orbits retrograde as direct.

Doubtless this is only approximately true, and when later and better statistical data are available it will be found that there is some preponderance of direct motion, and moreover, some preference for a particular zone, corresponding to the principal plane of the solar system. Such an outcome would seem to follow from the theory here adopted.

In addition to the comets which come to us from the original "Home" at the borders of the primordial solar nebula, there may be a very few of these masses which enter the solar system from the regions of the fixed stars. If such comets

exist they will no doubt have a distribution in space depending on the secular motion of the Sun, and of the nebulosity in the neighboring regions of space. It seems barely possible that both groups of comets will be found to be confused together; if so it will be some time before our knowledge is sufficiently advanced to enable us to disentangle them. Consequently the theory now suggested and more fully developed below must be regarded as provisional, but there can scarcely be any doubt that it rests upon a substantial basis of truth.

The Zodiacal Light and the Gegenschein Evidence of Cosmical Dust in Planetary Space.

The current theory that comets and other fragments of our primordial solar nebula are now disintegrated and scattered over the interior of our solar system seems to account for such phenomena as the Zodiacal Light and the Gegenschein. Both of these latter objects are composed of cosmical dust so situated in space and so well illuminated that these aggregations of particles become visible from the Earth.

The Zodiacal Light about the Sun is evidently produced by the fragments of comets, meteor swarms, and nebulosity, reflecting the Sun's light with sufficient intensity to be observable after sunset. The matter producing the Gegenschein is of similar character,* but perhaps finer, and certainly much nearer the Earth; and moreover, rendered visible to us by virtue of the darkness of the background of the sky at night.

These phenomena prove to us that a certain amount of cosmical dust pervades the whole of the space occupied by the solar system. This cosmical dust is the same kind of material as that seen in the meteors daily swept up by the Earth. In the course of ages it falls in such quantities as to cover up and largely obliterate the older craters on the Moon; and obviously the mass of the Earth as well as the masses of the other planets and satellites and of the Sun are also increased.

The deposit of cosmical dust increases the masses of all these bodies and accelerates their orbital motions. It may be said also that the Zodiacal Light and the Gegenschein establish the existence of vortices about the Sun and planets of the kind we have described; and this confirms the origin of the axial rotations of these bodies.

§ 318. *Historical Sketch of the Theories of the Origin of Comets.*

In § 178, p. 382, we have outlined very briefly the most probable theory of the origin of the comets. This subject is so important for a comprehensive view

* Compare also the theory of INNES, mentioned in the footnote on page 626.

of the solar system and of the nebula from which it developed, that we shall here consider it at somewhat greater length. KEPLER *declared that there are as many comets in the heavens as there are fish in the sea, but he does not seem to have developed any theory of the origin of comets.* NEWTON *recognized that they range all over the heavens in eccentric orbits and are a sort of planet* (*Principia*, Lib. III, Prop. XLI, Prop. XXI), *yet he assigned no reason why the comets should depart from the Zodiac in which the planets move, and the illustrious author of the Principia probably had been unable to develop a satisfactory theory of comets.* In his original formulation of the nebular hypothesis, 1796, LAPLACE regarded the comets as foreign to the solar system; and for a long time this view was generally held by astronomers.

KANT, on the other hand, at an earlier period, 1755, declared that there was no essential difference between planets and comets, and that the two classes of bodies would be found to be merged together by insensible gradations; so that the differences at last analysis disappear. He saw the increase of the eccentricity of *Saturn's* orbit, on the outer parts of the solar system as then known, and thence inferred that the remoter planets would have eccentricities approaching those of the comets. "We cannot make out of the comets a special species of heavenly body which is entirely distinct from the family of the planets. Nature works here, as elsewhere, by gradual steps, and whilst she runs through all degrees of change, she unites by means of a chain of connecting links the remoter properties with the nearer ones."

In his celebrated "Astronomical Observations Relating to the Construction of the Heavens, Arranged for the Purpose of a Critical Examination, the Results of Which Appear to Throw Some Light Upon the Organization of the Celestial Bodies," read before the Royal Society, June 20, 1811, and published in the *Philosophical Transactions* for that year, SIR WILLIAM HERSCHEL devotes Section 22, p. 306, to nebulae that have a cometic appearance: "Among the numerous nebulae I have seen, there are many that have the appearance of telescopic comets. . . . . It seems that this species of nebulae contains a somewhat greater degree of condensation than that of the round nebulae of the last article, and might perhaps not very improperly have been included in their number. Their great resemblance to telescopic comets, however, is very apt to suggest the idea that, possibly, such small telescopic comets as often visit our neighborhood may be composed of nebulous matter, or may in fact be such highly condensed nebulae."

The views of HERSCHEL as thus set forth were approved and adopted by LAPLACE in the *Connaissance des Temps* for 1816, p. 213, but this did not greatly modify the views already announced in the *Système du Monde* twenty years before.

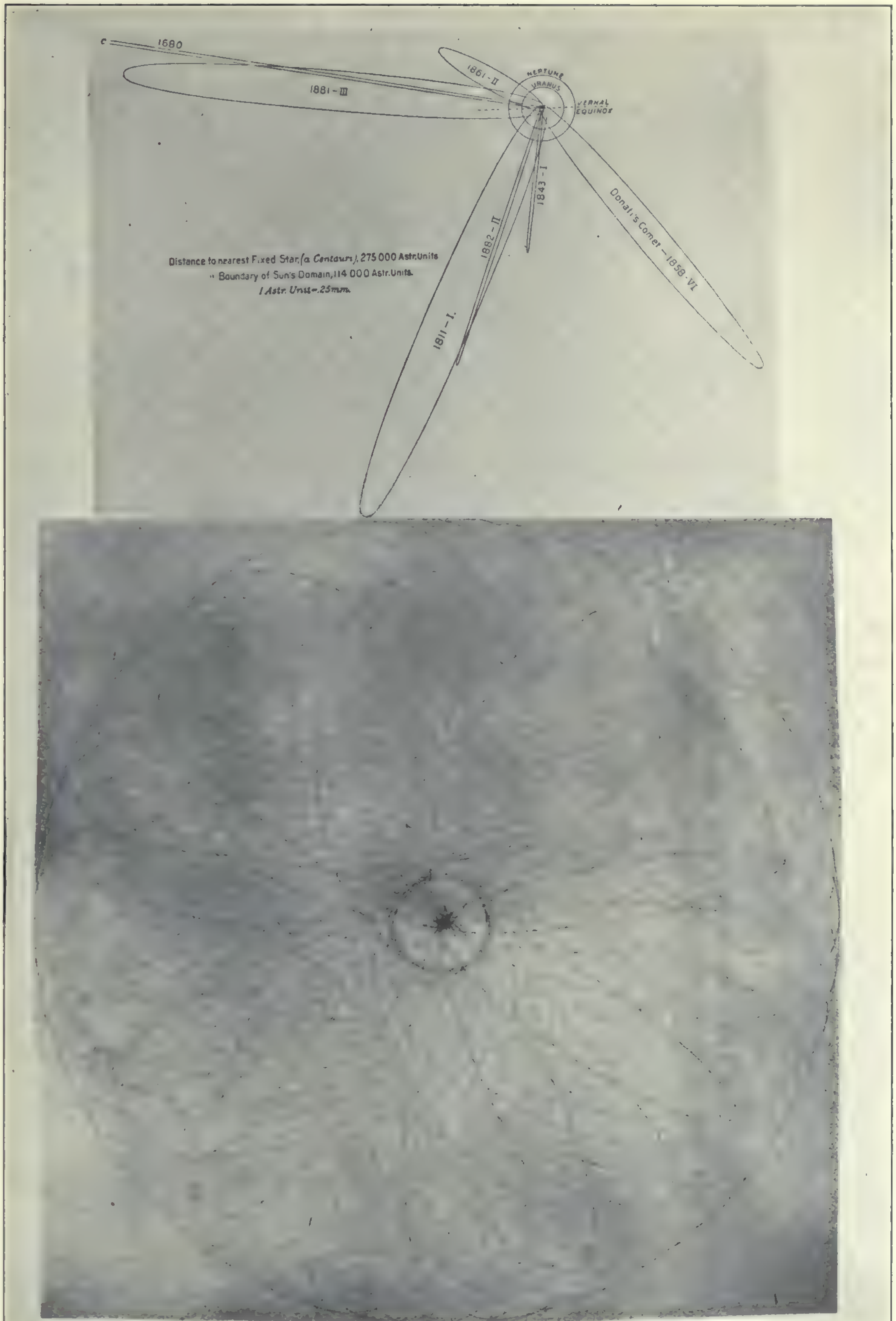

PLATE XXVIII. ILLUSTRATIONS OF THE NEW THEORY OF COMETS DEVELOPED BY T. J. J. SEE.

THE LOWER FIGURE ILLUSTRATES THE VAST SYSTEM OF SMALL BODIES CIRCULATING ABOUT THE SUN AND CONSTITUTING THE OUTER SHELL OF THE PRIMORDIAL SOLAR NEBULA. THE INNER PORTION OF THE ANCIENT NEBULA HAS BEEN GRADUALLY EATEN OUT AND RENDERED VACANT IN THE FORMATION OF THE PLANETARY SYSTEM, LEAVING ONLY THE OUTER PART FOR THE SUPPLY OF COMETS. A FEW OF THESE TINY WISPS FROM TIME TO TIME DROP DOWN AND COME WITHIN THE RANGE OF OUR VISION, WHILE THE VAST MAJORITY REMAIN FOREVER INVISIBLE. THIS CONFIRMS KEPLER'S VIEW, THAT THERE ARE AS MANY COMETS IN THE HEAVENS AS THERE ARE FISH IN THE SEA. THE UPPER FIGURE, FROM LOWELL'S SOLAR SYSTEM, SHOWS THE ORBITS OF A FEW ACTUAL COMETS WHICH HAVE APPEARED IN THE SHORT INTERVAL SINCE NEWTON'S FAMOUS COMET OF 1680.

The difficulty of accounting for the different types of cometary orbits has always been considerable, and the supposed existence of hyperbolic, parabolic and elliptic paths has not unnaturally increased the mystery attaching to comets. If LAPLACE's theory that the comets are foreign bodies were true, we should expect to have a great number of cases of hyperbolic motion, unless the hyperbolic comets pass so far from the Sun as to escape observation, which was the view tentatively adopted by the great French mathematician. This theory of LAPLACE is discussed in § 68 of this volume of *Researches*, pp. 126-129.*

At the present time only about a dozen comets are known which were supposed to have described hyperbolas, and even in these cases the true state of fact is very uncertain, in all except two or three instances, owing to the inferiority of the older observations. And as for the comets assumed to be describing parabolas, it is now well known that all or very nearly all of these paths are really ellipses with very long major axes.

We may therefore dismiss the hyperbolic comets as doubtful, and consider that all comets move in ellipses with great eccentricities and very long periods, so that they resemble parabolas in the parts of the orbits near the Sun over which the observations extend.

One great object of astronomy to-day is to follow the comets as far into space as possible. For this pursuit we need telescopes of enormous light-grasp, on account of the feebleness of the Sun's light in the remoter depths of space, and because of the shrinkage of the head and disappearance of the tail of a comet as soon as it recedes more than a very few astronomical units from the Sun. The insignificance of the supposed solid nuclei of comets and the darkness of the outer regions of our system renders the pursuit of these small masses to great distance extremely difficult.

In the investigation of the nature of comets, the theory of foreign origin proposed by LAPLACE has long been prominent, though of late years astronomers have begun to reject it. About thirty years ago PROCTOR added fresh interest to an old subject by proposing an ejection theory — that the comets are small masses of matter ejected from the heavenly bodies by eruptions, as in the ejection of prominences from the Sun. There may be some mechanical difficulty in conceiving the comets as ejected from distant suns, but since the development of the theory of the repulsion of cosmical dust by the radiation pressure of light and by

* Since this was written I find that conclusions similar to those adopted by the author, as to the nature of comet orbits, have been reached by SCHIAPARELLI as far back as 1871 (*Mem. dell' Istit. Lombardo*, t. XII, p. 164), and more recently by FABRY, who published an important memoir on the subject in 1893 ("Étude sur la Probabilité des Comètes Hyperboliques," *Annales de la Faculté des Sciences de Marseille*, 1893, p. 158). The reader is referred to the memoir of FABRY for a full treatment of the problem of the entrance of comets into our system, which is more satisfactory than the theory of LAPLACE on this subject.

electric forces, such arbitrary ejections of matter as PROCTOR imagined are altogether unnecessary, because it is simpler and more natural to suppose the comets formed by the precipitation and aggregation of the cosmical dust diffused from the stars under the action of Repulsive Forces.

Thirty or forty years ago it began to be realized that most of the comets belong to the solar system, and the "Home of the Comets" was a subject of investigation by PROFESSOR BENJAMIN PEIRCE, the celebrated mathematician of Harvard University. Other astronomers before and since have attempted to solve the problem of the origin of comets, without, however, attaining entire success. PEIRCE correctly held that they probably come from a "Home" conceived as a shell of nebulous matter, at a great distance from the Sun, but he could not assign any definite reason why the comets should have such a "Home."

§ 319. *Confirmation of the Origin of Comets from the Solar Nebula.*

As it has been proved in this work that our planetary system arose from a spiral nebula, and that the planetary orbits were originally much larger and much more eccentric than they are now — having in the course of ages been greatly reduced in size and rounded up by moving in the nebular resisting medium formerly pervading the system — it is now known that our primordial nebula once extended at least a thousand and may be ten thousand times the distance of the Earth from the Sun. Accordingly it would appear that this remote shell of the old solar nebula is exactly where the comets come from.

We observe in the heavens great numbers of spiral nebulae of vast extent and extreme tenuity produced by the settlement and coiling up of unsymmetrical clouds of cosmical dust under the action of universal gravitation. In some cases these coils close around and produce a nearly complete ring like that observed in the famous ring nebula in *Lyra*; in others the nebulosity is scattered somewhat equally over the whole surface of the sphere, as in the *planetary nebulae* so much studied by SIR WM. HERSCHEL. This suggests to us a very good conception of the shell of dark, nebulous matter still surviving about the solar system and furnishing us an abundant supply of comets, which come to us from all directions in space. Let us examine the question a little more carefully.

If we conceive a nebula to be made up of cosmical dust gathering from various directions in space, it is clear that in the settlement under gravitation the usual unsymmetrical figure will produce a rotation about some axis, and this will give rise to a spiral structure. A cosmical system, such as that of the Sun and planets,

will in time develop in the centre, while the orbits will become smaller and smaller and more and more circular, until the system occupies but an infinitesimal part of the sphere of the original nebula. Yet, since the Sun controls the outer shell as well as the inner sphere into which most of the matter has been collected, the remoter wisps of nebulosity also will circulate about the Sun with extreme slowness, owing to the great distance, and usually remain out of our range of vision. Those wisps which, for any cause, drop down into the central system, however, will have their orbits transformed by the action of the planets, as in the observed case of the comets. This explains the shell of dark, nebulous matter giving the mysterious comet-dropping envelope surrounding the Sun, the nature of which heretofore has proved so bewildering to astronomers.

Accordingly the study of the spiral nebulae enables us to make out the nature of the "Home of the Comets" and to understand the character of their orbits, which have engaged the attention of mathematicians for centuries.

In the *Philosophical Transactions* for 1789, SIR WM. HERSCHEL has a "Catalogue of a Second Thousand of New Nebulae and Clusters of Stars," in which he examines the constitution of clusters, and (on page 219) the evidence they present of the action of central powers which have given them the observed globular forms. He cites the globular figures of the planets, and the spherical figures of the nebulae, with increasing brightness towards the centre as the effect of gravity, and estimates the age of a cluster by the state of its condensation. HERSCHEL again examines these questions at length in his "Astronomical Observations Relating to the Construction of the Heavens," in the *Philosophical Transactions* for 1811, and there shows (p. 299) that the tendency to spherical arrangement observed in so many nebulae can be explained only by the action of universal gravitation (cf. also *Phil. Trans.*, 1789, p. 225).

Now the solar system is known to be very old, and if the dust forming the primordial nebula was originally assembled from all parts of the heavens, the outer shell, where hydrostatic pressure has never operated, ought still to have its elements arranged in a haphazard manner, so that the wisps of nebulosity, dropping down to us as comets, would come from all directions indiscriminately. And if the comet-dropping envelope at first were not entirely symmetrical, it would tend to become more so in time, from the effects of gravitation, as in the case of the clusters and nebulae considered by HERSCHEL. *For the cometary masses surrounding our system at a distance of thousands of radii of the Earth's orbit, would be essentially a feeble cluster, though quite devoid of luminosity, and with the lapse of ages thus tend to become essentially globular.*

Accordingly it appears probable that all the inner parts of the nebula have been cleared away in producing the Sun, planets and satellites, but that many small masses

still revolve as survivals in the outer shell of the old nebula, and these are the comets. Hence they move in long ellipses, and in former times it was not unnaturally supposed that the orbits are nearly parabolic. This opinion has been changed by later researches, and the proof of elliptic orbits has led us to the true significance of comets and of the comet-dropping envelope about the Sun. And although it has taken astronomers nearly three centuries* of research to develop a consistent theory of the comets, the result is worth all the effort which has been put on it. Nor has the difficulty been greater than might have been expected in dealing with the most mysterious of all the heavenly bodies, which were long believed to defy the tendency to law and order found to characterize the motions of the planets.

§ 320. *Why Comets Have Tails.*

The discovery and proof of the origin of the comets, from the outer parts of the ancient nebula which formed the solar system, carries with it also the explanation of why comets have tails. It is shown in these *Researches* that the nebulae are formed by the expulsion of fine dust from the stars. This cosmical dust is repelled by the radiation-pressure of the Sun's light, and by electric forces, and thus drifts about hither and thither till it collects into clouds called nebulae, which settle down under the secular action of universal gravitation and form cosmical systems. The spiral form in the nebulae is the outcome of this gradual settlement. Now all the matter in a nebula was once expelled from the stars, and therefore any wisp of nebulosity will have a volatile part. This is easily vaporized when the comet comes near the Sun, and therefore some of it will again be driven away by the Sun's repulsive forces.

If the comet should stay permanently near the Sun it would soon lose all its volatile matter, and thenceforth have no more tail than a satellite or an asteroid has now. But as the comets move in long ellipses, they are near the Sun but a comparatively short time, and the tails then developed cease to grow as soon as the comets recede. When the comet is far away in space it gathers up some more volatile matter before the next return, so that when it approaches the Sun it can again produce another tail. Most of the short-period comets, such as ENCKE's Comet, which has been near the Sun ever since its capture by

* As illustrating the unsatisfactory state of our knowledge of the origin of comets up to the present time, it may be mentioned: (1) That in the HALLEY Lecture at the University of Oxford, May 10, 1910, DR. HENRY WILDE, F.R.S., chose as his subject "Celestial Ejectamenta," and maintained that the comets originated within the solar system, by explosive discharges from the planets, especially the larger planets, in the process of cooling; (2) That in *Scientia* for April, 1910, DR. A. C. D. CROMMELIN of the Royal Observatory, Greenwich, considers the same theory and says that it cannot be summarily rejected. He mentions also the theories that the comets may be ejected from the Sun, and that they may be fragments of the solar nebula, but is unable to reach a definite conclusion on the subject.

Jupiter ages ago, now have no tails, and this is the cause of it; which confirms the present theory, and shows that the comets, asteroids and satellites are all related and simply different products of our ancient nebula.

The comets develop tails on approaching the Sun because they still contain volatile matter which is repelled when they are near the perihelia of their eccentric orbits; whereas the asteroids and satellites have lost all their volatile matter long ago. Yet it is not to be doubted that, if an asteroid or satellite had its orbit so changed as to come very near the Sun, as a comet does, it, too, would develop a tail like a comet. The connection thus established between comets, asteroids and satellites, and the relationship of all these masses to the fine dust expelled by the stars to form the nebulae, completes our theory of the evolution of the heavenly bodies, and gives us beyond doubt one of the greatest laws of nature.

Heretofore we have had no satisfactory theory of the origin of the nebulae, nor of the comets; but all these masses have been treated as distinct and isolated groups of bodies without mutual relationship. And the problem of why comets develop tails has remained almost as mysterious to astronomers as the objects themselves have been to the generality of mankind. The complete theory of the universe involves the mutual interaction of attractive and repulsive forces,* which is so beautifully illustrated by the phenomena of comets, moving in orbits with wide range of eccentricity and developing tails of amazing length and variety.

The great comet of 1811 was calculated by Sir Wm. Herschel to have a solid nucleus 428 miles in diameter, and therefore probably equal to the largest asteroids in mass. Not all comets are without solid nuclei; but the fact that many of them are is the chief reason why we have treated of the capture of satellites, etc., rather than of the capture of comets, the impact of comets, etc., in the theory of the lunar craters and other phenomena. All these bodies are parts of the ancient nebula which formed our system, and they should not be viewed separately and in isolation, but in their mutual relation as different parts of the solar nebula. This was strongly suspected by Stephen Alexander in 1851,† and more recently by H. A. Newton and Callandreau, in the case of the asteroids and periodic comets; but the subject is so important that it deserves renewed emphasis in connection with the larger problem of cosmical evolution.

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* In regard to the intensity of the repulsive forces observed in the tails of particular comets, compared to gravitational attraction, it may be stated that the results vary widely. Thus Brédikhine finds a value of 36, Jaegermann 80; while direct measures on the tail particles of Comet Morehouse (1908*c*) led Eddington to values ranging from 100 to 800 (cf. *Monthly Notices* of the Royal Astronomical Society for March, 1910). Mr. Eddington also obtained values ranging from 180 to 19,000, and thus enormously greater than had been previously supposed from the measures of the tail material of comets. This is a subject on which we still have much to learn.

† "On the Similarity of Arrangement of the Asteroid and the Comets of Short Period, and the Possibility of Their Common Origin," in Gould's *Astronomical Journal*, No. 19, p. 147, and No. 20, p. 181, 1851.

§ 321. *Concluding Considerations on the Origin of Comets.*

(1) Owing to the high eccentricities of their orbits, it has never been held that comets were thrown off from the Sun or other bodies of the solar system, except possibly by the process of ejection, imagined by PROCTOR.

(2) The comets were not ejected from our Sun, because their perihelia usually are too remote from the supposed points of ejection in the Sun's globe, to which they would necessarily return at the end of a revolution; and from the wide range of the aphelia it follows that they are not ejected from the planets.

(3) The comets are not strangers coming from other fixed stars, because nearly all the orbits are sensibly elliptical, and the motion periodic, though the periods often are of great length.

(4) Being neither strangers from other systems nor ejections from the Sun and planets, the comets can be nothing else than wisps of nebulosity from the outer parts of our ancient nebula.

(5) As the periods are very long they may revolve safely for many millions of years, and would long escape destruction; nevertheless it seems probable that their orbits would be gradually transformed and the comets finally disintegrated and reduced to meteoric dust.

(6) And in the same way new comets now invisible from the Earth, owing to their great perihelion distances, may from time to time be brought within the range of our vision, to take the place of the comets which are gradually destroyed by secular action of the Sun and planets.

(7) This supply of fresh nebular material from the outer parts of the system gives us meteorites moving with essentially parabolic velocity, which might be inferred to be hyperbolic in certain cases, and others with elliptical velocity appropriate to the solar system.

(8) Thus we have a simple and natural explanation of comets, meteors and kindred phenomena, which have long proved bewildering to astronomers; and the whole conception is a necessary result of the development of our system from a nebula of vast extent.

§ 322. *Spiral Movement Among the Nebulae, Clusters and the Milky Way.**

The spiral character of the movement in nebulae is impressively illustrated by the plates given in this work. We have also discussed quite fully the theory of this movement, and shown that it is due to settlement of streams towards a centre. No other theory of the whirlpool nebulae is tenable. In this settlement with whirling motion about the centre we see the operation of one of nature's greatest laws.

Stars and planets form in the streams as they coil up, and eventually the orbits are reduced in size and rounded up by the secular effects of the resistance encountered. These whirlpool nebulae give us much light on the primordial state of the solar system; and our knowledge of the solar system in turn enables us to throw a clearer light on the ultimate fate of such a whirlpool nebula.

It is fortunate for Science that we live in a mature solar system and can combine our knowledge of its mode of formation with that of other systems just beginning to develop and therefore still in the nebular stage. If we lived in a nebula our vision of the stellar universe would be obstructed by surrounding nebulosity, and since the planetary systems among the fixed stars necessarily would be invisible, as at present, we could form no correct idea of the essential nature of a mature planetary system. But living in a highly finished planetary system, we have been able to make out its mode of formation and also discover its primitive state from the study of the spiral nebulae, which are visible on account of their continuous streams of nebulosity, showing the nature of the movement, even when a lifetime is too short an interval in which to observe it.

The movement in the clusters and in the Milky Way is likewise of spiral character; that is, the bodies move in streams, and there is rotation of the whole mass about an axis and motion of parts along the stream lines. The elements of nebulosity move along the tangent of the streams very gradually, while the stream itself is wound closer and closer by the whirling about the centre. The nature of the streams is different in different cases, so that a great variety of spiral movement results.

In the Milky Way the scale of every sub-galactic system is so immense that the circuits can hardly ever be closed, nor is it easy to make out the star streams which make up a sidereal vortex. In the clusters, however, the limits of the

* The theory of the spiral arrangement of the Milky Way adopted in this volume was completed in February, 1910. Sir David Gill, President of the Royal Astronomical Society, at the meeting of June 10, 1910, has since expressed views so very similar that we quote them as given in the *Observatory* for July, 1910, p. 272:

"For these and other reasons I cling somewhat at present to the idea that, if we were to view our universe from a sufficiently distant point in space situated in a line nearly at right angles to the plane of our Milky Way, we should see the Milky Way as something like the great nebula in *Andromeda* or the spiral nebula in *Canes Venatici*, and that our Sun, together with its surrounding cluster of stars, would appear like one of the condensations in these nebulae. There is yet no proof of this speculation, it is merely thrown out as a suggestion or guide in future research."

systems are more definitely fixed, and the circuits are essentially closed, though we cannot trace them except in the case of nebulous clusters. Yet the nature of the movement will always be of this type; and the same principle applies to the immense movements of the Milky Way, though we may never be able to make out the details of their true character with much certainty.

All Stars Rotate on Their Axes, and Those Which Appear to be Single Have Planetary Systems Revolving About Them.

We have found that the planets were formed at a great distance from the Sun, and have since approached the great central body by which they are governed, warmed and lighted. From the nature of the mechanical process at work in condensation, as illustrated by the vortices observed in the spiral nebulae, it follows that every star has a motion of rotation about an axis. It is absolutely impossible for the nebulous matter to form a star, by collecting together from a state of wide diffusion into one of much compression, without producing a whirling vortex as it condenses towards that centre. In this original nebula planets begin to form, and in the condensing vortex some of them will grow by accretion and survive; and when mature they will have captured systems of satellites like those captured by the planets of the solar system.

It should be noticed, on the one hand, that nebulosity in the finest known state is expelled from the stars and drifts hither and thither through the universe, till it collects into cosmical clouds or nebulae; and, on the other, that these in turn condense and form fixed stars surrounded by cosmical systems. Thus a star captures and develops a system of planets, and the planets in turn capture systems of satellites. *This is the inevitable outcome of the condensation of the cosmical dust expelled from the stars; while the dust collecting into nebulae is originally dispersed by the intensity of repulsive forces which may be traced to the high temperature and intensity of the light and electric forces operating in the stars.*

§ 323. *On the Tendency to the Development of Oblateness in a Globular Nebula with Haphazard Motion of Its Elements but Having a Resultant Moment of Momentum About Some Axis and Losing Energy by Collision and Radiation.*

With the conditions here specified the nebula is assumed to be devoid of true hydrostatic pressure, and while the external figure of the nebula is taken to be essentially spherical, it is not supposed that the velocities of the individual elements,

the internal distribution of density, etc., are such as to give the whole nebula no resultant moment of momentum about an axis; on the contrary, it is assumed that such internal irregularities, with a resultant moment of momentum, do exist, and that the nebula is therefore slowly acquiring a motion of rotation and losing energy by collisions among the particles as they freely circulate in all manner of directions. The paths are not strictly re-entrant, owing to the attraction of neighboring masses, but for our present purposes we may take them to be temporarily elliptical orbits, slowly changing their elements by collisions, nebular resistance and the disturbance of neighboring masses. Here are the most important results of the motion of such a non-conservative system:

(1) The total moment of momentum of the entire system remains unchanged, however much energy is lost by collision and radiation, and whatever transfers of moment of momentum may take place between the different bodies of the system.

(2) If the shrinking globular nebula be contracting, from loss of energy by friction in collision, the particles will fall towards the centre of attraction of the whole mass, which usually is not the centre of figure, but some point in the plane of maximum areas or equatorial plane, and thereby acquire accelerated angular velocity, in order to keep the areas constant, under the central forces, which may vary in any manner.

(3) Those particles which do not move near the fundamental plane of maximum areas will by mutual collision from either side have their motions compounded, while retrograde motions are destroyed, so that most of them will come to move near the plane of maximum areas. Thus the globular nebula will become an oblate spheroid by the effects of perturbation, collision, and secular settlement.*

(4) The energy lost by radiation from the system is potential energy, due to condensation, and it comes about by the particles dropping down towards the centre, and towards the plane of maximum areas where most of them revolve. The dominant attractive force is not central strictly, but always directed to some point in the fundamental plane, where the particles accumulate to form the oblate disc into which the globular nebula gradually settles by development of rotation.

(5) Therefore at the same time that the resisted particles drop nearer the centre, they drop also nearer the plane of the equator; and by successive collisions and compositions of motion are made to take orbits nearer and nearer that plane.

* This is owing to successive transformations of the orbits of small bodies by larger ones, as in the case of the comets which pass near *Jupiter*. Their orbits are finally made to have but slight inclinations to the plane of the disturbing planet; and as many of these small masses are finally absorbed by the disturbing body, its mass is thereby increased, and thus there is a tendency for the total mass to accumulate in the fundamental plane of the system.

By the resisting medium of diffuse nebulosity the motion is gradually communicated to all the particles; and satellites and particles are drawn more and more into the equators of the planets.

(6) The general explanation here given enables one to grasp more easily both the oblate figures of the nebulae shown by photography, and the planar arrangement of the solar system treated in § 182, page 392, and it aids one in understanding also the equatorial preference of the satellites observed in the solar system. The satellites are captured bodies which have acquired their present positions by the wear and tear of nebular vortices revolving about the planets once much denser than they are at present. Most of the nebulosity has been absorbed into the planets, though a little has gone into their satellites by collisions, such as we see traces of in the battered face of the Moon.

§ 324. *Life a General Phenomenon of Nature, and Almost as Universal in Its Distribution as Matter Itself.*

If therefore the laws of nature are such as to form planetary systems of the cosmical dust expelled from the stars, through precipitation, condensation and falling together under the attraction of gravitation; while on the other hand the dust itself in a finer condition is originally expelled from the stars, by the action of the repulsive forces arising from high temperature, intense radiation-pressure and powerful electric charges, it follows that there is a cyclic process by which stars and systems arise from nebulae, while nebulae in turn are formed from the stars. On this point there does not seem to be the slightest doubt; and we may regard this cyclic process as perhaps the greatest of all the laws of nature. Indeed, it seems to operate on a stupendous scale throughout the sidereal universe.

Since therefore the starry heavens are shown to be filled with many millions of planetary systems, and an indefinite number of habitable worlds, is it not obvious that these worlds as a rule are also inhabited?* *From the uniformity of the laws of nature, it would seem that this must be true, and, so far as one may now judge, this is the most inspiring message yet delivered to mankind by Modern Science.*

Let us see on what foundation this conclusion rests: (1) Gravitation operates according to the same laws in other parts of the sidereal universe as upon the Earth;

* In his thoughtful address at the dedication of the Flower Observatory, Philadelphia, May 12, 1897, Professor Newcomb discusses the plurality of worlds as follows:

"There is one question connected with these studies of the universe on which I have not touched, and which is, nevertheless, of transcendent interest. What sort of life, spiritual and intellectual, exists in distant worlds? We cannot for a moment suppose that our own little planet is the only one throughout the whole universe on which may be found the fruits of civilization, warm firesides, friendship, the desire to penetrate the mysteries of creation." Again, in the article "Stars," *Encyclopedia Americana*, he remarks that the stars in clusters may have planets revolving around them.

(2) The velocity of light, and electricity, and no doubt of other physical agencies, is the same in all parts of space; (3) The chemical elements are the same everywhere, whether the light involved comes from a flame in our laboratory or from one of the stars; (4) Mechanical laws are the same in the solar system and among the nebulae and fixed stars, and this makes possible the development of cosmical systems of the same general type throughout nature; (5) Electronic, atomic, molecular, gravitational and electric and other repulsive forces are the same everywhere; (6) Life depends in some way for its physical basis on electronic, atomic and molecular forces, and as these forces and the elements on which they act are the same everywhere, and the universe is shown to be full of habitable worlds made up of the same elements subjected to the same forces as in the case of our own planets revolving around the Sun, it follows incontestably that life is a general phenomenon of the physical universe, and almost as universal as matter itself.

It is true that the psychical and spiritual element of life is not yet fully understood, but whatever be its character, it can flourish elsewhere in nature quite as well as on the planet called the Earth in the solar system. Our Sun is simply a fixed star of very ordinary magnitude, and the Milky Way includes hundreds of millions of such centres of planetary systems. Accordingly, in view of the established uniformity of nature's processes throughout the immensity of space, who can doubt that life is a general phenomenon ordained by the Deity from the Creation of the World, and destined to develop wherever planets are forming and the stars are shining? Whatever be the nature of life, it has as much right to develop as planetary systems or combinations of atoms; it is indeed the bloom of nature, the culmination of the highest creative forces. *To hold any other views than those here announced would be to violate the doctrine of uniformity, which lies at the basis of Natural Philosophy as formulated by* NEWTON *in the Principia (Lib. III); and moreover lead to the conclusion that life upon the Earth was an accident and a mistake, in violation of the usual order of Nature,* which is infinitely improbable and, in fact, impossible for a philosopher to admit.*

If therefore life is as universal as the stars in space, it is evident that when we

* If life on the Earth exists by a mere accident and in violation of natural laws, is it likely that it would have shown such power of propagation and of resistance to adverse conditions as it is known to have possessed throughout Geological History? It seems to have been a veritable spark which simply could not be extinguished, and must therefore have been burning on and flourishing, not in violation of, but in accordance with, natural laws. Those who believe that life is an accident and a mistake, a noxious development flourishing in violation of the laws of nature, may with consistency deny the existence of life throughout the universe. But having shown that habitable planets revolve everywhere about the fixed stars, in orbits which are nearly circular, and rotate so as to give alternation of day and night, as on the Earth, it seems to me more philosophical to follow the example of SIR WILLIAM HUGGINS, in regard to the chemical elements, and declare that life exists wherever there is a sun to warm and light its attendant planets, and therefore wherever a star twinkles in the depths of space. The other view, that life is an accident, leads to a *reductio ad absurdum* fully as conclusive as those employed in Geometry.

behold the starry heavens and contemplate the glorious arch of the Milky Way on a cloudless night, we receive from distant suns and worlds ethereal vibrations which tell us at the same time of living beings throughout immensity. Let us, therefore, quietly rejoice, when we survey the starry heavens in all their splendor, and remember this sublime message telling us that we are not mere dust confined to this dark planet, but a part of the flower of the visible creation, which blooms everywhere with the cosmic order, and is as universal as the stars which illuminate the depths of immensity.

Without the sublime researches of SIR WM. HERSCHEL, we should have a very inadequate conception of the profundity to which our telescopes can penetrate into the blackness of unilluminated space, and thus could poorly interpret the message of the universe. But this great astronomer showed that the stars extend principally in the direction of the Milky Way, and light up that region so brilliantly that we can extend our explorations to a distance which it would take the light millions of years to traverse. Thus the Milky Way is like a great but somewhat narrow corridor lighted up by the stars to the remotest regions to which our telescopes can penetrate, with no indication of an end to the starry stratum. To realize on good, substantial and indisputable scientific grounds that life accompanies the stars to the remotest depths of space, and that we can look out upon such countless worlds from our tiny abode near the Sun, and thus connect the feeble life of our globe with the universal life in the endless order of inhabited spheres, is not the least inspiring message in the Epic Poem of Science. It is indeed a message from the stars. And it seems to me that if Astronomy had achieved no other result than this, it would more than justify all the labors which have been bestowed upon it from the earliest ages.

This message from the stars passeth not away, but endureth unto all generations. As ageless as the heavens from which it comes, it will continue to travel downward with the starlight,* and thus awaken new life and hope in the hearts of mankind. For it is absolutely impossible for this order of mind, life and intelligence as widespread as the stars in space, to have been established throughout Nature without design and abiding great and good purpose; and therein lies the proof of the existence of the Deity. The teachings of true science are therefore among the most sacred which have ever been delivered, and they deserve the veneration which is always due to Ultimate Truth.

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* "Were a star quenched on high,
For ages would its light,
Still traveling downward from the sky,
Shine on our mortal sight."— *Longfellow.*

§ 325. *Origin of the Double and Multiple Stars and Clusters.*

It will have been perceived that while we do not deny the occasional existence of partial *fluid fission* for the separation of double and multiple stars, we yet believe the general process of nature to be one of *nebular fission*, according to which small masses drop down and collect in the larger centres, under the secular effects of resistance, but are not thrown off by rotation as was so long believed prior to the development of the Capture Theory. The resisting medium aids in the process of capture, and one or both centres of attraction grow by the capture of satellites.

The distribution of the satellites may be such as to give two principal masses, as in binary stars; but in general, it is probable that such systems contain multitudes of smaller bodies, unless they have finally been destroyed by capture and absorption into the large masses.

In binary systems with two powerful centres of attraction, planets might revolve about either large mass, or about both together; but the motion would not endure for an indefinite time. The large star and companion are formed and grow larger and larger by the capture and absorption of the smaller bodies; and as the large bodies were not set revolving in round orbits at the beginning, the eccentric character of the orbit is especially favorable to the swallowing up of the smaller masses.

The explanation here given applies to the spectroscopic as well as to the visual binaries. And in the case of the triple and multiple stars, the process is similar, the principal centres in such nebulae having formed far apart, and their companions developing close about them. The secondary companions are within the HILL surfaces for the principal components just as in the case of the satellites of the solar system. Owing to the immense extent of the nebulae, and the resulting possibility of the development of many nuclei, the origin of the multiple stars presents no difficulty. In general, the companions were originally nebular nuclei, but they afterwards grow larger by the same process of capturing satellites and smaller elements of nebulosity.

The development of star clusters is similar to that of multiple stars, except that the primordial nebula is correspondingly larger, and the process of capture more extended in time and space. The star clouds of the Milky Way are the final product of the capture theory, where whole streams and masses of stars are concerned and so circumstanced that they pass under a central control. This leads to the highest form of cosmical system, with an organization too complex to be understood, but yet capable of existing for millions of ages, owing to the conservative tendency of projectile forces, as was long ago remarked by HERSCHEL.*

* cf. *Phil. Trans.*, 1785, p. 217.

§ 326. *Cause of the Variability in the Light of the Stars.*

Variable stars have long been an object of research, and at length vast masses of observational data have been accumulated by labor and persevering industry. But in spite of the labors of many eminent men, it has been difficult, if not impossible, to assign the cause of stellar variability. The causes assigned are many, and of the most varied and unsatisfactory character. While it would not be admissible to suppose that there is always and everywhere but a single cause at work, it yet seems certain there is only *one principal cause* to which the phenomenon of variability should be referred. This is indicated by the periodicity of the light fluctuations, and by the groups into which variables may be arranged; and by the striking prevalence of certain types of variability as well as by the connections shown to exist between the different classes of variables.

It is not permissible to go into much detail here, and we shall therefore simply remark that an examination of the subject indicates that the resisting medium is the principal cause of stellar variability. Orbital motion of one or more companions through such a medium will explain the nearly constant periods of the variables, as well as the fluctuations of period and amplitude occasionally encountered. It unites the *Algol* and the *Ant-Algol* variables, and connects variables with the spectroscopic and visual binaries, by recognizing orbital motion as the principal circumstance leading to the fluctuation in brightness. The *Ant-Algol* variables abound in certain clusters, and may usually be explained by close companions moving in comparatively eccentric orbits, so that the brightness suddenly increases near periastron passage. Various situations of the orbits, size of companions, and variation in the law of density of the medium, will account for the fluctuations from the normal types. The importance of the resisting medium in dealing with variables has been recognized by several recent writers, so that the present explanation is not entirely new, though it has not been generalized before.

But the important and even paramount part which the medium is shown to play in the development of all cosmical systems, may be said to add enormously increased weight to the theory now advanced. If the resisting medium has greatly transformed the orbits of the heavenly bodies, it necessarily follows that the resistance encountered would often give rise to effects which are sensible to observation. This is the exact state of fact. The light of every star is changeable, but it is only in certain cases that the fluctuations are conspicuous enough to justify the designation *variable*. The details of this large subject must be reserved for future investigation, but it seemed necessary to call attention to the most general cause affecting the changes in the light of the stars.

TABLE OF VARIABLE STARS IN CLUSTERS, FROM *Harvard College Observatory Circular*, No. 33.

Designation	Position, 1900 R.A. ; Decl.	No. of Stars Examined	Area Examined in Square Minutes.	No. of Variables	Proportion	
					Fraction	1 in.
	h m ; ° ′					
N.G.C. 104, 47 *Toucanae*	0 19.6 ; −72 38	2000	1257	6	0.003	333
362	0 58.9 ; −71 23	675	314	14	0.021	48
869 } Double Cluster in 884 } *Perseus* (HIPPARCHUS)	2 12.0 ; +56 41 } 2 15.4 ; +56 39 }	1050	10800	1	0.001	1050
1904, MESSIER 79	5 20.1 ; −24 37	200	79	5	0.025	40
3293	10 29.6 ; −57 40	724	314	0	0.000	....
4755, *κ Crucis*	12 47.7 ; −59 48	555	314	0	0.000	....
5139, *ω Centauri*	13 20.8 ; −46 47	3000	1257	125	0.042	24
5272, MESSIER 3	13 37.6 ; +28 53	900	1237	132	0.147	7
5904, MESSIER 5	15 13.5 ; + 2 27	900	1257	85	0.094	11
5986	15 39.5 ; −37 26	289	314	1	0.003	289
6093, MESSIER 80	16 11.1 ; −22 44	145	79	2	0.014	72
6205, MESSIER 13	16 38.1 ; +36 39	1000	177	2	0.002	500
6266, MESSIER 62	16 54.9 ; −29 58	960	218	26	0.027	37
6397	17 32.5 ; −53 37	487	218	2	0.004	244
6626, MESSIER 28	18 18.4 ; −24 55	900	314	9	0.010	100
6656, MESSIER 22	18 30.3 ; −23 59	1550	218	16	0.010	97
6723	18 52.8 ; −36 46	900	314	16	0.018	56
6752	19 2.0 ; −60 8	600	218	1	0.002	600
6809, MESSIER 55	19 33.7 ; −31 10	440	218	2	0.005	220
7078, MESSIER 15	21 25.2 ; +11 44	900	1257	51	0.057	18
7089, MESSIER 2	21 28.3 ; − 1 16	600	218	10	0.017	60
7099, MESSIER 30	21 34.7 ; −23 38	275	218	3	0.011	92
Totals for 23 Clusters		19050	20380	509		

The proportion of Variables in different clusters is very different, MESSIER 3 being the richest, with one in every seven stars. The tendency to variability depends on the presence of nebulosity sufficiently dense to act as a resisting medium, though it is too close to the stars to be seen in a telescope or on a photograph of a cluster. Accordingly those clusters which are filled with variables are still *nebulous*, while those more free from variables are further advanced in their development. This is a point of deep interest in the general theory of clusters, and it gives us a criterion which seems likely to prove of great value in the future study of these dense masses of stars. At the great distances of the clusters it might never be possible to detect the presence of nebulosity by any process of direct observation, but we can demonstrate its presence indirectly, through the phenomena presented by the cluster variables. I venture to think that this is not the least interesting result of Modern Astronomy. It opens a new field for the development of the "Astronomy of the Invisible," of much profounder import than that due to the companions attending such stars as *Sirius* and *Procyon*, which was justly considered so wonderful by BESSEL seventy years ago.

§ 327. *The Relative Probability of Collisions Within a System and Between Separate Systems.*

In Chapter VI we have considered the theory of collisions between the stars, which was first developed by PROFESSOR A. W. BICKERTON, of New Zealand, and afterwards adopted by LORD KELVIN and PROFESSORS RUCKER and ARRHENIUS and many others. In fact, of late years, the collision theory has been widely adopted by writers on Cosmogony, and one can scarcely take up a popular work on Astronomy without finding in it something about the formation of cosmical systems by collision.

It would be going too far to say that such catastrophes as stellar collisions never occur, but we have adduced solid grounds for holding that such events, as depending on the mutual approach and passage of separate stars, under difference of proper motion, must be very rare indeed. In fact, such collisions would be so rare that it is impossible to believe that the physical effects would be recognizable among the visible phenomena of the sidereal universe. A very few nebulae might arise in this way, but not the vast numbers of these objects now known to exist in the depths of the heavens.

The origin of the nebulae noticed to be in process of condensation evidently is the same as that of the thin veil of hazy nebulosity seen to be diffused so generally over the background of the sky; and this latter can be nothing else than the expulsion of finely divided matter from the stars under the secular action of repulsive forces. When this widely diffused nebulosity collects here and there into dense masses, it passes into real cosmical clouds, shining with a whitish or greenish light; and this is the origin of the nebulae, from which cosmical systems are eventually developed.

But while collisions between separate stars and systems are so rare as to be practically negligible, *this is not true of the numerous small bodies forming within a nebula.* Here there are produced an indefinite number of moons, planets, and comets, by the condensation of nebulosity at many centres; and as the system condenses, these bodies traverse their orbits millions of times in comparative proximity; and under the influence of perturbations the orbits are slowly transformed, so that collisions become inevitable. Millions of comets, satellites and planets are swallowed up by the larger planets and the central sun of every cosmical system.*

The new or temporary stars noticed suddenly to blaze forth in the sky are no doubt due to the collisions of smaller bodies with larger ones; and we have pointed

* A few of the colliding bodies are large enough to produce the Novae or Temporary Stars, which occasionally light up the regions near the Milky Way.

out that collision with any mass is greatly facilitated by the expanded state characteristic of a nebula. Moreover, there are many dark bodies of planetary size wandering in space. If one of these should enter a small nebula the result might be the blazing forth of a new star, such as was observed by HIPPARCHUS, 134 B.C.

The Novae are of such short duration that we can scarcely attribute them to the bodily collision of globes of stellar size. Even if the small planetary globes be thousands of times more numerous than the self-luminous stars, it would still be difficult to account for the Novae by collisions, except these impacts occur within systems, where millions of small bodies revolve close together, in slowly shifting orbits, giving in time countless collisions, some of which might produce enough light to be visible to the inhabitants of our planet.

Accordingly, with this necessary modification, we may adopt a part of BICKERTON's theory of collisions; but deny the interstellar collisions heretofore supposed to occur between separate stars. The grounds for rejecting the theory of interstellar collisions is that the stars are too far apart and the chances of impact too small. Thus the theory of impact becomes very important *within a cosmical system* composed of a great multitude of small bodies resulting from the condensation of a nebula; but is scarcely applicable to the interaction of separate stars upon one another.

The lunar craters impressively illustrate satellite collision within our solar system. The spiral nebulae, however, evidently have not resulted from the impact of separate systems, as has been incorrectly inferred by several writers who have made very superficial study of the sidereal universe. The promulgation of these latter views is unjustifiable and wholly detrimental to the cause of science. The collision theory, therefore, must be accepted with considerable reserve and only under limitations which discriminate sharply between interstellar collisions and collisions occurring within systems composed of an indefinite number of bodies crowded into a very limited space and repeating their motions in slowly changing paths many millions of times. Interstellar catastrophes scarcely ever occur, while collisions among the bodies of a system are inevitable, and actually illustrated by the blazing forth of Novae, several of which have appeared in nebulae. It is not by chance that the Nova of May 18, 1860, appeared in the nebula *Messier* 80; that of August 16, 1885, in the Great Nebula of *Andromeda*; and that of July 18, 1895, in a nebula in *Centaurus*. Moreover, it will be remembered that during SIR JOHN HERSCHEL's sojourn at the Cape of Good Hope *Eta Argus* blazed forth with almost the splendor of a nova of the first magnitude, exceeding the brightness of *Canopus* and rivaling *Sirius* in 1843. It too is in the midst of a great nebula; and this celebrated outburst seems to give a suggestive connecting link between

Novae and variables of very long period. The phenomena of *Nova Persei* No. 2, 1901, point in the same direction, and the observed transformation of the spectra of Novae into those of planetary nebulae shows that the connection between Novae and nebulae is undeniable.

§ 328. *The Two Streams Among the Stars Due to Spiral Movement in the Milky Way.*

We have traced the two streams recently discovered among the stars to the spiral movement in the Milky Way. This gives a natural and simple explanation of the phenomenon, and shows that our Sun is immersed in these streams. Difference in velocity of the Sun and the stars produces the observed effect, which is also modified by our position in the streams. It may be a long time before we shall be able to make out the details of this movement; but there can scarcely be any doubt that it will be traced to the spiral movement in that portion of the Milky Way which is comparatively near the Sun. Our Sun is somewhat eccentrically placed, being much nearer the body of the stars in *Sagittarius* than in *Monoceros*; and hence it is natural to think of the whirling movement as about a centre in *Sagittarius*, *Aquila* or *Cygnus*. This would account for the preponderance of the stream towards *Sagittarius*, and the lesser stream towards *Orion*. As the sidereal stratum is comparatively thin the radiant to which the stars converge should be fairly definite; and yet it ought to be expected that some stars would depart from these general tendencies, owing to the special position of our Sun or to the action of neighboring groups of stars unsymmetrically placed, the Sun having a proper motion in our cluster, and thereby sensibly influencing the course of the stars.*

§ 329. *Arrangement of the Milky Way.*

We have found that the preference of the clusters for the Milky Way is due mainly to an effect of perspective incident to the great extent of the starry stratum in the plane of the Galaxy. The clusters thus seem to follow the Milky Way, whereas they exist throughout the starry stratum; but owing to the thinness of that layer, are usually projected upon the path of the Milky Way, or follow its borders very closely.

* An interesting discussion of the work of Hough and Halm occurred at the meeting of the Royal Astronomical Society on June 10, 1910, reported in the *Observatory* for July, 1910. The two star-streams are confirmed, and it is shown that the irregular distribution of these streams explains certain discordances in the values of the Precession Constant found by Newcomb and Struve. Sir David Gill summed up the results by saying that stream No. I seems to be an expression of the motion of our Sun through a chaotic star-cluster, of which our Sun is a member, while stream No. II probably is an indication of the motion of that cluster as a whole amongst the other stars of the Galaxy. This is essentially the author's conception, as embodied in Plate τ, and may be strongly commended to the attention of the reader.

The white nebulae, on the other hand, are remote from the Galaxy, probably owing to their formation from matter expelled from the Milky Way; and they are thus collected in preponderant groups near the poles of the Galaxy, while the irregular large and diffuse nebulae, being formed of matter recently expelled from the stars, naturally tend to follow the general course of the Galaxy. That this interpretation of the universe rests upon a substantial basis of truth can scarcely be doubted. It accords with the obvious indications of nature, and is not contradicted by any known law, but supported by true physical causes known to be at work.

The clustering power noticed by the elder HERSCHEL is gradually breaking up the Milky Way, and forming vacancies and dark rifts and holes in it, and clouds of stars of various degrees of density. Whether this process of clustering together continues indefinitely or dispersion again follows from the continual interpenetration of streams of stars with different tendencies cannot at present be determined; but it seems likely that the clustering power is largely overcome at long intervals by an opposing tendency analogous to the operation of repulsive forces; by which the stars are again dispersed and finally scattered with some average degree of uniformity over the immensity of the sidereal heavens. Unless there were some such corrective process at work, it seems unlikely that the clustering power would not already have made greater progress in the complete breaking up of the Milky Way than is indicated by observation.

This eventual breaking up of the Milky Way was a subject of much solicitude to SIR WILLIAM HERSCHEL in the latter years of his life. Assuming the process to be continuous and working uninterruptedly in the same direction, he correctly concluded that the clustered aspect of the stars might be used as a kind of chronometer for measuring the relative ages of the clusters and star-clouds composing the Milky Way. The grandeur of this conception is worthy of the genius of HERSCHEL, and constitutes indeed an everlasting monument to the sublimity of his thought; but it still remains for us to discover whether there is not some imperfectly understood cyclic process such as nature so often presents, by which these cumulative tendencies in a given direction are finally corrected, and a permanence assured to the mean state of the sidereal universe, as the matter is incessantly transformed, producing alternately stars and nebulae and an infinite succession of cosmical systems.

§ 330. *The Extent of the Sidereal Universe and the Extinction of Light in Space.*

In Chapter XXII, which is devoted to the Milky Way, we have seen that the sidereal universe is greatly extended in the plane of the Galaxy, and much more

restricted in the direction perpendicular to that plane. The exact ratio of the distances to which our telescopes can penetrate in the two directions is not known, but cannot be less than 3 to 1 and is more likely as 5 to 1, or larger.

Except for the illumination due to the stratum of stars, the universe is veiled in darkness. The stars act as torches for lighting up the corridors of the universe, and this enables us to look out to a vast distance in the plane of the Milky Way. The stratum of stars appears to extend on indefinitely; but, as we have seen in Chapter XXIII, the amount of cosmical dust in space is so large that at great distances the light of the stars is cut off, as by a haze or fog in our atmosphere.

The result is that there is a limit to which a telescope can penetrate, and that is less than the theoretical distance, and the number of stars observed less than that calculated on the hypothesis that light decreases inversely as the square of the distance. This result was deduced by STRUVE from HERSCHEL's gauges, and has not been changed by more modern research.

The important work of SEELIGER shows that the HERSCHEL stars obey a different law from the *Durchmusterung* stars, and we have seen that this indicates the indefinite extension of the universe and the extinction of light in space.

The preference of the Novae for the Milky Way points in the same direction, and confirms by new evidence of independent character the theory that the stratum of stars is enormously extended in the direction of the Milky Way. All the evidence derived from the density, distribution and brightness of the stars, and from the observed distribution of Novae concurs in establishing the great extent of the sidereal stratum in the plane of the Galaxy and also the extinction of light in space.

If we had a telescope of about 14-feet aperture, we could see stars of the 20th magnitude, and if the law found to hold for the HERSCHEL stars should be confirmed, it would give us further extension of the sidereal universe amounting to 65 per cent. of the depth heretofore fathomed in the plane of the Galaxy. Such an exploration of the now invisible regions of the universe is greatly to be desired; and until this work can be carried out, there will necessarily be some uncertainty as to the indefinite extension of the starry stratum. This great work is to be commended to the telescope builders of the future. It is obvious that its importance can hardly be overestimated. The completion of a survey of the stars to the 20th magnitude would give us a second epoch in sidereal astronomy comparable to that made by the elder HERSCHEL, which has been the basis of nearly all our modern theories of the structure and arrangement of the universe.

§ 331. *Remarks on the Processes of Discovery.*

With regard to many of the important problems treated in this volume we must for the present preserve an open mind, and earnestly seek for more light on the great secrets of nature. In this effort we shall gain but little knowledge of the true system of the universe by giving undue deference to the inadequate philosophy heretofore current in intellectual circles. Our dominant scientific thought obviously has been lacking in breadth, and so one-sided that it failed to give us even a faint glimpse of nature's greatest laws. Perhaps this should, after all, occasion no surprise to thoughtful persons. Many writers are so greatly swayed by current opinion, and by the inducements which are always held out to follow the beaten path, that they do little for the advancement of true physical science.

If the pioneers in the history of Science had done likewise, perhaps unconsciously preferring temporary popularity to the Eternal Truth, Science itself would not have survived to tell the story. It may be, indeed, an unwelcome task to attack and overthrow antiquated theories which have outlived their usefulness and are now stifling scientific effort. But it is obvious that this has to be done, in order that progress* may not perish from the Earth, and a Stationary Period come over us, as happened during the Middle Ages and during the supremacy of the Arabians.

The discovery of the laws of Cosmical Evolution has been one of the great aims of natural philosophers ever since the foundations of Physical Science were laid by the Greek sages at Athens two thousand three hundred years ago. In view of the great historical importance of this effort of the centuries, it is clear that we ought to make an earnest endeavor to reduce Cosmogony to the basis of an exact science, and thus greatly extend what is undeniably the sublimest portion of human knowledge.

Our age is a peculiar one, in that with the progress of Astronomy vast masses of observational data are accumulated by the persevering industry of self-denying men of science; but so long as these data cannot be put together to yield us the long-sought laws of cosmical evolution, the outcome is almost as vain as the weaving of PENELOPE's web. Natural philosophers believe, however, that the time is now auspicious for a great advance, not merely in the details but also in the laws and principles of exact Astronomical Science. One of the ultimate aims of the Physical Sciences in all ages has been the discovery of the laws of cosmical evolution. If with the modern improvements in the mathematical treatment of the

* In Physical Science to-day nothing is more needed than a revival of the sublime Natural Philosophy of SIR WILLIAM HERSCHEL, and we have therefore labored to give the reader some conception of his penetrating remarks on the nature of the Sidereal Universe.

problem of three bodies, and the observational data derived from photographic study of the nebulae and clusters, as well as from the visual and spectroscopic binary stars, this progress be not possible in our time, it is difficult to see how better results can be expected in the future. Nearly all the reasoning in Cosmogony since the time of LAPLACE has been based on a false premise to the effect that the revolving bodies have been detached from the central masses which now govern their motion by acceleration of rotation. When this false premise was discovered and confirmed by exact calculations based on BABINET's criterion, I thought it worth while to attempt to reach solid ground on which a real Science could be established.

§ 332. *The Process of Cosmical Evolution Concealed from Mortal Sight, and Therefore Most Difficult to Discover.*

Whatever be the exact process involved in the formation of the solar system it seems clear that when the planetary order was essentially complete the Architect of the Universe removed the scaffolding employed in the construction, and allowed us mortals to behold chiefly the finished structure, without any obvious suggestion as to how this beautiful and orderly development had come about. The laws connected with the evolution of planets and satellites were withheld from our sight, and it almost seems as if Nature challenged the mathematician and natural philosopher to unravel her securely hidden mysteries. For a long time this part of the riddle of the Universe remained unread, and many have declared that the secret could never be penetrated.

The faith of the writer, however, remained unshaken, and in May, 1908, the time finally came when attention was turned from other finished researches in natural philosophy to certain mathematical problems much studied by the ancient geometers, and particularly by ARCHIMEDES of Syracuse. While examining the original works of this great geometer I became deeply interested in the spiral of ARCHIMEDES. As vast multitudes of spiral nebulae were known, but no adequate theory of them yet developed, it occurred to me to apply this celebrated spiral to the figures of the nebulae as revealed by astronomical photography.

After considerable attention had been given to this comparison, and the conviction gained that the observed coils of the nebulae are chance spirals, and not true mathematical figures of geometric regularity, it was easy to imagine how the condensation of such masses gradually coiling up under the influence of their mutual attraction might produce various kinds of bodies. It then remained to work out the details of the process involved in the suggested cosmical evolution. The greatest obscurity still attached to the evolution of the planetary system, with

nearly circular orbits and very small attendant bodies revolving about much larger central masses.

For several weeks the writer brooded over the difficulties encountered by previous investigators. All their criticisms lay clear before him, and he labored incessantly to reconcile them with known mechanical laws, and with the general law of spirality observed among the nebulae in the immensity of space. In the course of this inquiry every premise in the chain of reasoning was questioned, every prejudice and preconceived notion freely given up. Early in July, 1908, many of the difficulties began to give way, and it was evident that a solution was near at hand. But the difficulty of accounting for the roundness of the orbits of the solar system still remained and it seemed stupendous. Finally on July 14, the thought of a resisting medium producing the observed roundness of the orbits of the planets and satellites came to my relief; and I saw at once that the chief difficulty had been overcome. In a flash I saw *Jupiter* and *Saturn* moving against a resisting medium and having their orbits reduced in size and at the same time transformed into nearly perfect circles. With this idea firmly conceived, the darkness and confusion previously existing rapidly cleared away, and the objections to the new nebular hypothesis vanished almost immediately.

Remembering the well-known secular effects of the action of a resisting medium it was now clear how the planets and satellites by moving for long ages against such a resistance had acquired such perfectly round orbits; that the orbits had formerly been much more eccentric and also much larger than at present; that other remote planets of the solar system might revolve unseen beyond *Neptune*, so that the full extent of our system was not yet realized; that the asteroids were a vast collection of satellites gathered within the orbit of *Jupiter*, those which extended to the regions of the major planets having been absorbed by collision and reduced by the perturbing action of this giant planet and finally thrown into regions of stability within *Jupiter's* orbit; that much of such waste matter had gone into the Sun to make up its immense mass of seven hundred and forty-six times that of all the planets combined, and hence the central part of the system was at length cleared of nebular wreckage; that the orbits of the terrestrial planets likewise were originally much larger than at present, and also much more eccentric; and that while these four grains of cosmical dust, from the interior region of the solar nebula, had survived, hundreds of thousands of similar satellites or small planets had been swallowed up to produce the Sun's preponderant mass for governing, and heat and light for warming and lighting, the rest of the system; that the planets which survive are only a few of the much vaster number of small bodies which have been destroyed to lay the foundation of our system on which the planets and satellites

are dependent; that even the comets are certainly survivals of the solar nebula; that the satellites too are survivals of the system of small planets, just as the large planets are of the small planets and nebulosity condensing into the Sun; that the retrograde satellites of *Jupiter* and *Saturn* had been captured while moving against a resisting medium, as shown by the surviving large eccentricities of their orbits; that the resisting medium is verified by the observed motion of *Phobos* and the three inner Galilean satellites of *Jupiter*, so prophetically pointed out by LAPLACE a century ago; that the equatorial acceleration of the Sun and planets gives evidence of the vortices of cosmical dust still revolving about these bodies.

In short, there was not a single phenomenon in the solar system the explanation of which now offered any great difficulty. And moreover the formation of our solar system was seen to harmonize with the general law of spiral movement exhibited by the photographs of nebulae scattered abundantly throughout the immensity of space. It was noticed that if the solar nebula was essentially free from hydrostatic pressure, the objections of BABINET, KIRKWOOD and PEIRCE, would all disappear. And as this would explain both the small size of the attendant bodies and the circularity of their orbits — the two leading characteristics of the solar system — there could no longer be any hesitation in concluding that the true law of the formation of our system had been discovered and confirmed beyond all doubt. This verification of the new theory was practically complete on July 14, but the details of the argument have been worked out since, and the mature Capture Theory arranged in form suitable for publication.

Such was the essential order of the writer's thought in the development and verification of the Capture Theory, which he records with some hesitation and diffidence, but chiefly in the hope that it may be useful to others. That which relates to the process of discovering truth is as valuable as truth itself.

The Genesis of All Classes of the Heavenly Bodies and the Relationship of the Nebulae to the Milky Way Explained.

Under the action of attractive and repulsive forces — the heavy bodies always drifting towards centres of attraction, and light bodies being dispersed to act as a resisting medium and finally precipitate into small masses which are used in building up larger ones, in accordance with the Capture-Theory — it will be seen that we have succeeded in explaining all classes of the heavenly bodies:

(1) *The Stars*, when the condensation of nebulosity produces central suns, with the resulting formation of cosmical systems revolving about them.

(2) *The Double and Multiple Stars,* when the circumstances are such that the attendant multitude of small bodies divides so as to build up two or more comparable suns.

(3) *Planetary Systems,* when the attendant small bodies revolving about central nuclei unite into many minor globes of planetary size, and thus produce what appear to be single stars.

(4) *Systems of Satellites,* when some of the smaller planets are captured by the larger ones, as in our solar system.

(5) *Asteroids and Comets,* when small fragments from the outer shell of the primordial nebula drop down to visit the central sun, and thus describe orbits which are nearly parabolic, a few of which are transformed into movements of short period, as by the action of *Jupiter.*

(6) *Diffuse Nebulosity,* when the dust expelled from the stars at length becomes a faint haze spread over whole constellations.

(7) *Nebulae,* when the nebulosity condenses into cosmical clouds, which may be *irregular, spiral, annular, elliptical,* or *planetary,* according to the circumstances of condensation and the stage of development attained.

(8) *Variable Stars,* when some of the attending companions are so situated as to produce eclipses at regular periods, or revolve in eccentric orbits and encounter enough resistance near periastron to produce periodic fluctuations of the starlight.

(9) *New or Temporary Stars,* when some of the companions corresponding to planets, satellites or comets come into actual collision with their central stars, and thus produce sudden conflagrations.

(10) *Clusters of Stars,* when the condensation of vast nebulae, and the gathering in of neighboring stars under the clustering power of gravitation, produce highly compressed groups of stars.

(11) *Star Clouds of the Milky Way,* when millions of suns come to be aggregated together and more or less dominated by their mutual gravitation, but show less regularity of form and less accumulation towards their centres than do the clusters.

(12) *The Milky Way,* when the grand aggregate of all these swarms of stars is taken together; and thus the Galaxy appears to traverse the heavens as a clustering stream, here and there attaining a perfect blaze from the intensity of the accumulated starlight.

The action of repulsive forces explains also the apparent antipathy of the Nebulae to the Milky Way, this primordial star-dust having been expelled originally from the Starry Stratum, and therefore inevitably drifting away from it, and gathering with maximum accumulation near the poles of the Galaxy — a phenom-

enon of great significance in the theory of the construction of the sidereal heavens, but heretofore unexplained and equally bewildering to the astronomer, the geometer and the natural philosopher.

When the nebulae condense into stars, it is probable that by an unknown circulatory process they again drift back towards the medial plane of the Galaxy. Thus there is an expulsion of dust from the starry stratum, and a subsequent recovery of this matter in the form of mature stars, some of which in time are absorbed by other stars, so that in the long run the total number of stars and of nebulae remains about the same. The gravitational attraction of the starry stratum fixes a limit beyond which the dust is not driven by repulsive forces, and this natural balance between the opposing tendencies is what gives the sidereal universe its observed aspects.

§ 333. *Lesson Taught by the Pioneer Discoverers Who Laid the Foundation of the True System of the World.*

As this is the culmination of continued labor extending over more than a quarter of a century the following words of KEPLER, regarding the discovery and verification of the third law of the planetary motions, are not without interest:

"What I prophesied two and twenty years ago as soon as I had discovered the five solids among the heavenly bodies; — what I firmly believed before I had seen the *Harmonics of* PTOLEMY; — what I promised my friends in the title of this book (On the Most Perfect Harmony of the Celestial Motions) which I named before I was sure of my discovery;— what sixteen years ago I regarded as a thing to be sought; — that for which I joined TYCHO BRAHÉ, for which I settled in Prague, for which I have devoted the best part of my life to astronomical contemplations; — at length I have brought to light, and have recognized its truth beyond my most sanguine expectations."

For in spite of the perils and obstacles which always attend pioneer effort the true process involved in the formation of the solar system is now established for the first time. *Moreover the laws of Cosmical Evolution, by the process of capture, thus brought to light are proved to be universal and shown to be applicable to the various types of nebulae and stellar systems observed to constitute the sidereal universe.*

Discoverers always have to overcome not only the inertia of received opinion, but also the active opposition of the established schools of thought, which often are indisposed to welcome further progress *brought about by others.* Thus, in publishing his great work, *De Revolutionibus Orbium Celestium*, 1543, COPERNICUS says: "Though I know that the thoughts of a philosopher do not depend on the judgment

of the many, his study being to seek out truth in all things as far as that is permitted by God to human reason: yet when I considered how absurd my doctrine would appear, I long hesitated whether I should publish my book, or whether it were not better to follow the example of the Pathagoreans and others, who delivered their doctrines only by tradition and to friends."

In defense of COPERNICUS against the charge of disrespect to the ancients, RHETICUS writes to SCHEINER: "He was very far from rashly rejecting the opinions of ancient philosophers, except for weighty reasons and irresistible facts, through any love of novelty. His years, his gravity of character, his excellent learning, his magnanimity and nobleness of spirit, are very far from having any liability to such a temper, which belongs either to youth, or to ardent and to light tempers, or else to those *τῶν μέγα φρονούντων ἐπὶ θεωρίᾳ μικρῇ*, 'who think much of themselves and know little,' as ARISTOTLE says."

WHEWELL remarks that this deference to the great men of the past, joined with the talent of seizing the spirit of their methods when the letter of their theories is no longer tenable, is undoubtedly the true mental constitution of discoverers (*Hist. of Inductive Sciences*, Vol. I, p. 376).

There need, therefore, be no surprise that COPERNICUS was so indifferent to unjust criticism of his work: "If there be *μεταιολόγοι*, vain babblers, who knowing nothing of mathematics, yet assume the right of judging on account of some place of Scripture perversely wrested to their purpose, and who blame and attack my undertaking; I heed them not, and look upon their judgments as rash and contemptible."

The attitude of the true philosopher, as thus described, may properly be assumed not only in the time of COPERNICUS, but also in our own time, which is so filled with extreme specialization, half-learning and superficial and incompetent criticism as to make real progress exceedingly difficult. Those who have failed to solve the problems of Cosmogony, after having written much on the subject, will naturally be very loth to admit that another author could be more fortunate, and will seek to maintain that the questions under discussion are still unsettled. After the mathematical and observational demonstrations given in this work it is felt that they will be welcome to all the satisfaction they may be able to derive from this kind of argument.

The True Object of Scientific Criticism.

In order to be of any value, scientific criticism should have two principal objects in view: (1) The correction of errors and fallacies still current among contem-

porary writers and investigators, in the hope of improving the state of Science; (2) The doing of exact justice to investigators of our own and of former ages, to whom we owe the advancement of our understanding of the physical universe. If the work is well done this enables the whole truth to be brought clearly before the reader. It must be left to others to judge to what extent this object has been attained in the present volume. In estimating the effort here made, however, it should be remembered that the labors of the author have necessarily been those of a pioneer. *Whatever defects may be found to exist in this treatment of a hitherto undeveloped subject, it is only right to say that the criticisms made herein are deemed fair and just, and have been put forth solely with a view of improving our knowledge; and if any other claim is made by those who are not equal to the high philosophic standard set by* HIPPARCHUS 2000 *years ago,* who unjustly blame my undertaking, without understanding it, I too shall heed them not, but look upon their judgments as rash and contemptible.*

For after long and careful meditation I have concluded that unless some one has the courage to brush aside the erroneous doctrines heretofore current, as one would the accumulated dust and cobwebs of ages, we shall never be able to cut loose from antiquated traditions and make lasting progress in reducing Cosmogony to a scientific basis. It has already been remarked that owing to false premises our efforts heretofore have been almost as vain as the weaving of PENELOPE'S web; and the necessity for getting rid of this dull tread-mill of stationary effort, has appeared to justify a stand not one whit less resolute than that which was taken by COPERNICUS when he laid the foundations of the true system of the world.

Concluding Remarks and Acknowledgments.

In the course of the publication of this large volume several important subjects have been developed which were but slightly touched upon in the Introduction. It scarcely seems necessary to dwell upon these now, beyond remarking that whatever differences of opinion may arise from the study of details of the treatment here given, there will be a general agreement that these sublime subjects are properly brought within the domain of the science of Natural Philosophy. Until this is done our theories of the Universe will necessarily remain very incomplete; and the Laws of the Starry Heavens appear to be lacking in that perfect uniformity and continuity which should characterize the true order of Nature.

The scope of this volume is so comprehensive that the author would be sanguine indeed if he dared to anticipate that all parts of it would prove to be entirely

* PTOLEMY justly describes HIPPARCHUS as *ἀνήρ φιλόπονος καὶ φιλαλήθης*, "labor-loving and truth-loving man." The present work is of course intended only for persons who are interested in truth.

satisfactory, and yet he entertains the belief that at least a fair start has been made in the right direction, and indulges the hope that whatever is found to be defective may be not so much reprehended as kindly supplied by the researches of others who may hereafter take up the treatment of these profound questions. *Many of the illustrations included in this volume have been published by other astronomers before, but the interpretation here given is so different from that heretofore current that a new light is thrown upon the most important phenomena of the heavens. It is one thing to have photographs of celestial phenomena, however perfect and satisfactory in themselves; quite another to study them in connection with a rational interpretation. In this work we have accompanied the illustrations by an interpretation based on a simple and natural hypothesis. Nothing is more certain than that heretofore we have had the most unsatisfactory basis for our Theories of Cosmical Evolution.*

The Capture Theory is so overwhelmingly indicated by the most diverse phenomena of the Starry Heavens, that I cannot doubt that it represents an ultimate truth of the very first order of importance. The repulsion of cosmical dust from the stars supplies the diffuse primordial matter which collects here and there into clouds and forms the nebulae; the settlement of the irregular figures of the nebulae thus arising produces chance vortices or whirlpool nebulae, and the result is the development of cosmical systems from the condensation of these spiral masses. The most varied celestial phenomena indicate that the grand order thus disclosed probably is the sublimest of all cosmical processes.

Solid globes, such as moons and planets, have such large masses relatively to their surfaces, that they are scarcely affected by repulsive forces, and therefore they always drift towards the powerful centres of attraction, and are finally captured or absorbed; while the fine dust arising from the vaporization and disintegration of the various masses is so powerfully repelled by all central stars that it is diffused throughout immensity and gives rise to the formation of nebulae, most of which finally settle down and take the spiral form as the result of the secular action of Universal Gravitation.

No slight importance is attached to the theory of repulsive forces as here developed and applied to the sublimest phenomena of Nature. The cause of the congregation of the nebulae near the poles of the Milky Way — one of the greatest and most fundamental facts of the Starry Heavens — has not been assigned before. Indeed, this relation of the nebulae to the Galaxy has completely bewildered astronomers ever since the law first became imperfectly known to Sir William Herschel in 1785, and was more fully confirmed by Sir John Herschel and others about the middle of the 19th century. The profound significance of this great order of the sidereal universe is not to be doubted; and the present simple

explanation, based on the secular effects of repulsive forces, is the outgrowth of ten years of careful thought, a first outline of which was given in an address entitled "Repulsive Forces in Nature," delivered at the University of Cincinnati, by invitation of the President, DR. HOWARD AYERS, the eminent biologist, April 7, 1902; and published in *Popular Astronomy* for December of that year.

Not the least significant fact about the Capture Theory of Cosmical Evolution is the way in which each part of the theory confirms and supports the other, so as to make a finished edifice at once imposing and strong enough to weather the storms of the ages. Thus the established origin of comets gives an immediate connection with the original state of the embryo planets when they were developing at a great distance from the Sun; and a similar mutual support is afforded by every other part of the theory, which may therefore be regarded as firmly established.

In concluding this volume the author acknowledges gratefully his indebtedness to a number of eminent astronomers not sufficiently mentioned in the Introduction: the late SIR WILLIAM HUGGINS, one of the most illustrious astronomers of all time, whose long life was devoted wholly to the advancement of Truth, and who took an abiding interest in the progress of this work from the first publication twenty years ago; PROFESSOR E. E. BARNARD, of the Yerkes Observatory, for his characteristic liberality in granting the use of the magnificent photographs of the Milky Way; PROFESSOR FRANK SCHLESINGER, Director of the Allegheny Observatory, for valuable data on spectroscopic binary stars; PROFESSOR W. W. CAMPBELL, Director of Lick Observatory, for permission to use the Lick photographs of nebulae; DR. H. D. CURTIS, of the Lick Observatory, for the photograph of the Cluster *Omega Centauri*; MR. S. S. HOUGH, H. M. Astronomer at the Cape of Good Hope, for several photographs of Southern Clusters; DR. P. H. COWELL, recently of the Royal Observatory, Greenwich, now Superintendent of the British Nautical Almanac, for useful suggestions regarding the Moon's secular acceleration; MR. J. K. FOTHERINGHAM, M.A., of Oxford, for valuable suggestions regarding Ancient Eclipses; PROFESSOR E. W. BROWN, the eminent Lunar Theorist, of Yale University, for several suggestions regarding the motion of the Moon, and the outstanding uncertainty of its positions in ancient times; CAPTAIN A. W. DODD, U.S.N., of Mare Island, who verified the impact theory of the Lunar Craters, experimentally; Rear ADMIRAL C. H. DAVIS, U.S.N., Superintendent, and PROFESSOR S. J. BROWN, U.S.N., Astronomical Director, under whose administration of the Naval Observatory at Washington, I was enabled to measure all the planets and satellites of the solar system, and to detect the faint belts on *Neptune*.

An incident in the rounding out of the Capture Theory should be noted here. After I had developed the theory of the capture of the satellites, with perhaps the exception of the Moon, from the data supplied by BABINET'S criterion, I had the opportunity of discussing the whole development with my friend CAPTAIN CHAS. E. FOX, U.S.N., at Mare Island, December 10, 1908. When he had heard the argument regarding the development of the planets and satellites, he remarked that the Moon could not well be an exception to the general rule of our system; and he urged me to make a further examination of this particular case before the new theory was published. I entered upon the work at once, but owing to grave illness beginning January 11, 1909, was unable to complete the examination of the subject till May 24th, when the result was cabled to the *Astronomische Nachrichten* and published in No. 4325.

Whilst the author is deeply indebted to these gentlemen and others who have been good enough to aid him by helpful suggestions, *it must be distinctly understood that not one of them, directly or indirectly, is responsible for anything contained in this volume*; the author alone assumes full responsibility for the theory here developed and for such defects as it may be found to contain. In one or two instances the names of advisers on particular topics still in academic dispute have been indicated, not with any thought of throwing responsibility on others, but simply as the only means of making a faithful record of contemporary activity. The author did not wish credit for the first rejection of certain erroneous doctrines to be given to himself, when in fact this honor belonged to others. This point should not be misunderstood by those who read this book.

It would be a poor appreciation of the kind interest shown in the author's scientific efforts for many years if he failed to record his indebtedness to several eminent public men, but more especially to the friend of his youth, the HONORABLE D. R. FRANCIS, of St. Louis, the foremost citizen of Missouri, and to the late illustrious SENATOR W. B. ALLISON, of Iowa, who for thirty-five years was so great an ornament to our Senate, and whose enlightened statesmanship contributed so substantially to the development of the scientific life of our country.

MR. LAWRENCE TIERNAN, the efficient assistant at the Naval Observatory, Mare Island, has likewise contributed to the completion of this volume; and I wish to mention also the valuable services of MR. W. R. SMITH, draftsman, in the preparation of several plates, and also of MR. JOHN HAROLD, by whom most of the manuscript was copied for the printer.

It is a great pleasure to speak of the untiring labors of the Publishers, MESSRS. THOS. P. NICHOLS & SONS, for the perfection of this work. Whatever excellence of appearance the volume may present must be ascribed largely to their zeal,

steadfast devotion and personal attention to the art of printing, for which this well established firm is so justly celebrated.

Acknowledgments are due also to the Binner-Wells Company, of Chicago, by whom most of the engravings were made. Their work speaks for itself and requires no comment. The Andersen-Lamb Photogravure Company, of New York, has produced the excellent photogravure plates of the Nebula and Clusters of Stars, and thereby contributed materially to the perfection of the work.

Finally, I wish to record my indebtedness to my Brother, Mr. M. F. See, for an appreciative interest in this work which contributed greatly to its proper publication. But of all the persons to whom I am indebted, I owe most to my wife, Mrs. Frances Graves See. Without her devotion through a dangerous illness, the author could scarcely have survived to finish the work, and without her constant support and encouragement the steadfast labor and sacrifices required for the development and publication of this large volume could not have been undertaken. If it contains any important discoveries I wish it always to be remembered that she contributed in an eminent degree to their development and presentation to the scientific world.

T. J. J. SEE.

Starlight, Blue Ridge on Loutre,
Montgomery City, Missouri,
July 14, 1910.

Ἐν μὲν γαῖαν ἔτευξ', ἐν δ' οὐρανόν, ἐν δὲ θάλασσαν,
ἠέλιόν τ' ἀκάμαντα σελήνην τε πλήθουσαν,
ἐν δὲ τὰ τείρεα πάντα, τά τ'οὐρανὸς ἐστεφάνωται,
Πληιάδας θ' Ὑάδας τε τό τε σθένος 'Ωρίωνος
ἄρκτον θ', ἣν καὶ ἄμαξαν ἐπίκλησιν καλέουσιν,
ἥ τ' αὐτοῦ στρέφεται καί τ' 'Ωρίωνα δοκεύει,
οἴη δ' ἄμμορός ἐστι λοετρῶν 'Ωκεανοῖο·

Therein he wrought the Earth, and the Heavens, and the Sea,
The unwearied Sun and the full Moon,
And all the constellations with which the Heavens are crowned,
The *Pleiades*, the *Hyades*, the strength of *Orion*,
And the *Bear*, which they also call by the appellation of the *Wain*,
Which there revolves and watches *Orion*,
But is alone unwashed by Ocean's briny bath.

—ILIAD, XVIII, 483–489.

— FINIS. —